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A Tutorial to Approximately Invert the Sumudu Transform

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Abstract

Unlike the traditional Laplace transform, the Sumudu transform of a function, when approximated as a power series, may be readily inverted using factorial-based coefficient diminution. This technique offers straightforward computational advantages for approximate range-limited numerical solutions of certain ordinary, mixed, and partial linear differential and integro-differential equations. Furthermore, discrete convolution (the Cauchy product), may also be utilized to assist in this approximate inversion method of the Sumudu transform. Illustrative examples are provided which elucidate both the applicability and limitations of this method.

Keywords

Sumudu Transform, Double Series, Discrete Convolution, Cauchy Product

1. Introduction

Transform mathematics has traditionally been utilized for obtaining solutions of differential and integro-differential equations (DEs and IDEs) which arise in control theory, engineering, and related areas such as pharmacologic and mathematical modeling [1] [2] [3]. In general, transform mathematics allows for the conversion of differentiation and integration into algebraic processes which yield a preliminary solution that is expressed within the transform domain. Subsequent inversion of this transform function then produces the actual solution of the original DE or IDE which is represented in terms of moments.

The Laplace transform is classically utilized for this purpose [4] [5]. Typically, the inversion of a Laplace transform is accomplished using pre-existing tables which facilitate this process. Partial fraction expansion may be helpful, in simplifying an expression within the Laplace domain, but is not always reliable in generating a readily invertible result.

Therefore, when a tabular result is unavailable, inversion of the Laplace transform may be difficult; requiring integration within the complex plane. Approximate Laplace transform inversion techniques exist which are based upon numerical methods [6], Fourier analysis [7], or repetitive symbolic differentiation [8].

Relatively recently, alternative mathematical transforms have been developed. Specifically, the Sumudu transform is defined as [9]:

$$G(u)=\frac{1}{u}\int_0^\infty g(t)\mathrm{e}^{\frac{-t}{u}}\,\mathrm{d}t. \tag{1}$$

It should be noted that the Sumudu transform of a real function yields results, within the u-domain, which are also real. Additionally, the Sumudu transform "preserves dimensions." Thus, whatever physical dimension $g(t)$ has $G(u)$ will also have [9] [10] [11]. This is particularly helpful when checking for algebraic accuracy.

For simplicity, the Sumudu transform process will be referred to using the $\boldsymbol{S}$ operator:

$$G(u)=\boldsymbol{S}\left[g(t)\right]. \tag{2}$$

Whereas the inversion process will be referred to as:

$$g(t)=\boldsymbol{S}^{-1}\left[G(u)\right]. \tag{3}$$

Note that the traditional Laplace transform is defined using the $\boldsymbol{L}$ operator:

$$\boldsymbol{L}\left[g(t)\right]=F(s)=\int_0^\infty g(t)\mathrm{e}^{-st}\,\mathrm{d}t. \tag{4}$$

Inspection of (1) and (4) demonstrates that the conversion, from a Laplace to a Sumudu transform, is obtained by using the substitution of $\frac{1}{u}$ for s with the subsequent multiplication of the Laplace transform by $\frac{1}{u}$:

$$\underbrace{G(u)=\boldsymbol{S}\left[g(t)\right]}_{\text{Sumudu}}=\frac{1}{u}\underbrace{\boldsymbol{L}\left[g(t)\right]}_{\text{Laplace}}\Bigg|_{s=\frac{1}{u}}=\frac{1}{u}\underbrace{F(s)}_{\text{Laplace}}\Bigg|_{s=\frac{1}{u}}. \tag{5}$$

A "duality" between the Laplace and Sumudu transforms therefore exists in terms of similar mathematical properties regarding linearity, convolution, differentiation, and integration. These topics have been explored and discussed previously [10] [11]. Furthermore, tables of Sumudu transforms are available which allow for straightforward conversions, between the u-domain and t-domains, for many commonly used functions [10] [11].

If t in the above equation has units of time, then s consequently has units of complex frequency, whereas u would also have the dimension of time. As previously stated, the Sumudu transform "preserves units."

The purpose of this paper is to demonstrate an approximate inversion process of the Sumudu transform utilizing a geometric power series technique which uses non-negative integer values for n:

$$G(u) \approx \sum_{n=0}^{N} (a_n) u^n. \tag{6}$$

Furthermore, this approximate inversion process may only apply within a narrow numerical range owing to the limitations of a truncated geometric power series as a means of representing a function.

Note that $G(u)$ in (6) would also be continuously differentiable or "smooth." In addition, a_n is a constant coefficient. Moreover, this geometric power series method can also be combined, utilizing superposition, with single or multiple known "pre-existing" Sumudu transform or transforms; such as those available from a table:

$$G(u) \approx \left(\boldsymbol{S}_1 \left[g_1(t) \right] + \boldsymbol{S}_2 \left[g_2(t) \right] + \boldsymbol{S}_3 \left[g_3(t) \right] + \cdots \right) + \sum_{n=0}^{N} (a_n) u^n. \tag{7}$$

Multiple geometric power series may also be combined in an additive or multiplicative manner.

The Sumudu Transform of a Power Series

The Sumudu transform of a power series in the t-domain is a "factorial-based" *amplified* power series in the u-domain (*See* **Appendix A**) [9]. Therefore:

$$\boldsymbol{S}\left[t^0 \right] = 0! u^0 = 1, \tag{8}$$

$$\boldsymbol{S}\left[t^1 \right] = 1! u^1 = u, \tag{9}$$

$$\boldsymbol{S}\left[t^2 \right] = 2! u^2 = 2u^2, \tag{10}$$

$$\boldsymbol{S}\left[t^3 \right] = 3! u^3 = 6u^3, \tag{11}$$

$$\boldsymbol{S}\left[t^4 \right] = 4! u^4 = 24u^4. \tag{12}$$

Thus, for integer values of $n \ge 0$:

$$\boldsymbol{S}\left[t^n \right] = n! u^n. \tag{13}$$

A summation of multiple power *terms*, in the time domain, could then be expressed as a power *series*:

$$g(t) \approx \sum_{n=0}^{N} g(t,n) = \underbrace{a_0 t^0}_{g(t,0)} + \underbrace{a_1 t^1}_{g(t,1)} + \cdots + \underbrace{a_N t^N}_{g(t,N)} = \sum_{n=0}^{N} a_n t^n. \tag{14}$$

The corresponding Sumudu transform of the above equation would therefore be:

$$G(u) \approx \sum_{n=0}^{N} G(u,n) = \underbrace{a_0 (0!) u^0}_{G(u,0)} + \underbrace{a_1 (1!) u^1}_{G(u,1)} + \cdots + \underbrace{a_N (N!) u^N}_{G(u,N)} = \sum_{n=0}^{N} a_n (n!) u^n. \tag{15}$$

Use of *factorial-based coefficient diminution* (FBCD) subsequently yields the inversion of (15) "back to" (14):

$$\sum_{n=0}^{N} g(t,n) = \left. \sum_{n=0}^{N} \frac{G(u,n)}{n!} \right|_{t=u}. \tag{16}$$

Note that t has to be substituted for u on the RHS of the above equation. Therefore, functions which can be approximated with a geometric power series, expressed with the form of (6) in the *u*-domain, may be readily inverted, back to the *t*-domain, using the aforementioned technique. Thus, approximate solutions of certain types of both linear differential equations, as well as linear integro-differential equations, can be generated. However, these approximate solutions may also be range-limited.

In addition, a Sumudu transform is frequently a rational function expressed within the *u*-domain:

$$G(u)=\frac{P(u)}{Q(u)}. \tag{17}$$

However, $G(u)$ may sometimes be expressed utilizing transcendental functions. Nonetheless, approximating $G(u)$ as a geometric power series in the form of (6) can often be accomplished using commonly-known mathematical techniques and algorithms. Computer-based symbolic processors can also be utilized. *As previously stated, geometric power series representations of functions may have limitations which restrict the acceptable accuracy of this approximation to that of a relatively narrow numerical range*.

Lastly, a Sumudu transform may sometimes be expressed as a product of two or more rational functions:

$$G(u)=\left[\frac{P_n(u)}{Q_n(u)}\cdot\frac{P_{n+1}(u)}{Q_{n+1}(u)}\cdots\frac{P_N(u)}{Q_N(u)}\right]=\prod_{n=0}^{N}\left(\frac{P_n(u)}{Q_n(u)}\right). \tag{18}$$

Each rational function would then be approximated as a unique geometric power series. Consequently, $G(u)$ would be represented as the product of two or more infinite series:

$$G(u)\approx\left[\sum_{m=0}^{\infty}{}^{(n)}a_m u^m\cdot\sum_{m=0}^{\infty}{}^{(n+1)}a_m u^m\cdots\sum_{m=0}^{\infty}{}^{(N)}a_m u^m\right]. \tag{19}$$

So that:

$$G(u)\approx\prod_{n=0}^{N}\left(\sum_{m=0}^{\infty}{}^{(n)}a_m u^m\right). \tag{20}$$

where:

$$\frac{P_n(u)}{Q_n(u)}\approx\sum_{m=0}^{\infty}{}^{(n)}a_m u^m, \tag{21}$$

$$\frac{P_{n+1}(u)}{Q_{n+1}(u)}\approx\sum_{m=0}^{\infty}{}^{(n+1)}a_m u^m, \tag{22}$$

and:

$$\frac{P_N(u)}{Q_N(u)}\approx\sum_{m=0}^{\infty}{}^{(N)}a_m u^m. \tag{23}$$

Note that: ${}^{(n)}a_m$, ${}^{(n+1)}a_m$, and ${}^{(N)}a_m$ all represent series-specific constant coefficients. Furthermore, the above product of multiple series can be combined using discrete convolution (the Cauchy product) [12] [13]. However, each series must be truncated.

As will be shown later, this technique can also be utilized as an approximation-based means to invert a partial fraction expansion from the u-domain back to the t-domain.

2. Methods

Numerical analysis and the conversion of rational functions into geometric power series were accomplished using Mathcad (PTC Corporation, MA, USA). Graphs were prepared using Excel (Microsoft Corporation, WA, USA).

3. Preliminary Examples of the Exponential, Sine, and Cosine Functions

3.1. Exponential Function

The Sumudu transform of the exponential function is [9]:

$$\boldsymbol{S}\left[\mathrm{e}^{at}\right]=\frac{1}{u}\int_0^{\infty}\mathrm{e}^{at}\mathrm{e}^{\frac{-t}{u}}\mathrm{d}t=\frac{1}{u}\int_0^{\infty}\mathrm{e}^{\left(a-\frac{1}{u}\right)t}\mathrm{d}t,\quad |ua|<1. \tag{24}$$

Thus:

$$\boldsymbol{S}\left[\mathrm{e}^{at}\right]=\frac{1}{u\left(a-\frac{1}{u}\right)}\mathrm{e}^{\left(a-\frac{1}{u}\right)t}\Bigg|_0^{\infty}=0-\frac{1}{(ua-1)}=\frac{1}{(1-au)}. \tag{25}$$

Therefore, the Sumudu transform of the exponential function subsequently expressed as a power series is:

$$\boldsymbol{S}\left[\mathrm{e}^{at}\right]=\frac{1}{(1-au)}\approx 1+au+a^2u^2+a^3u^3+\cdots \tag{26}$$

Use of FBCD with substitution of t for u then yields the well-known Taylor's series of an exponential function. Thus, the above Sumudu transform is approximately inverted without the need for integration within the complex plane:

$$\mathrm{e}^{at}\approx\frac{1}{0!}+\frac{at}{1!}+\frac{a^2t^2}{2!}+\frac{a^3t^3}{3!}+\cdots\approx\sum_{n=0}^{\infty}\frac{a^nt^n}{n!}. \tag{27}$$

3.2. Sine Function

Using (1) the Sumudu transform of the sine function is [9]:

Expressing the above as a power series:

$$\boldsymbol{S}\left[\sin(at)\right]=\frac{au}{\left(1+a^2u^2\right)}. \tag{28}$$

$$\frac{au}{\left(1+a^2u^2\right)}\approx au-a^3u^3+a^5u^5-a^7u^7+a^9u^9+\cdots \tag{29}$$

By using FBCD, the above Sumudu transform, expressed as a power series, is approximately inverted. This yields the Taylor series of a sine function. Note that u has been replaced by t:

$$\sin(at) \approx \frac{at}{1!} - \frac{a^3t^3}{3!} + \frac{a^5t^5}{5!} - \frac{a^7t^7}{7!} + \frac{a^9t^9}{9!} + \cdots \tag{30}$$

Equivalently:

$$\sin(at) \approx \sum_{n=0}^{\infty} (-1)^n \frac{(at)^{(2n+1)}}{(2n+1)!}. \tag{31}$$

3.3. Cosine Function

The Sumudu transform of the cosine function is found in a manner similar to that of the sine function [7]:

$$\boldsymbol{S}\left[\cos(at)\right] = \frac{1}{\left(1+a^2u^2\right)}. \tag{32}$$

Therefore:

$$\frac{1}{\left(1+a^2u^2\right)} \approx 1 - a^2u^2 + a^4u^4 - a^6u^6 + a^8u^8 + \cdots \tag{33}$$

The Taylor's series for the cosine function is then approximated using FBCD along with substitution of t for u within the above equation:

$$\cos(at) \approx \frac{1}{0!} - \frac{a^2t^2}{2!} + \frac{a^4t^4}{4!} - \frac{a^6t^6}{6!} + \frac{a^8t^8}{8!} + \cdots \tag{34}$$

So that:

$$\cos(at) \approx \sum_{n=0}^{\infty} (-1)^n \frac{(at)^{(2n)}}{(2n)!}. \tag{35}$$

4. Differentiation and Integration within the Sumudu Domain

Differentiation and integration, with the Sumudu transform, have an "inverse resemblance" to that of Laplace transforms [10] [11]. The first derivative is:

$$\boldsymbol{S}\left[\frac{\mathrm{d}g}{\mathrm{d}t}\right] = \frac{G(u)}{u} - \frac{g(0)}{u}. \tag{36}$$

Whereas the second derivative is:

$$\boldsymbol{S}\left[\frac{\mathrm{d}^2g}{\mathrm{d}t^2}\right] = \frac{G(u)}{u^2} - \frac{g(0)}{u^2} - \frac{\left(\left.\frac{\mathrm{d}g}{\mathrm{d}t}\right|_{t=0}\right)}{u}. \tag{37}$$

Higher-order derivatives can also be determined:

$$\boldsymbol{S}\left[\frac{\mathrm{d}^ng}{\mathrm{d}t^n}\right] = \frac{G(u)}{u^n} - \frac{g(0)}{u^n} - \cdots - \frac{\left(\left.\frac{\mathrm{d}^{(n-1)}g}{\mathrm{d}t^{(n-1)}}\right|_{t=0}\right)}{u}. \tag{38}$$

Furthermore, integration within the Sumudu transform domain can be summarized as:

$$\boldsymbol{S}\left[\underbrace{\int\cdots\iint}_{n} g(t)(\mathrm{d}t)^{n}\right]=u^{n}G(u). \tag{39}$$

Sumudu Shift Theorem

The Sumudu transform of a function $t\cdot g(t)$ can be expressed using the Sumudu shift theorem and $G(u)$:

$$\boldsymbol{S}\left[t\cdot g(t)\right]=u\left[G(u)+u\frac{\mathrm{d}}{\mathrm{d}u}G(u)\right]. \tag{40}$$

Making use of the product rule yields:

$$\boldsymbol{S}\left[t\cdot g(t)\right]=u\frac{\mathrm{d}\left[uG(u)\right]}{\mathrm{d}u}. \tag{41}$$

5. Approximate Solutions to Certain Linear Differential Equationsand Integro-Differential Equations

5.1. Example 1

Consider the fourth-order linear inhomogeneous differential equation with all initial conditions equal to zero:

$$\frac{\mathrm{d}^4 g}{\mathrm{d}t^4}+g(t)=1. \tag{42}$$

Using the Sumudu transform technique:

$$\frac{G(u)}{u^4}+G(u)=1. \tag{43}$$

Which is equivalent to:

$$G(u)+u^4G(u)=u^4. \tag{44}$$

Thus:

$$G(u)=\frac{u^4}{\left(1+u^4\right)}. \tag{45}$$

Expressing the above equation as a power series yields:

$$G(u)\approx u^4-u^8+u^{12}-u^{16}+u^{20}-u^{24}+u^{28}-u^{32}+\cdots \tag{46}$$

Thus:

$$G(u)\approx\sum_{n=0}^{\infty}(-1)^n u^{4(n+1)}. \tag{47}$$

The approximate solution is readily obtained by means of FBCD and substitution of t for u:

$$g(t)\approx\frac{t^4}{4!}-\frac{t^8}{8!}+\frac{t^{12}}{12!}-\frac{t^{16}}{16!}+\frac{t^{20}}{20!}-\frac{t^{24}}{24!}+\frac{t^{28}}{28!}-\frac{t^{32}}{32!}. \tag{48}$$

Therefore:

$$g(t) \approx \sum_{n=0}^{\infty} (-1)^n \frac{t^{4(n+1)}}{[4(n+1)]!}. \tag{49}$$

The fourth derivative of the above is:

$$\frac{d^4 g}{dt^4} \approx 1 - \frac{t^4}{4!} + \frac{t^8}{8!} - \frac{t^{12}}{12!} + \frac{t^{16}}{16!} - \frac{t^{20}}{20!} + \frac{t^{24}}{24!} - \frac{t^{28}}{28!} + \frac{t^{32}}{32!} + \cdots \tag{50}$$

Thus:

$$\frac{d^4 g}{dt^4} \approx \sum_{n=0}^{\infty} (-1)^n \frac{t^{4n}}{(4n)!}. \tag{51}$$

Inspection of (48) and (50) demonstrates the solution:

$$\frac{d^4 g}{dt^4} = 1 - g(t). \tag{52}$$

The numerical results of this are graphically illustrated in **Figure 1**.

5.2. Example 2

The FBCD technique can also be used to determine the approximate solution to this inhomogeneous first order integro-differential equation (IDE):

$$\frac{dg}{dt} + \int g(t)\,dt = t^3. \tag{53}$$

Note that a straightforward ordinary differential equation (ODE) results:

$$\frac{G(u)}{u} + uG(u) = 3!u^3. \tag{54}$$

Assume that $g(0) = 0$. Solving for $G(u)$:

$$G(u) = \frac{6u^3}{\left(\frac{1}{u} + u\right)} = \frac{6u^4}{1+u^2}. \tag{55}$$

Expressing the above as a power series:

$$G(u) \approx 6\left(u^4 - u^6 + u^8 - u^{10} + u^{12} - u^{14} + u^{16} + \cdots\right). \tag{56}$$

Representing the above using series notation:

$$G(u) \approx 6\sum_{n=0}^{\infty} (-1)^n u^{(2n+4)}. \tag{57}$$

By applying FBCD and substituting t for u results in the inversion of $G(u)$ to $g(t)$:

$$g(t) \approx 6\left(\frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} - \frac{t^{10}}{10!} + \frac{t^{12}}{12!} - \frac{t^{14}}{14!} + \frac{t^{16}}{16!} + \cdots\right). \tag{58}$$

Therefore:

$$g(t) \approx 6\sum_{n=0}^{\infty} (-1)^n \frac{t^{(2n+4)}}{(2n+4)!}. \tag{59}$$

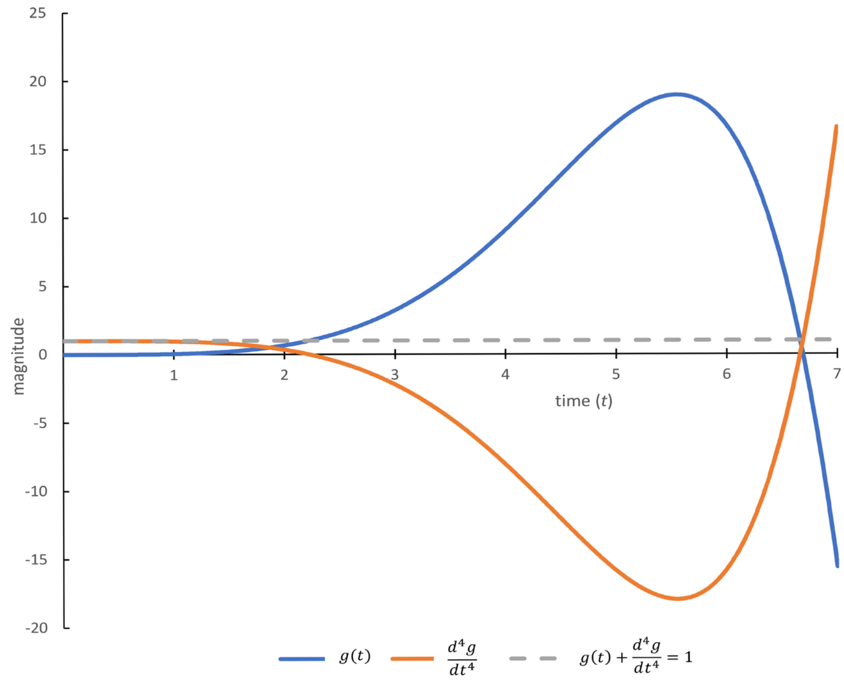

Figure 1. The numerical solution to $\frac{\mathrm{d}^4 g}{\mathrm{d}t^4}+g(t)=1$ is displayed.

And

$$\frac{\mathrm{d}g}{\mathrm{d}t} \approx 6\sum_{n=0}^{\infty}(-1)^n \frac{t^{(2n+3)}}{(2n+3)!}. \tag{60}$$

And

$$\int g(t)\,\mathrm{d}t \approx 6\sum_{n=0}^{\infty}(-1)^n \frac{t^{(2n+5)}}{(2n+5)!}. \tag{61}$$

This approximate solution is shown in Figure 2 whereas the "components" of the equation are illustrated in Figure 3.

5.3. Example 3

Consider the following integro-differential equation (IDE) which incorporates the sine integral (Si) function:

$$\frac{\mathrm{d}g}{\mathrm{d}t}+g(t)=\int_0^t \frac{\sin(\tau)}{\tau}\mathrm{d}\tau. \tag{62}$$

where: $\mathrm{Si}(t)=\int_0^t \frac{\sin(\tau)}{\tau}\mathrm{d}\tau$. Note that that $g(0)=0$. The Sumudu transform of the above IDE is:

$$\frac{G(u)}{u}+G(u)=\mathrm{atan}(u). \tag{63}$$

where $\boldsymbol{S}\left[\int_0^t \frac{\sin(\tau)}{\tau}\mathrm{d}\tau\right]=\mathrm{atan}(u)$. This is derived in **Appendix B**. Rearranging (63):

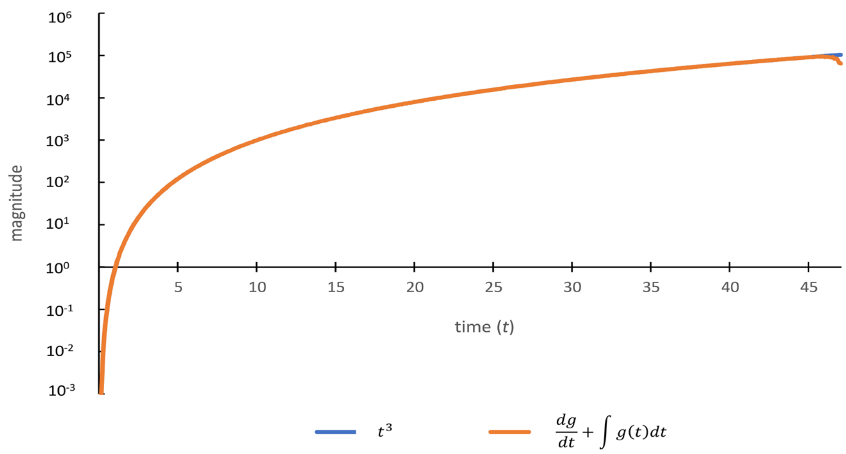

Figure 2. An approximate solution to: $\frac{\mathrm{d}g}{\mathrm{d}t} + \int g(t)\mathrm{d}t = t^3$ is illustrated.

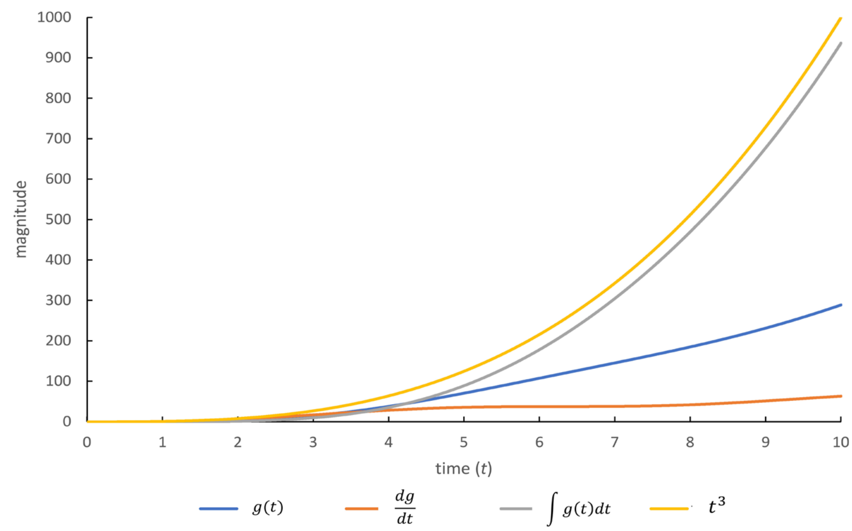

Figure 3. The above graph displays $g(t)$, $\frac{\mathrm{d}g}{\mathrm{d}t}$, $\int g(t)\mathrm{d}t$, and t^3 from Example 2.

$$G(u)\cdot\left[\frac{1}{u}+1\right] = \operatorname{atan}(u). \tag{64}$$

The above can be expressed as:

$$G(u)\cdot\left[\frac{u+1}{u}\right] = \operatorname{atan}(u). \tag{65}$$

Thus:

$$G(u) = \left[\frac{u}{u+1}\right]\operatorname{atan}(u). \tag{66}$$

Note that [14]:

$$\operatorname{atan}(u) \approx u - \frac{u^3}{3} + \frac{u^5}{5} - \frac{u^7}{7} + \cdots \tag{67}$$

Therefore:

$$\operatorname{atan}(u) \approx \sum_{n=0}^{\infty} (-1)^n \frac{u^{(2n+1)}}{(2n+1)}. \tag{68}$$

Furthermore,

$$\frac{u}{u+1} \approx \sum_{k=0}^{\infty} (-1)^k u^{(k+1)}. \tag{69}$$

Thus, the product of the above two power series, (68) and (69), yields an approximation for $G(u)$:

$$G(u) \approx \left[\sum_{k=0}^{\infty} (-1)^k u^{(k+1)}\right] \cdot \left[\sum_{n=0}^{\infty} (-1)^n \frac{u^{(2n+1)}}{(2n+1)}\right]. \tag{70}$$

Use of the Cauchy product, or discrete convolution with two truncated series, results in a double or "nested" series:

$$G(u) \approx \sum_{k=0}^{m}\sum_{n=0}^{m} \left\{\left[(-1)^k u^{(k+1)}\right] \cdot \left[(-1)^{(m-n)} \frac{u^{(2(m-n)+1)}}{(2(m-n)+1)}\right]\right\}. \tag{71}$$

This allows for a single expression for u as a power function and ultimately a power series. *Thus, the two series, which have undergone convolution, can also be algebraically combined*:

$$G(u) \approx \sum_{k=0}^{m}\sum_{n=0}^{m} \left\{(-1)^{(k+m-n)} \frac{u^{(2(m-n)+k+2)}}{(2(m-n)+1)}\right\}. \tag{72}$$

Use of FBCD and substitution of t for u, yields the approximate inversion of $G(u)$:

$$g(t) \approx \sum_{k=0}^{m}\sum_{n=0}^{m} \left\{(-1)^{(k+m-n)} \frac{t^{(2(m-n)+k+2)}}{(2(m-n)+1)\cdot(2(m-n)+k+2)!}\right\}. \tag{73}$$

In addition, a double series which resulted from the discrete convolution of two series, is readily integrated and differentiated utilizing the same "term-by-term" methodology as a single series:

$$\frac{\mathrm{d}g}{\mathrm{d}t} \approx \sum_{k=0}^{m}\sum_{n=0}^{m} \left\{(-1)^{(k+m-n)} \frac{t^{(2(m-n)+k+1)}}{(2(m-n)+1)\cdot(2(m-n)+k+1)!}\right\}. \tag{74}$$

Using a value of $m = 50$ results in range-limited approximation of the Si function as shown in **Figure 4**. Furthermore, $g(t)$ and $\frac{\mathrm{d}g}{\mathrm{d}t}$ are graphically displayed in **Figure 5**. *It should be noted that the value of* $m = 50$ *was utilized owing to the upper limits of the Mathcad factorial function.*

5.4. Example 4

Consider the following third-order differential equation in which all initial conditions are equal to zero:

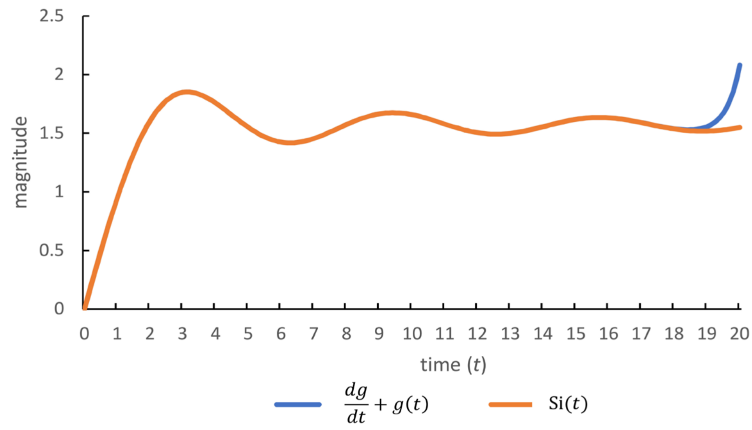

Figure 4. The sine integral function is approximated by $\frac{\mathrm{d}g}{\mathrm{d}t}+g(t)$ from Example 3.

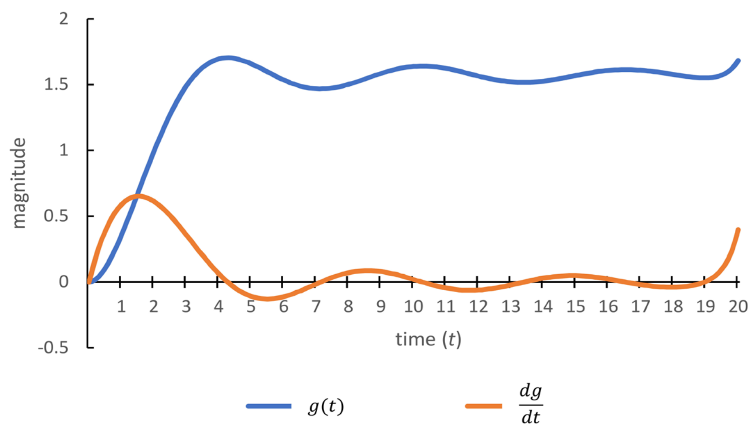

Figure 5. The above graph illustrates $g(t)$ and $\frac{\mathrm{d}g}{\mathrm{d}t}$ from Example 3.

$$\frac{\mathrm{d}^3 g}{\mathrm{d}t^3}+\frac{\mathrm{d}^2 g}{\mathrm{d}t^2}+\frac{\mathrm{d}g}{\mathrm{d}t}+g(t)=\frac{t^2\mathrm{e}^{-t}}{2}. \tag{75}$$

It is expressed within the Sumudu transform domain as:

$$\frac{G(u)}{u^3}+\frac{G(u)}{u^2}+\frac{G(u)}{u}+G(u)=\frac{u^2}{(1+u)^3}. \tag{76}$$

Note that the following relationship: $\boldsymbol{S}\left[\frac{t^2\mathrm{e}^{-t}}{2}\right]=\frac{u^2}{(1+u)^3}$ is from a pre-existing Sumudu transform table [8] [9].

Solving for $G(u)$:

$$G(u)=\frac{u^2}{\left(\frac{1}{u^3}+\frac{1}{u^2}+\frac{1}{u}+1\right)(1+u)^3}. \tag{77}$$

5.4.1. Solution Method 1: Series Representation of the Partial Fraction Expansion

Use of the partial fraction expansion method results in the following expression:

$$G(u)=\frac{-u}{4(u^2+1)}+\frac{5}{4(u+1)}-\frac{11}{4(u+1)^2}+\frac{2}{(u+1)^3}-\frac{1}{2(u+1)^4}. \tag{78}$$

The above equation is then represented as a sum of five series:

$$G(u)\approx\sum_{h=1}^{5}G_h(u). \tag{79}$$

where:

$$G_1(u)=\frac{-u}{4(u^2+1)}\approx\left(\frac{-1}{4}\right)\sum_{j=0}^{\infty}(-1)^j u^{(2j+1)}, \tag{80}$$

$$G_2(u)=\frac{5}{4(u+1)}\approx\left(\frac{5}{4}\right)\sum_{k=0}^{\infty}(-1)^k u^k, \tag{81}$$

$$G_3(u)=\frac{-11}{4(u+1)^2}\approx\left(\frac{-11}{4}\right)\sum_{l=0}^{\infty}(-1)^l(l+1)u^l, \tag{82}$$

$$G_4(u)=\frac{2}{(u+1)^3}\approx\sum_{m=0}^{\infty}(-1)^m(m+1)(m+2)u^m, \tag{83}$$

and:

$$G_5(u)=\frac{-1}{2(u+1)^4}\approx\left(\frac{1}{12}\right)\sum_{n=0}^{\infty}(-1)^{(n+1)}(n+1)(n+2)(n+3)u^n. \tag{84}$$

Use of FBCD and substitution of t for u yields the following five power series:

$$g_1(t)\approx\left(\frac{-1}{4}\right)\sum_{j=0}^{\infty}(-1)^j\frac{t^{(2j+1)}}{(2j+1)!}, \tag{85}$$

$$g_2(t)\approx\left(\frac{5}{4}\right)\sum_{k=0}^{\infty}(-1)^k\frac{t^k}{k!}, \tag{86}$$

$$g_3(t)\approx\left(\frac{-11}{4}\right)\sum_{l=0}^{\infty}(-1)^l(l+1)\frac{t^l}{l!}, \tag{87}$$

$$g_4(t)\approx\sum_{m=0}^{\infty}(-1)^m(m+1)(m+2)\frac{t^m}{m!}, \tag{88}$$

and:

$$g_5(t)\approx\left(\frac{1}{12}\right)\sum_{n=0}^{\infty}(-1)^{(n+1)}(n+1)(n+2)(n+3)\frac{t^n}{n!}. \tag{89}$$

So that:

$$g(t)\approx\sum_{h=1}^{5}g_h(t). \tag{90}$$

5.4.2. Solution Method 2: Use of Discrete Convolution (The Cauchy Product)

Reiterating the original Sumudu transform equation:

$$G(u)=\frac{u^2}{\left(\frac{1}{u^3}+\frac{1}{u^2}+\frac{1}{u}+1\right)(1+u)^3}. \tag{91}$$

Algebraic rearrangement subsequently yields the product of two rational functions:

$$G(u)=\frac{u^5}{\left(1+u+u^2+u^3\right)}\cdot\frac{1}{(1+u)^3}. \tag{92}$$

Each of these rational functions can be approximated using infinite series representations:

$$\frac{u^5}{\left(1+u+u^2+u^3\right)}\approx\left(u^5-u^6+u^9-u^{10}+u^{13}+\cdots\right), \tag{93}$$

and

$$\frac{1}{(1+u)^3}\approx\left(1-3u+6u^2-10u^3+15u^4+\cdots\right). \tag{94}$$

Therefore:

$$G(u)\approx\left(u^5-u^6+u^9-u^{10}+u^{13}+\cdots\right)\cdot\left(1-3u+6u^2-10u^3+15u^4+\cdots\right). \tag{95}$$

Note that:

$$\left(u^5-u^6+u^9-u^{10}+u^{13}+\cdots\right)=\left[\sum_{k=0}^{\infty}\left(u^{(4k+5)}\right)-\sum_{n=0}^{\infty}\left(u^{(4n+6)}\right)\right]. \tag{96}$$

and:

$$\left(1-3u+6u^2-10u^3+15u^4+\cdots\right)=\left[\frac{1}{2}\sum_{p=0}^{\infty}(-1)^p(p+1)(p+2)u^p\right]. \tag{97}$$

Therefore:

$$G(u)\approx\left[\sum_{k=0}^{\infty}\left(u^{(4k+5)}\right)-\sum_{n=0}^{\infty}\left(u^{(4n+6)}\right)\right]\cdot\left[\frac{1}{2}\sum_{p=0}^{\infty}(-1)^p(p+1)(p+2)u^p\right]. \tag{98}$$

Making use of the distributive property:

$$\begin{aligned}G(u)\approx\Bigg\{&\frac{1}{2}\left[\sum_{k=0}^{\infty}\left(u^{(4k+5)}\right)\right]\cdot\left[\sum_{p=0}^{\infty}(-1)^p(p+1)(p+2)u^p\right]\\&-\frac{1}{2}\left[\sum_{n=0}^{\infty}\left(u^{(4n+6)}\right)\right]\cdot\left[\sum_{p=0}^{\infty}(-1)^p(p+1)(p+2)u^p\right]\Bigg\}.\end{aligned} \tag{99}$$

Applying discrete convolution (the Cauchy product) with two truncated series, each of m terms, yields:

$$\begin{aligned}G(u)\approx\Bigg\{&\frac{1}{2}\sum_{k=0}^{m}\sum_{p=0}^{m}\left[u^{(4k+5)}(-1)^{(m-p)}(m-p+1)(m-p+2)u^{(m-p)}\right]\\&-\frac{1}{2}\sum_{n=0}^{m}\sum_{p=0}^{m}\left[u^{(4n+6)}(-1)^{(m-p)}(m-p+1)(m-p+2)u^{(m-p)}\right]\Bigg\}.\end{aligned} \tag{100}$$

Utilizing further algebraic simplification:

$$\begin{aligned}G(u)\approx\Bigg\{&\frac{1}{2}\sum_{k=0}^{m}\sum_{p=0}^{m}\left[u^{(4k+5+m-p)}(-1)^{(m-p)}(m-p+1)(m-p+2)\right]\\&-\frac{1}{2}\sum_{n=0}^{m}\sum_{p=0}^{m}\left[u^{(4n+6+m-p)}(-1)^{(m-p)}(m-p+1)(m-p+2)\right]\Bigg\}.\end{aligned} \tag{101}$$

The use of FBCD and substitution of t for u yields the approximate solution:

$$g(t) \approx \left\{ \frac{1}{2} \sum_{k=0}^{m} \sum_{p=0}^{m} \left[\frac{t^{(4k+5+m-p)}}{(4k+5+m-p)!} (-1)^{(m-p)} (m-p+1)(m-p+2) \right] - \frac{1}{2} \sum_{n=0}^{m} \sum_{p=0}^{m} \left[\frac{t^{(4n+6+m-p)}}{(4n+6+m-p)!} (-1)^{(m-p)} (m-p+1)(m-p+2) \right] \right\}. \tag{102}$$

"Term-by-term" differentiation results in the following:

$$\frac{\mathrm{d}g}{\mathrm{d}t} \approx \left\{ \frac{1}{2} \sum_{k=0}^{m} \sum_{p=0}^{m} \left[\frac{t^{(4k+4+m-p)}}{(4k+4+m-p)!} (-1)^{(m-p)} (m-p+1)(m-p+2) \right] - \frac{1}{2} \sum_{n=0}^{m} \sum_{p=0}^{m} \left[\frac{t^{(4n+5+m-p)}}{(4n+5+m-p)!} (-1)^{(m-p)} (m-p+1)(m-p+2) \right] \right\}, \tag{103}$$

$$\frac{\mathrm{d}^2 g}{\mathrm{d}t^2} \approx \left\{ \frac{1}{2} \sum_{k=0}^{m} \sum_{p=0}^{m} \left[\frac{t^{(4k+3+m-p)}}{(4k+3+m-p)!} (-1)^{(m-p)} (m-p+1)(m-p+2) \right] - \frac{1}{2} \sum_{n=0}^{m} \sum_{p=0}^{m} \left[\frac{t^{(4n+4+m-p)}}{(4n+4+m-p)!} (-1)^{(m-p)} (m-p+1)(m-p+2) \right] \right\}, \tag{104}$$

$$\frac{\mathrm{d}^3 g}{\mathrm{d}t^3} \approx \left\{ \frac{1}{2} \sum_{k=0}^{m} \sum_{p=0}^{m} \left[\frac{t^{(4k+2+m-p)}}{(4k+2+m-p)!} (-1)^{(m-p)} (m-p+1)(m-p+2) \right] - \frac{1}{2} \sum_{n=0}^{m} \sum_{p=0}^{m} \left[\frac{t^{(4n+3+m-p)}}{(4n+3+m-p)!} (-1)^{(m-p)} (m-p+1)(m-p+2) \right] \right\}. \tag{105}$$

The results of this are graphically illustrated in **Figure 6** and **Figure 7**. Note that a value of $m = 30$ has been utilized owing to the upper limits of the Mathcad factorial function. *Although not illustrated, both method* 1 *and method* 2 *yielded numerical results with similar range-limited accuracy.*

5.5. Example 5

The following double integral equation can be expressed as an ordinary differential equation (ODE) using a Sumudu transform and the shift theorem:

$$\iint g(t)\,\mathrm{d}t\mathrm{d}t + t \cdot g(t) = 0. \tag{106}$$

Thus:

$$u^2 G(u) + u\left[G(u) + u \frac{\mathrm{d}G}{\mathrm{d}u} \right] = 0. \tag{107}$$

Rearranging:

$$G(u)\left[u^2 + u \right] = -u^2 \frac{\mathrm{d}G}{\mathrm{d}u}. \tag{108}$$

Separating variables and simplifying:

$$-\left[1 + \frac{1}{u} \right] \mathrm{d}u = \frac{\mathrm{d}G}{G}. \tag{109}$$

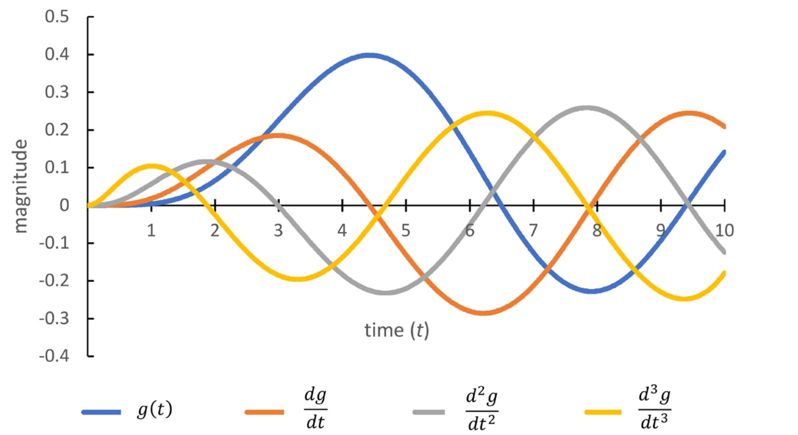

Figure 6. The function $g(t)$ and its first, second, and third derivatives are illustrated from Example 4.

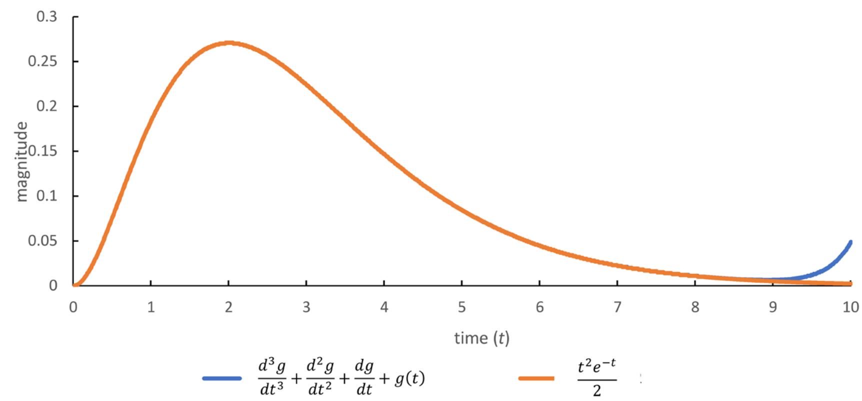

Figure 7. The approximate and exact solutions for Example 4 are illustrated.

Integrating both sides:

$$-\int\left[1+\frac{1}{u}\right]\mathrm{d}u=\int\frac{\mathrm{d}G}{G}. \tag{110}$$

This subsequently yields:

$$-\left[u+\ln(u)+C\right]=\ln\left\{G(u)\right\}. \tag{111}$$

Exponentiation results in:

$$G(u)=\mathrm{e}^{-\left[u+\ln(u)+C\right]}=\mathrm{e}^{-u}\mathrm{e}^{\ln\left(\frac{1}{u}\right)}\mathrm{e}^{-C}. \tag{112}$$

Allowing the constant of integration, C, to equal zero yields:

$$G(u)=\frac{1}{u}\mathrm{e}^{-u}. \tag{113}$$

The above equation can subsequently be expressed as an infinite series:

$$G(u)\approx\frac{1}{u}\left(1+\frac{(-u)}{1!}+\frac{(-u)^2}{2!}+\frac{(-u)^3}{3!}+\frac{(-u)^4}{4!}+\cdots\right). \tag{114}$$

Distribution of the $\frac{1}{u}$ term yields:

$$G(u) \approx \left(\frac{1}{u} - \frac{1}{1!} + \frac{u}{2!} - \frac{u^2}{3!} + \frac{u^3}{4!} + \cdots \right). \tag{115}$$

Thus:

$$G(u) \approx \frac{1}{u} + \sum_{n=0}^{\infty} (-1)^{(n+1)} \frac{u^n}{(n+1)!}. \tag{116}$$

The above Sumudu transform can be inverted by separating it into two distinct functions within the u-domain. Firstly:

$$\boldsymbol{S}^{-1}\left[\frac{1}{u}\right] = \delta(t). \tag{117}$$

where $\delta(t)$ is the *Dirac delta function*.

Secondly, combining the Sumudu transform $\boldsymbol{S}^{-1}\left[\frac{1}{u}\right] = \delta(t)$ along with FBCD for the remaining terms of (116) yields the following:

$$g(t) \approx \delta(t) + \sum_{n=0}^{\infty} (-1)^{(n+1)} \frac{t^n}{(n!)\cdot(n+1)!}, \quad t > 0. \tag{118}$$

Additionally, t has replaced u in the above equation as well. Subsequent integration yields:

$$\int g(t)\,\mathrm{d}t \approx H(t) + \sum_{n=0}^{\infty} (-1)^{(n+1)} \frac{t^{(n+1)}}{\left[(n+1)!\right]^2}, \quad t > 0. \tag{119}$$

where $H(t)$ is the *Heaviside step function* which is the indefinite integral of the *Dirac delta function*. The double integral of (118) is:

$$\iint g(t)\,\mathrm{d}t\mathrm{d}t \approx t + \sum_{n=0}^{\infty} (-1)^{(n+1)} \frac{t^{(n+2)}}{(n+2)!\cdot(n+1)!}. \tag{120}$$

Note that the variable t is the indefinite integral of the *Heaviside step function*. Furthermore, t is also the double indefinite integral of the *Dirac delta function*. Lastly, the constants of integration for (119) and (120) are equal to zero. **Figure 8** and **Figure 9** illustrate the solution of this example.

5.6. Example 6

The aforementioned techniques can also be applied to certain partial differential equations (PDEs). As an example:

$$\frac{\partial g}{\partial x} - \frac{\partial g}{\partial t} = x. \tag{121}$$

The above equation can be readily converted to an ODE using the Sumudu transform:

$$\frac{\mathrm{d}G}{\mathrm{d}x} - \frac{G}{u} = x. \tag{122}$$

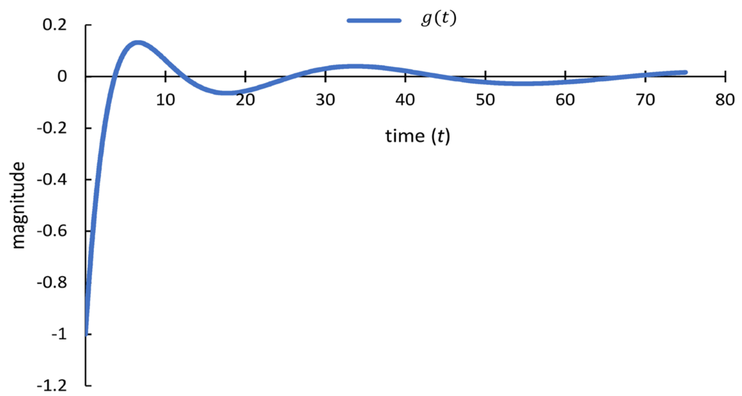

Figure 8. The function $g(t)$, from Example 5, is graphically displayed.

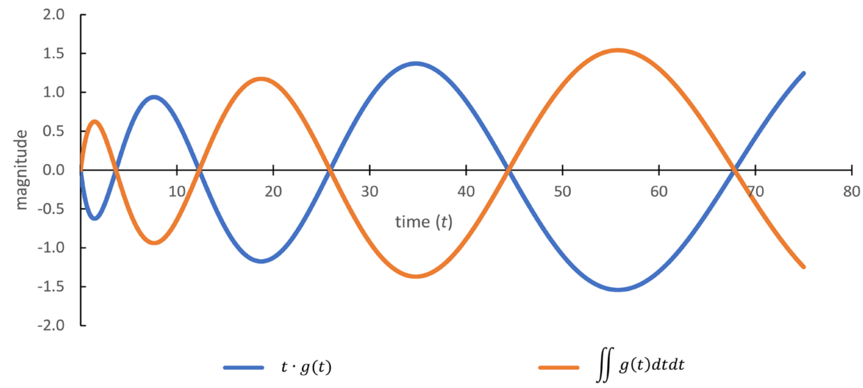

Figure 9. Based on Example 5, $t \cdot g(t)$ and $\iint g(t)\mathrm{d}t\mathrm{d}t$ are additive inverses.

An integrating factor is then utilized to take advantage of the product rule:

$$\mathrm{e}^{\int\left(\frac{-1}{u}\right)\mathrm{d}x} = \mathrm{e}^{\left(\frac{-x}{u}\right)}. \tag{123}$$

Thus:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(G(x,u)\mathrm{e}^{\left(\frac{-x}{u}\right)}\right) = x\mathrm{e}^{\left(\frac{-x}{u}\right)}. \tag{124}$$

Expressing the above equation as two separate indefinite integrals:

$$\int \mathrm{d}\left(G(x,u)\mathrm{e}^{\left(\frac{-x}{u}\right)}\right) = \int x\mathrm{e}^{\left(\frac{-x}{u}\right)}\mathrm{d}x. \tag{125}$$

This yields an expression such that the integrating factor now cancels from both sides of the resulting equation:

$$G(x,u)\mathrm{e}^{\left(\frac{-x}{u}\right)} = -u^2\mathrm{e}^{\left(\frac{-x}{u}\right)} - ux\mathrm{e}^{\left(\frac{-x}{u}\right)}. \tag{126}$$

The Sumudu transform is then simplified:

$$G(x,u) = -\left[ux + u^2\right]. \tag{127}$$

Use of FBCD and substitution of t for u yields:

$$g(x,t) = -\left(\frac{t}{1!}x + \frac{t^2}{2!}\right) = -\left(tx + \frac{t^2}{2}\right). \tag{128}$$

The partial derivatives of (128) demonstrate the solution:

$$\frac{\partial g}{\partial x} = -t \tag{129}$$

and

$$\frac{\partial g}{\partial t} = -x - t = -(x+t). \tag{130}$$

5.7. Example 7

The Sumudu transform, with FBCD, and can also be utilized with this mixed partial integro-differential equation (IDE):

$$\frac{\partial^2 g}{\partial x \partial t} + \int g(x,t)\,\mathrm{d}t = 0. \tag{131}$$

Expressing the above results in an ODE within the Sumudu domain:

$$\left(\frac{\mathrm{d}G}{\mathrm{d}x}\right)\frac{1}{u} + uG = 0. \tag{132}$$

Separating variables yields:

$$\frac{\mathrm{d}G}{G} = -u^2\,\mathrm{d}x. \tag{133}$$

Integrating both sides:

$$\int \frac{\mathrm{d}G}{G} = -\int u^2\,\mathrm{d}x. \tag{134}$$

This results in an expression which utilizes the natural log function:

$$\ln(G) = -u^2 x + C. \tag{135}$$

Exponentiation then yields $G(u,x)$:

$$G(u,x) = \mathrm{e}^{-u^2 x + C}. \tag{136}$$

With $C = 0$ the above equation can then be approximated as a power series:

$$G(u,x) \approx \frac{1}{0!} - \frac{u^2 x}{1!} + \frac{u^4 x^2}{2!} - \frac{u^6 x^2}{3!} + \cdots \tag{137}$$

Equivalently:

$$G(u,x) \approx \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n} x^n}{n!}. \tag{138}$$

Utilizing FBCD and substitution of t for u results in the following expression:

$$g(t,x) \approx \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n} x^n}{n! \cdot (2n)!}. \tag{139}$$

Therefore:

$$\frac{\partial g}{\partial t} \approx \sum_{n=1}^{\infty} (-1)^n \frac{t^{(2n-1)} x^n}{n! \cdot (2n-1)!}. \tag{140}$$

and

$$\frac{\partial^2 g}{\partial t \partial x} \approx \sum_{n=1}^{\infty} (-1)^n \frac{t^{(2n-1)} x^{(n-1)}}{(n-1)! \cdot (2n-1)!}. \tag{141}$$

Note that both infinite series, in (140) and (141), use indices starting at $n=1$ to avoid a negative factorial. Furthermore:

$$\int g(t,x)\,\mathrm{d}t \approx \sum_{n=0}^{\infty} (-1)^n \frac{t^{(2n+1)} x^n}{n! \cdot (2n+1)!}. \tag{142}$$

Figure 10 and **Figure 11** illustrate two solutions to this example.

5.8. Example 8

Consider a partial IDE:

$$g(x,t) + \frac{\partial g(x,t)}{\partial x} + \int g(x,t)\,\mathrm{d}t = 0. \tag{143}$$

Use of a Sumudu transform leads to an ODE:

$$G + \frac{\mathrm{d}G}{\mathrm{d}x} + uG = 0. \tag{144}$$

Rearrangement yields:

$$G(1+u) = -\frac{\mathrm{d}G}{\mathrm{d}x}. \tag{145}$$

Separating variables:

$$\frac{\mathrm{d}G}{G} = -(1+u)\,\mathrm{d}x. \tag{146}$$

Integrating both sides:

$$\int \frac{\mathrm{d}G}{G} = -\int (1+u)\,\mathrm{d}x. \tag{147}$$

Thus:

$$\ln(G) = -(1+u)x + C. \tag{148}$$

Exponentiation yields:

$$G(x,u) = \mathrm{e}^{-(1+u)x}\mathrm{e}^{C}. \tag{149}$$

Allowing C to equal zero:

$$G(x,u) = \mathrm{e}^{-(1+u)x} = \mathrm{e}^{-x}\mathrm{e}^{-ux}. \tag{150}$$

Note that $\mathrm{e}^{-(1+u)x}$ cannot be readily represented as a power series with the form of (6). However, $\mathrm{e}^{-x}\mathrm{e}^{-ux}$ can be approximated as the product of two power series each of which uses the format of (6):

$$G(x,u) \approx \sum_{n=0}^{\infty}\left[(-1)^n \frac{x^n}{n!}\right] \cdot \sum_{p=0}^{\infty}\left[(-1)^p \frac{x^p u^p}{p!}\right]. \tag{151}$$

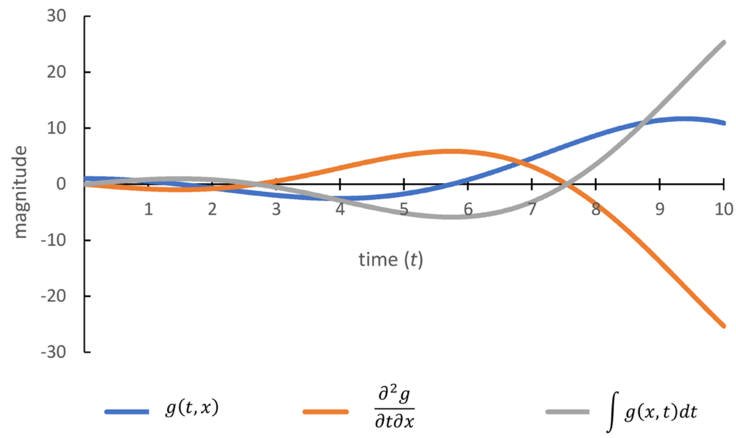

Figure 10. Examination of the above graph demonstrates how $\frac{\partial^2 g}{\partial t \partial x}$ and $\int g(x,t)\mathrm{d}t$ are additive inverses from Example 7. This graph is based upon $x=1$ and $0 \le t \le 10$.

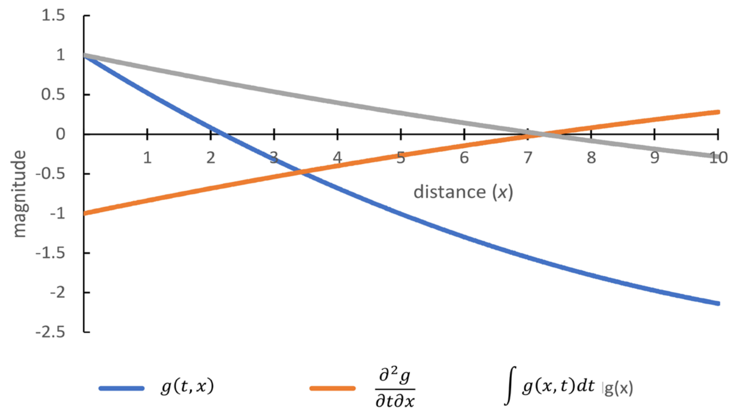

Figure 11. Inspection of the above graph, from Example 7, illustrates how $\frac{\partial^2 g}{\partial t \partial x}$ and $\int g(x,t)\mathrm{d}t$ are again additive inverses. In this case, $t=1$ and $0 \le x \le 10$.

Discrete convolution (the Cauchy product) can then be utilized to combine the product of the above two series into a truncated double series:

$$G(x,u) \approx \sum_{n=0}^{m} \sum_{p=0}^{m} \left\{ \left[(-1)^n \frac{x^n}{n!} \right] \cdot \left[(-1)^{(m-p)} \frac{x^{(m-p)} u^{(m-p)}}{(m-p)!} \right] \right\}. \tag{152}$$

Algebraically combining terms yields:

$$G(x,u) \approx \sum_{n=0}^{m} \sum_{p=0}^{m} \left\{ (-1)^{(n+m-p)} \frac{x^{(n+m-p)} u^{(m-p)}}{(n!) \cdot (m-p)!} \right\}. \tag{153}$$

Use of FBCD, with substitution of t for u, produces the solution:

$$g(x,t) \approx \sum_{n=0}^{m} \sum_{p=0}^{m} \left\{ (-1)^{(n+m-p)} \frac{x^{(n+m-p)} t^{(m-p)}}{(n!) \cdot ((m-p)!)^2} \right\}. \tag{154}$$

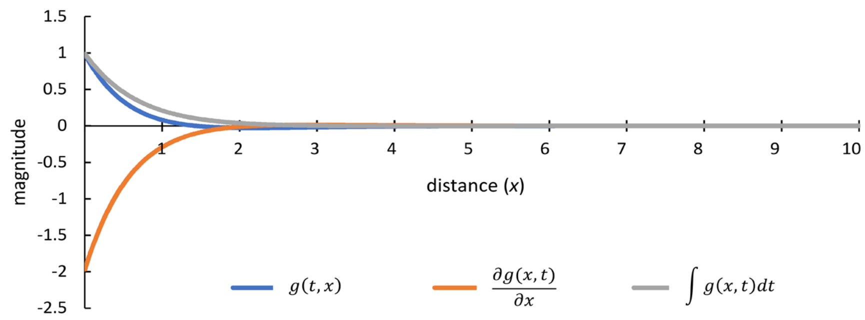

Figure 12. Using Example 8, sum of $g(t,x)$, $\frac{\partial g(x,t)}{\partial x}$, and $\int g(x,t)\mathrm{d}t$ is equal to zero. Note that: $t=1$ and $0 \le x \le 10$.

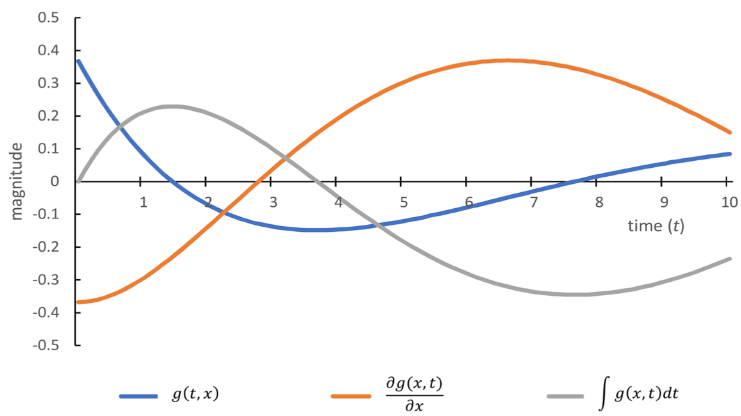

Figure 13. Again, using Example 8, the sum of $g(t,x)$, $\frac{\partial g(x,t)}{\partial x}$, and $\int g(x,t)\mathrm{d}t$ is also equal to zero. Note that: $x=1$ and $0 \le t \le 10$.

Therefore:

$$\frac{\partial g(x,t)}{\partial x} = \sum_{n=0}^{m}\sum_{p=0}^{m}\left\{(-1)^{(n+m-p)}\frac{x^{(n+m-p-1)}t^{(m-p)}}{(n!)\cdot((m-p)!)^2}(n+m-p)\right\}, \tag{155}$$

and:

$$\int g(x,t)\mathrm{d}t \approx \sum_{n=0}^{m}\sum_{p=0}^{m}\left\{(-1)^{(n+m-p)}\frac{x^{(n+m-p)}t^{(m-p+1)}}{(n!)\cdot((m-p)!)^2(m-p+1)}\right\}. \tag{156}$$

Figure 12 and Figure 13 demonstrate the solution. Note that a value of $m=50$ has been utilized oweing to the numerical limits of Mathcad.

6. Conclusion

By taking advantage of the relationship between a geometric power series in the u-domain and its inversion back to the t-domain utilizing FBCD, an approximate range-limited solution to certain differential equations as well as integro-differential equations may be obtained. This process may also be facilitated with the utilization of convolution. Further research and applications of this technique, particularly with non-linear and fractional differential equations, may be warranted.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix A: Determining the Sumudu Transform of t^n Using Its Laplace Transform

The Laplace transform of t^n is:

$$\boldsymbol{L}\left[t^n\right] = n!s^{-(n+1)}. \tag{A1}$$

Using (5) the corresponding Sumudu transform is therefore:

$$\boldsymbol{S}\left[t^n\right] = n!\left(\frac{1}{u}\right)\left(\frac{1}{u}\right)^{-(n+1)} = n!\left(\frac{1}{u}\right)(u)^{(n+1)}. \tag{A2}$$

Simplifying the above:

$$\boldsymbol{S}\left[t^n\right] = n!(u)^n. \tag{A3}$$

For integer values of $n \geq 0$. Therefore, the use of FBCD with substitution of t for u, leads to inversion of the above Sumudu transform:

$$\frac{\boldsymbol{S}\left[t^n\right]}{n!} = \left.\frac{n!(u)^n}{n!}\right|_{t=u} = t^n. \tag{A4}$$

Appendix B: The Sumudu Transform of the Sine Integral (Si) Function

Note that the Si function is based upon the following:

$$\operatorname{sinc}(x) = \frac{\sin(x)}{x} \approx \frac{1}{x}\sum_{n=0}^{\infty}(-1)^n \frac{x^{(2n+1)}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{(2n)}}{(2n+1)!}. \tag{B1}$$

Using (31), $\sin(x)$ is approximated as:

$$\sin(x) \approx \sum_{n=0}^{\infty}(-1)^n \frac{x^{(2n+1)}}{(2n+1)!}. \tag{B2}$$

Therefore:

$$\operatorname{Si}(t) = \int_0^t \operatorname{sinc}(\tau)\,\mathrm{d}\tau = \int_0^t \frac{\sin(\tau)}{\tau}\mathrm{d}\tau \approx \sum_{n=0}^{\infty}(-1)^n \frac{t^{(2n+1)}}{(2n+1)!\cdot(2n+1)}. \tag{B3}$$

Use of "factorial-based" coefficient *amplification* yields the following Sumudu transform:

$$\boldsymbol{S}\left[\int_0^t \frac{\sin(\tau)}{\tau}\mathrm{d}\tau\right] = \boldsymbol{S}\left[\sum_{n=0}^{\infty}(-1)^n \frac{t^{(2n+1)}}{(2n+1)!\cdot(2n+1)}\right] = \sum_{n=0}^{\infty}(-1)^n \frac{u^{(2n+1)}}{(2n+1)}. \tag{B4}$$

Note that the factorial-based coefficient amplification results in a cancellation, in both the numerator and denominator, of the above expression:

$$\sum_{n=0}^{\infty}(-1)^n \frac{(2n+1)!u^{(2n+1)}}{(2n+1)!(2n+1)} = \sum_{n=0}^{\infty}(-1)^n \frac{u^{(2n+1)}}{(2n+1)}. \tag{B5}$$

It is well established that [14]:

$$\operatorname{atan}(u) \approx \sum_{n=0}^{\infty}(-1)^n \frac{u^{(2n+1)}}{(2n+1)},\ |u| < 1. \tag{B6}$$

Therefore:

$$\boldsymbol{S}\left[\int_0^t \frac{\sin(\tau)}{\tau}\mathrm{d}\tau\right] = \operatorname{atan}(u) \approx \sum_{n=0}^{\infty}(-1)^n \frac{u^{(2n+1)}}{(2n+1)}. \tag{B7}$$