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Non-Singular Trees, Unicyclic Graphs and Bicyclic Graphs

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Abstract

We called graph G non-singular if adjacency matrix $A(G)$ of G is non-singular. A connected graph with n vertices and $n-1$, n and $n+1$ edges are called the tree, the unicyclic graph and the bicyclic graph. Respectively, as we all know, each connected bicyclic graph must contain $\infty(a,s,b)$ or $\theta(p,l,q)$ as the induced subgraph. In this paper, by using three graph transformations which do not change the singularity of the graph, the non-singular trees, unicyclic graphs and bicyclic graphs are obtained.

Keywords

Adjacency Matrix, Non-Singular, Rank, Nullity

1. Introduction

This paper considers only finite undirected simple graphs. Let G be a graph with order n, the matrix A is defined as the adjacency matrix of graph G, which is defined as follows: $A(G)=(a_{ij})_{n\times n}$

$$a_{ij}=\begin{cases}1 & \text{if } i\sim j,\\ 0 & \text{others.}\end{cases}$$

Obviously, $A(G)$ is a real symmetric matrix with diagonal elements of 0 and other elements of 0 or 1, its eigenvalues are real number. The eigenvalues of $A(G)$ are also said to be the eigenvalues of the graph G, collection n eigenvalues of G to form the spectrum of this graph. The number of nonzero eigenvalues and zero eigenvalues in the spectrum of the graph G are called the rant and nullity of the graph G, and denoted by $r(G)$ and $\eta(G)$, respectively, obviously $r(G)+\eta(G)=n$.

The chemist discovered, $\eta(G)=0$ is a necessary condition for the chemical stability of the molecules shown in the graph G [1] [2]. 1957, in [2], Collatz and

Sinogowitz put forward a question that how to find out all singular graphs (equivalently, to show all the nonsingular graphs), namely, to describe all graphs with nullity are greater than zero, it was a very difficult problem only with some special results [3] [4] [5] [6] [7]. This has inspired a lot of researches on nullity of graph [8]-[14]. In this paper, by using three graph transformations which do not change the singularity, the non-singular trees, unicyclic graphs and bicyclic graphs are obtained. This method is different from previous.

A connected graph of order n, the graph respectively with size $n-1, n$ and $n+1$ are called the tree, the unicyclic graph and the bicyclic graph. As usual, P_n, C_n and K_n denoted respectively the path, the cycle and the complete graph on n vertices. K_0 denoted the null graph (namely, a graph without vertex). The graph obtained by bonding the two ends of the $P_s(s \geq 1)$ to one vertex on C_a and C_b is recorded as $\infty(a,s,b)$, when $s=1$, $\infty(a,1,b)$ denotes the graph obtained by bonding a vertex on C_a with a vertex on C_b. The graphs obtained by bonding the starting vertices and the end vertices of the P_l, P_p and P_q are respectively is recorded as $\theta(p,l,q)$, where $\min\{p,l,q\} \geq 2$ and one of them at most is 2 (as shown in **Figure 1**). As everyone knows, each connected bicyclic graph must contain $\infty(a,s,b)$ or $\theta(p,l,q)$ as the induced subgraph. $G \cup H$ denotes the union graph of G and H. An edge relates a vertex of degree 1, then the vertex is called a pendant vertex, and the vertex adjacent to the pendant vertex is called the quasi-pendant vertex. The notation and terminology that are not described here are provided in [1].

2. Some Lemmas

Let A, B be two n-order real symmetric matrices, if there is an invertible matrix P of order n such that $P^{\mathrm{T}}AP = B$, then we say that A is congruent to B, denoted by $A \simeq B$.

Lemma 2.1. [15] Let A, B be n-order real symmetric matrices of two congruent, then $r(A) = r(B)$, $\eta(A) = \eta(B)$.

Lemma 2.2. Let $G = G_1 \cup G_2 \cup \cdots \cup G_t$, where $G_i(i=1,2,\cdots,t)$ are connected components of G. Then $\eta(G) = \sum_{i=1}^{t} \eta(G_i)$. Equivalent G is non-singular if and only if every $G_i(i=1,2,\cdots,t)$ is non-singular.

Lemma 2.3. [1] Let G have a pendant vertex, H is the graph obtained by deleting the pendant vertex from graph G and the quasi-pendant vertex adjacent to it, then $\eta(G) = \eta(H)$. Equivalently, G is non-singular if and only if every H is non-singular.

Lemma 2.4. [16] Let G be a graph containing path with four vertices of degree 2, let the graph H be obtained from G by replacing this path with an edge (as shown in **Figure 2**). Then $\eta(G) = \eta(H)$. Equivalently, G is non-singular if and only if every H is non-singular.

Lemma 2.5. [16] Let G be a graph containing two vertices and four edges of a cycle of length 4, positioned as shown in **Figure 3**. Let the graph H be obtained from G by removing this cycle. Then $\eta(G) = \eta(H)$. Equivalently G is non-singular if and only if every H is non-singular.

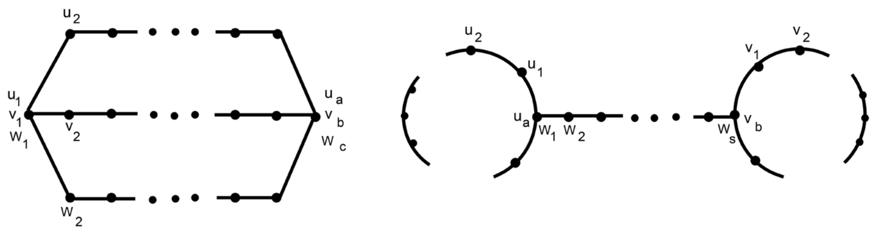

Figure 1. Graph $\theta(p,l,q)$ and $\infty(a,s,b)$.

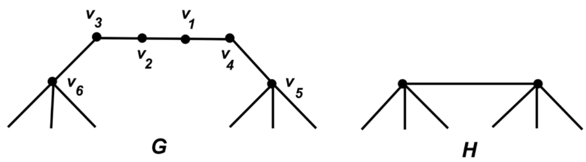

Figure 2. Graph transformation (*II*) which does not change the singularity of the graph.

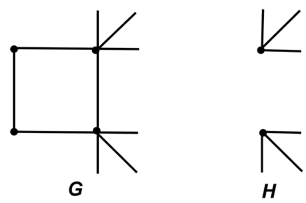

Figure 3. Graph transformation (*III*) which does not change the singularity of the graph.

We called the three graph transformations are given which does not change the singularity of the graph in Lemma 2.3, Lemma 2.4 and Lemma 2.5 are the graph transformation *I*, the graph transformation *I* and the graph transformation *III*, respectively.

We called the subgraph is elementary subgraph of G in which every component is K_2 or cycle. If a elementary subgraph of G contains all the vertices of G, then called this elementary subgraph that is the elementary spanning subgraph of G.

Lemma 2.6. [1] If G is a graph with n vertices and adjacency matrix is $A(G)$, then

$$det\left(A(G)\right)=(-1)^n\sum_{H\in\mathcal{H}}(-1)^{p(H)}2^{c(H)},$$

where $\mathcal{H}$ is the set of elementary spanning subgraph of G, $p(H)$ denotes the number of components of H and $c(H)$ denotes the number of cycles in H.

3. Main Theorems

We agree K_0 is non-singular and denoted by $\Omega_1=\{K_0\}$.

Theorem 3.1. If T is a tree with order n, then T is non-singular if and only if T is changed to K_0 by a series of the graph transformation *I*.

Proof. As we all know, T is changed to $H=rK_1$ by a series of the graph transformation *I*, T is non-singular if and only if H is non-singular. When $r>0$, then H is singular, when $r=0$, then $H=K_0$ is non-singular. So T is non-singular if and only if T is changed to K_0 by a series of the graph transformation *I*.

Corollary 3.1. If a tree is non-singular if and only if T has a perfect matching.

Lemma 3.1. If a cycle C_n is non-singular if and only if $4 \nmid n$.

Proof. According Lemma 2.6, we can easily calculate $det(A(C_3))=2$, $det(A(C_4))=0$, $det(A(C_5))=2$, $det(A(C_6))=-4$. Let $n=4q+r$, $q \geq 0$, $3 \leq r \leq 6$, C_n is changed to C_r by a series of the graph transformation *II*. Therefore C_n is non-singular if and only if C_r is non-singular and if and only if $4 \nmid n$.

Denoted by $\Omega_2=\{C_3,C_5,C_6\}$.

Theorem 3.2. Let G is a unicyclic graph, then G is non-singular if and only if it is changed to $H \in \Omega_1 \cup \Omega_2$ by a series of the graph transformation *I* and the graph transformation *II*.

Proof. As everyone knows, the unicyclic graph can always change to rK_1, C_n or $rK_1 \cup C_n$ through a series of the graph transformation *I*, and change to $H=rK_1$, C_s or $rK_1 \cup C_s \ (3 \leq s \leq 6)$ through a series of the graph transformation *II*. By the Theorem 3.1 and the Lemma 3.1, unicyclic graph G is non-singular if and only if H is non-singular, and if and only if $H \in \Omega_1 \cup \Omega_2$.

Denoted by

$$\begin{aligned}\Gamma_1=\{&\infty(3,1,3),\infty(3,2,3),\infty(3,3,3),\infty(3,4,3),\infty(3,5,3),\infty(3,2,5),\\&\infty(3,4,5),\infty(3,1,6),\infty(3,2,6),\infty(3,3,6),\infty(3,4,6),\infty(3,5,6),\\&\infty(5,1,5),\infty(5,2,5),\infty(5,3,5),\infty(5,4,5),\infty(5,5,5),\infty(5,1,6),\\&\infty(5,2,6),\infty(5,3,6),\infty(5,4,6),\infty(5,5,6),\infty(6,2,6),\infty(6,4,6)\}.\end{aligned}$$

Lemma 3.2. If $G=\infty(a,s,b)$, then G is non-singular if and only if G is changed to $H \in \Gamma_1$ by a series of the graph transformation *II*.

Proof. At first, we can prove graphs in the Γ_1 are non-singular. Taking $\infty(3,2,6)$ as an example, it has a elementary spanning subgraph $C_6 \cup C_3$; two $C_3 \cup 3K_2$, so by Lemma 2.6, we can easily calculate $det(A(\infty(3,2,6)))=(-1)^9\left[(-1)^2 2^2+(-1)^4 2\times 2\right]=-8$. The others are similar, so we have following:

$$\begin{aligned}&det(A(\infty(3,1,3)))=-4,\ det(A(\infty(3,2,3)))=3,\ det(A(\infty(3,3,3)))=4,\\&det(A(\infty(3,4,3)))=-3,\ det(A(\infty(3,5,3)))=-4,\ det(A(\infty(3,2,5)))=5,\\&det(A(\infty(3,4,5)))=-5,\ det(A(\infty(3,1,6)))=4,\ det(A(\infty(3,2,6)))=-8,\\&det(A(\infty(3,3,6)))=-4,\ det(A(\infty(3,4,6)))=8,\ det(A(\infty(3,5,6)))=4,\end{aligned}$$

$$\begin{aligned}&det(A(\infty(5,1,5)))=4,\ det(A(\infty(5,2,5)))=3,\ det(A(\infty(5,3,5)))=-2,\\&det(A(\infty(5,4,5)))=-3,\ det(A(\infty(5,5,5)))=4,\ det(A(\infty(5,1,6)))=-4,\\&det(A(\infty(5,2,6)))=-8,\ det(A(\infty(5,3,6)))=4,\ det(A(\infty(5,4,6)))=8,\\&det(A(\infty(5,5,6)))=-4,\ det(A(\infty(6,2,6)))=16,\ det(A(\infty(6,4,6)))=-16.\end{aligned}$$

Denoted by

$$\Delta_1 = \{\infty(3,1,4), \infty(3,2,4), \infty(3,3,4), \infty(3,4,4), \infty(3,5,4), \infty(3,1,5), \infty(3,3,5), \infty(4,1,4), \infty(4,2,4), \infty(4,3,4), \infty(4,4,4), \infty(4,5,4), \infty(4,1,5), \infty(4,2,5), \infty(4,3,5), \infty(4,4,5), \infty(4,5,5), \infty(4,1,6), \infty(4,2,6), \infty(4,3,6), \infty(4,4,6), \infty(4,5,6), \infty(6,1,6), \infty(6,3,6), \infty(6,5,6)\}.$$

Then we prove graphs in the Δ_1 are singular. For graph $\infty(4,s,b)$ or $\infty(a,s,4)$, we use the graph transformation *III*, will get a graph which contains K_1, yet K_1 provides a zero eigenvalue, so these kind of graphs are singular.

For the others, we have following:

$$det(A(\infty(3,1,5))) = 0,\ det(A(\infty(3,3,5))) = 0,$$
$$det(A(\infty(6,1,6))) = 0,\ det(A(\infty(6,3,6))) = 0,\ det(A(\infty(6,5,6))) = 0.$$

At last, let $a \ge 3, b \ge 3, s \ge 1$, and $a = 4i + a'(3 \le a' \le 6)$, $b = 4j + b'(3 \le b' \le 6)$, $s = 4k + a'(1 \le s' \le 5)$, for graph $\infty(a,s,b)$, we repeatedly use the graph transformation *II*, will get graph $\infty(a',s',b') \in \Gamma_1 \cup \Delta_1$. So $\infty(a,s,b)$ is non-singular if and only if $\infty(a',s',b')$ is non-singular, if and only if $\infty(a',s',b') \in \Gamma_1$.

Denoted by

$$\Gamma_2 = \{\theta(2,3,5), \theta(2,3,6), \theta(2,4,4), \theta(2,4,6), \theta(2,5,6), \theta(2,6,6), \theta(3,4,4), \theta(3,4,5), \theta(4,4,4), \theta(4,4,5)\}$$

Lemma 3.3. Let $G = \theta(p,l,q)$, then G is non-singular if and only if G is changed to $H \in \Gamma_2$ through a series of the graph transformation *II*.

Proof. First of all, we can prove that the graphs in the Γ_2 are non-singular. Taking $\theta(2,3,5)$ as an example, it has a elementary spanning subgraph C_6; two $3K_2$, so by Lemma 2.6, we can calculate $det(A(\theta(2,3,5))) = (-1)^6[(-1)^1 2 + (-1)^3 \times 2] = -4$. The others are similar, so we have following:

$$det(A(\infty(2,3,6))) = 4,\ det(A(\infty(2,4,4))) = -1,\ det(A(\infty(2,4,6))) = 1,$$
$$det(A(\infty(2,5,6))) = 4,\ det(A(\infty(2,6,6))) = -9,\ det(A(\infty(3,4,4))) = -4,$$
$$det(A(\infty(3,4,5))) = 4,\ det(A(\infty(4,4,4))) = 9,\ det(A(\infty(4,4,5))) = -4.$$

Denoted by

$$\Delta_2 = \{\theta(2,3,3), \theta(2,3,4), \theta(2,4,5), \theta(2,5,5), \theta(3,3,3), \theta(3,3,4), \theta(3,3,5), \theta(3,5,5), \theta(4,5,5), \theta(5,5,5)\}.$$

We can easily prove that graphs in the Δ_2 are singular.

At last, let $p \ge 2, l \ge 3, q \ge 3$, and $p = 4i + p'(2 \le p' \le 5)$, $l = 4j + l'(2 \le l' \le 5)$, $q = 4k + q'(2 \le q' \le 5)$, for graph $\theta(p,l,q)$, we repeatedly use the graph transformation *II*, will get graph $\theta(p',l',q') \in \Gamma_2 \cup \Delta_2$. So, $\theta(p,l,q)$ is non-singular if and only if $\theta(p',l',q')$ is non-singular, if and only if $\theta(p',l',q') \in \Gamma_2$.

Denoted by $\Omega_3 = \Gamma_1 \cup \Gamma_2$.

Theorem 3.3. Let G be a bicyclic graph, then G is non-singular if and only if G is changed to $H \in \Omega_1 \cup \Omega_2 \cup \Omega_3$ by a series of the graph transformation I and the graph transformation *II*.

Proof. As everyone knows, the bicyclic graph can always change to rK_1, C_n, $\infty(a,s,b)$, $\theta(p,l,q)$, $rK_1 \cup C_n$, $rK_1 \cup \infty(a,s,b)$ or $rK_1 \cup \theta(p,l,q)$ through a series of the graph transformation *I*, and change to $H = rK_1 \cup H'$, $H' \in \{K_0, C_r\,(3 \le r \le 6)\} \cup \Gamma_1 \cup \Delta_1 \cup \Gamma_2 \cup \Delta_2$ through a series of the graph transformation *II*. By the Lemma 2.3 and the Lemma 2.4, bicyclic graph G is non-singular if and only if H is non-singular, and by the Theorem 3.1, Theorem 3.2, Lemma 3.2 and the Lemma 3.3, H is non-singular if and only if $H = H' \in \Omega_1 \cup \Omega_2 \cup \Omega_3$.

4. Conclusion

By using three graph transformations which do not change the singularity of the graph, we found the non-singular trees, unicyclic graphs and bicyclic graphs. After writing this paper, I am very inspired; therefore, I want to do further research on the rank of graphs of more complex structures and so on.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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