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# On Exact Traveling Wave Solutions for (1 + 1) Dimensional Kaup-Kupershmidt Equation

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## Abstract

In this present paper, the Fan sub-equation method is used to construct exact traveling wave solutions of the (1 + 1) dimensional Kaup-Kupershmidt equation. Many exact traveling wave solutions are successfully obtained, which contain solitary wave solutions, trigonometric function solutions, hyperbolic function solutions and Jacobian elliptic function periodic solutions with double periods.

**Keywords:** Fan Sub-Equation Method, Kaup-Kupershmidt Equation, Exact Traveling Wave Solutions

## 1. Introduction

Nonlinear partial differential equations are widely used to describe complex phenomena in vary scientific fields and especially in areas of physics such as plasma, fluid mechanics, biology, solid state physics, nonlinear optics and so on. Therefore the investigation of the exact solutions to nonlinear equations plays an important role in the study of nonlinear science. Up to now, many powerful methods to seek for exact solutions to the nonlinear differential equations have been proposed. Among these are inverse scattering method [1], Lie group method [2,3], bifurcation method of dynamical systems [4-6], sine-cosine method [7,8], tanh function method [9-11], homogenous balance method [12], Weierstrass elliptic function method [13].

Recently, Fan [14] presented the Fan sub-equation method which is a unified algebraic method to obtain many types of traveling wave solutions based on an auxiliary nonlinear ordinary differential equation with constant coefficients called Fan sub-equation. The important feature of Fan' method is to, without much extra effort and without considering the integrability of nonlinear equations, directly get a series of exact solutions in a uniform way, which cover all results of tanh function method, extended function method, F-expansion method, etc. This method is a powerful technique to symbolically compute traveling wave solutions of nonlinear evolution equations and is widely used by many researcher such as in [15-17] and by the references therein.

In this paper, we will use the Fan sub-equation method to discuss the (1+1) dimensional Kaup-Kupershmidt equation [18] which can be shown in the form

$$-u_t + 5u^2 u_x + \frac{25}{2} u_x u_{xx} + 5uu_{3x} + u_{5x} = 0. \tag{1.1}$$

## 2. The Fan Sub-Equation Method

For a given nonlinear partial differential equation

$$F\left(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \cdots\right) = 0, \tag{2.1}$$

where $u = u(x,t)$ is an unknown function, $F$ usually is a polynomial in $u(x,t)$.

To seek exact solutions of (2.1), we outline the Fan sub-equation method. The main steps are given below [14].

**Step 1.** By using the traveling wave transformation

$$u(x,t) = u(\xi), \quad \xi = x - ct, \tag{2.2}$$

where $c$ is a wave speed, we can reduce (2.1) to an ordinary differential equation in the form

$$F\left(u, u', u'', u''', \cdots\right) = 0, \tag{2.3}$$

where the prime denotes the derivative with respect to $\xi$.

**Step 2.** Expand the solution of (2.3) in the form

$$u(\xi) = \sum_{i=0}^{n} a_i \phi^i(\xi), \tag{2.4}$$

where $a_i$ ($i$ = 1,2, ⋯, $n$) are constants to be determined later and the new variable $\phi(\xi)$ satisfies the following Fan sub-equation

$$\phi'(\xi) = \varepsilon\sqrt{\sum_{j=0}^{4} c_j \phi^j(\xi)}, \tag{2.5}$$

where $\varepsilon = \pm 1$ and $c_i$ are real constants.

**Step 3.** Determine $n$ in (2.4) by substituting (2.4) and (2.5) into (2.3) and balancing the highest order derivative terms with the highest order nonlinear terms.

**Step 4.** Substituting (2.4) and (2.5) into (2.3) again and collecting all coefficients like $\phi^k \sqrt{\left(\sum_{j=0}^{4} c_j \phi^j\right)^l}$ ($l$ = 0, 1; $k$ = 0, 1, ···, $n$), then setting these coefficients to zero will give a set of algebraic equations with respect to $a_i$ ($i$ = 1,2, ···, $n$) and $c$.

**Step 5.** Solve these algebraic equations to obtain $c$ and $a_i$. Substituting these results into (2.4) yields to the general form of traveling wave solutions.

**Step 6.** For each solution to (2.5) which depends on the special conditions chosen for $c_j$, it follows from (2.4) that the corresponding exact traveling wave solution of (2.1) can be constructed.

## 3. Exact Solutions for the (1 + 1) Dimensional Kaup-Kupershmidt Equation

The fifth order Kaup-Kupershmidt Equation (1.1) is one of the solitonic equations related to the integrable cases of the Henon-Heiles system and belongs to the completely integrable hierarchy of higher order KdV equations. Moreover the equation has infinite sets of conservation laws [19-22]. Let us find the exact traveling wave solutions of the (1 + 1) dimensional Kaup-Kupershmidt equation by using the Fan sub-equation method.

The traveling wave transformation (2.2) permits us to reduce (1.1) to an ODE in the form

$$cu' + 5u^2u' + \frac{25}{2}u'u'' + 5uu''' + u^{(5)} = 0. \tag{3.1}$$

According to Steps 1 and 2 in Section 2, by balancing $u^{(5)}$ and $u^2u'$ in (3.1), we obtain $n + 3 = 3n - 1$ and therefore give $n = 2$. Thus we can suppose that (3.1) has the following formal solutions

$$u(\xi) = a_0 + a_1\phi(\xi) + a_2\phi^2(\xi), \tag{3.2}$$

where $\phi(\xi)$ satisfies (2.5).

Substituting (3.2) and (2.5) into (3.1), collecting all terms with the same power in $\phi^k \sqrt{\sum_{j=0}^{4} c_j \phi^j}$ $(0 \le k \le 5)$, then setting all their coefficients to zero yields a set of simultaneous algebraic equations omitted here for the sake of brevity. Solving these algebraic equations with the help of Maple, we get the following two sets of solutions.

1) The first set of parameters is given by

$$\begin{aligned} &c_1 = 0,\ a_2 = -3c_4,\ a_1 = -\frac{3}{2}c_3,\ a_0 = \frac{3c_3^2 - 16c_2c_4}{16c_4}, \\ &c = \frac{240c_2c_3^2c_4 + 768c_0c_4^3 - 45c_3^4 - 256c_2^2c_4^2}{256c_4^2}, \end{aligned} \tag{3.3}$$

where $c_0,\ c_2,\ c_3,\ c_4 \neq 0$ are arbitrary constants.

2) The second set of parameters is given by

$$\begin{aligned} &c_1 = \frac{c_3\left(4c_2c_4 - c_3^2\right)}{8c_4^2},\ a_2 = -24c_4, \\ &a_1 = -12c_3,\ a_0 = \frac{3c_3^2 - 16c_2c_4}{2c_4}, \\ &c = \frac{11\left(144c_2c_3^2c_4 - 21c_3^4 + 768c_0c_4^3 - 256c_2^2c_4^2\right)}{16c_4^2}, \end{aligned} \tag{3.4}$$

where $c_0,\ c_2,\ c_3,\ c_4 \neq 0$ are arbitrary constants.

We may obtain many kinds of exact solutions depending on the special values chosen for $c_j$.

**Case 1.** If $c_0 = c_1 = c_3 = 0,\ c_2 < 0$ and $c_4 > 0$, Equation (2.5) admits a triangle solution

$$\phi(\xi) = \sqrt{-\frac{c_2}{c_4}}\ \sec\left(\sqrt{-c_2}\,\xi\right). \tag{3.5}$$

Substituting (3.5) along with (3.3) and (3.4) into (3.2) respectively yields two triangle solutions of (1.1)

$$u_1(x,t) = -c_2 + 3c_2\sec^2\left[\sqrt{-c_2}\left(x + c_2^2t\right)\right] \tag{3.6}$$

and

$$u_2(x,t) = -8c_2 + 24c_2\sec^2\left[\sqrt{-c_2}\left(x + 176c_2^2t\right)\right] \tag{3.7}$$

with $c_2 < 0$ being an arbitrary constant.

**Case 2.** If $c_0 = \frac{c_2^2}{4c_4},\ c_1 = c_3 = 0,\ c_2 > 0$ and $c_4 > 0$, we can find two periodic solutions of (2.5)

$$\phi(\xi) = \pm\sqrt{\frac{c_2}{2c_4}}\tan\left(\sqrt{\frac{c_2}{2}}\xi\right). \tag{3.8}$$

Substituting (3.8), (3.3) and (3.4) into (3.2) respectively yields two triangle solutions of (1.1)

$$u_3(x,t) = -c_2 - \frac{3}{2}c_2\tan^2\left[\sqrt{\frac{c_2}{2}}\left(x + \frac{c_2^2}{4}t\right)\right] \tag{3.9}$$

and

$$u_4(x,t) = -8c_2 - 12c_2\tan^2\left[\sqrt{\frac{c_2}{2}}\left(x + 44c_2^2t\right)\right] \tag{3.10}$$

with $c_2 > 0$ being an arbitrary constant.

 

**Case 3.** For $c_0 = c_1 = c_3 = 0,\ c_2 > 0,\ c_4 < 0$, we can gain the following hyperbolic function solution to (2.5)

$$\phi(\xi) = \sqrt{-\frac{c_2}{c_4}}\operatorname{sech}\left(\sqrt{c_2}\,\xi\right). \tag{3.11}$$

Similarly, we obtain two peak-shaped solitary wave solutions of (1.1)

$$u_5(x,t) = -c_2 + 3c_2\operatorname{sech}^2\left[\sqrt{c_2}\left(x + c_2^2 t\right)\right] \tag{3.12}$$

and

$$u_6(x,t) = -8c_2 + 24c_2\operatorname{sech}^2\left[\sqrt{c_2}\left(x + 176c_2^2 t\right)\right]. \tag{3.13}$$

with $c_2 > 0$ being an arbitrary constant.

To show the physical insight of these solitary wave solutions, here we take $u_5$ as an example. **Figure 1** shows the wave plot of the solution $u_5$ with $c_2 = 1$ and the initial status of $u_5$. Clearly the solution is a bell-shaped solitary wave with peak form and describes the traveling of wave in the negative $x$-direction.

**Case 4.** For $c_0 = \dfrac{c_2^2}{4c_4}, c_1 = c_3 = 0, c_2 < 0, c_4 > 0$, Equation (2.5) admits two following hyperbolic function solutions

$$\phi(\xi) = \pm\sqrt{\frac{-c_2}{2c_4}}\tanh\left(\sqrt{-\frac{c_2}{2}}\,\xi\right). \tag{3.14}$$

This in turn gives two peak-shaped solitary wave solutions of (1.1)

$$u_7(x,t) = -c_2 + \frac{3}{2}c_2\tanh^2\left[\sqrt{-\frac{c_2}{2}}\left(x + \frac{c_2^2}{4}t\right)\right] \tag{3.15}$$

and

$$u_8(x,t) = -8c_2 + 12c_2\tanh^2\left[\sqrt{-\frac{c_2}{2}}\left(x + 44c_2^2 t\right)\right] \tag{3.16}$$

with $c_2 < 0$ being an arbitrary constant.

**Case 5.** For $c_0 = c_1 = 0, c_3 = \pm 2\sqrt{c_2 c_4}, c_2 > 0, c_4 > 0$, one can find the following hyperbolic solutions of (2.5)

$$\phi(\xi) = \mp\frac{1}{2}\sqrt{\frac{c_2}{c_4}}\left[1 + \tanh\left(\frac{\sqrt{c_2}}{2}\xi\right)\right]. \tag{3.17}$$

Similar to Case 1, (1.1) has two peak-shaped solitary wave solutions.

$$u_9(x,t) = \frac{c_2}{2} - \frac{3c_2}{4}\tanh^2\left[\frac{\sqrt{c_2}}{32}\left(16x + c_2^2 t\right)\right] \tag{3.18}$$

and

$$u_{10}(x,t) = 4c_2 - 6c_2\tanh^2\left[\frac{\sqrt{c_2}}{2}\left(x + 11c_2^2 t\right)\right] \tag{3.19}$$

with $c_2 > 0$ being an arbitrary constant.

**Case 6.** For $c_0 = \dfrac{c_2^2 k_1^2\left(k_1^2 - 1\right)}{c_4 p_1^2},\ c_1 = c_3 = 0,\ c_2 > 0$ and $c_4 < 0$, Equation (2.5) admits the following Jacobian elliptic function solution

$$\phi(\xi) = \sqrt{\frac{-k_1^2 c_2}{c_4 p_1}}\operatorname{cn}\left(\sqrt{\frac{c_2}{p_1}}\xi, k_1\right), \tag{3.20}$$

where $p_1 = 2k_1^2 - 1$ and $k_1 \in \left(\sqrt{2}/2, 1\right)$ is an arbitrary constant.

This in turn gives the following two doubly periodic wave solution of (1.1)

$$u_{11}(x,t) = -c_2 + \frac{3c_2 k_1^2}{p_1}\operatorname{cn}^2\left[\sqrt{\frac{c_2}{p_1}}\left(x + \frac{q_1 c_2^2}{p_1^2}t\right), k_1\right] \tag{3.21}$$

and

$$u_{12}(x,t) = -8c_2 + \frac{24c_2 k_1^2}{p_1}\operatorname{cn}^2\left[\sqrt{\frac{c_2}{p_1}}\left(x + \frac{176 q_1 c_2^2}{p_1^2}t\right), k_1\right], \tag{3.22}$$

where $q_1 = k_1^4 - k_1^2 + 1$.

To demonstrate the physical insight of the new solutions, we take $u_{11}$ as an example. Obviously the solution is a Jacobi elliptic function with two periods and describes the traveling of wave in the negative $x$-direction with the wave velocity $c_2^2 q_1 / p_1^2$. **Figure 2** shows the wave plot of the solution $u_{11}$ to (1.1) with $c_2 = 0.5$, $k_1 = 0.9$ and the initial status of $u_{11}$.

**Case 7.** For $c_0 = \dfrac{c_2^2\left(1 - k_2^2\right)}{c_4 p_2^2}, c_1 = c_3 = 0, c_2 > 0, c_4 < 0,$ we can obtain one Jacobian elliptic function solution of (2.5)

$$\phi(\xi) = \sqrt{\frac{-c_2}{c_4 p_2}}\operatorname{dn}\left(\sqrt{\frac{c_2}{p_2}}\xi, k_2\right), \tag{3.23}$$

where $p_2 = 2 - k_2^2$ and $k_2 \in (0,1)$ is an arbitrary constant.

Thus we can give two corresponding periodic traveling wave solutions of (1.1)

$$u_{13}(x,t) = -c_2 + \frac{3c_2}{p_2}\operatorname{dn}^2\left[\sqrt{\frac{c_2}{p_2}}\left(x + \frac{q_2 c_2^2}{p_2^2}t\right), k_2\right] \tag{3.24}$$

and

$$u_{14}(x,t) = -8c_2 + \frac{24c_2}{p_2}\operatorname{dn}^2\left[\sqrt{\frac{c_2}{p_2}}\left(x + \frac{176 q_2 c_2^2}{p_2^2}t\right), k_2\right], \tag{3.25}$$

where $q_2 = k_2^4 - k_2^2 + 1$.

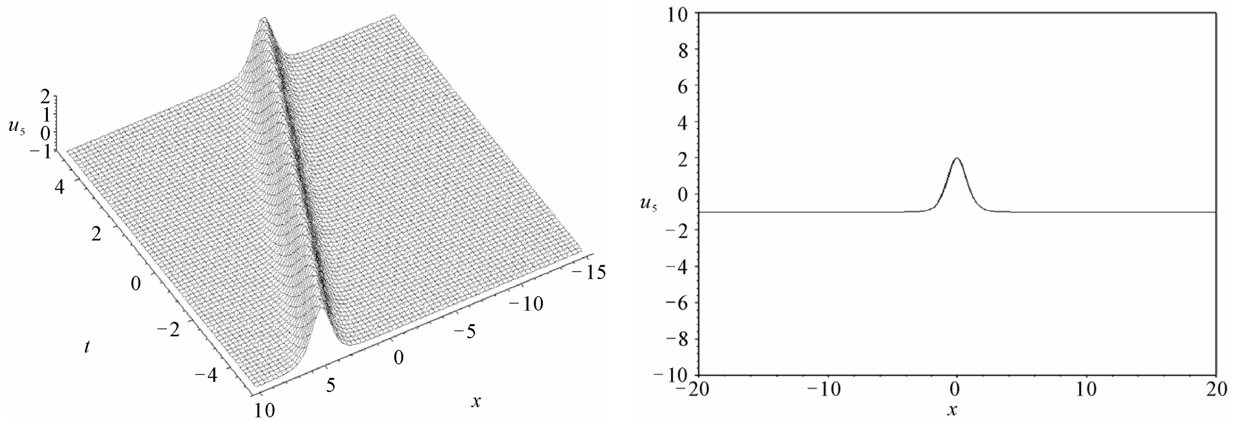


**Figure 1. The plot of the peak-shaped solitary wave solution $u_5$ to (1.1) with $c_2$ = 1 and the initial status of $u_5$.**

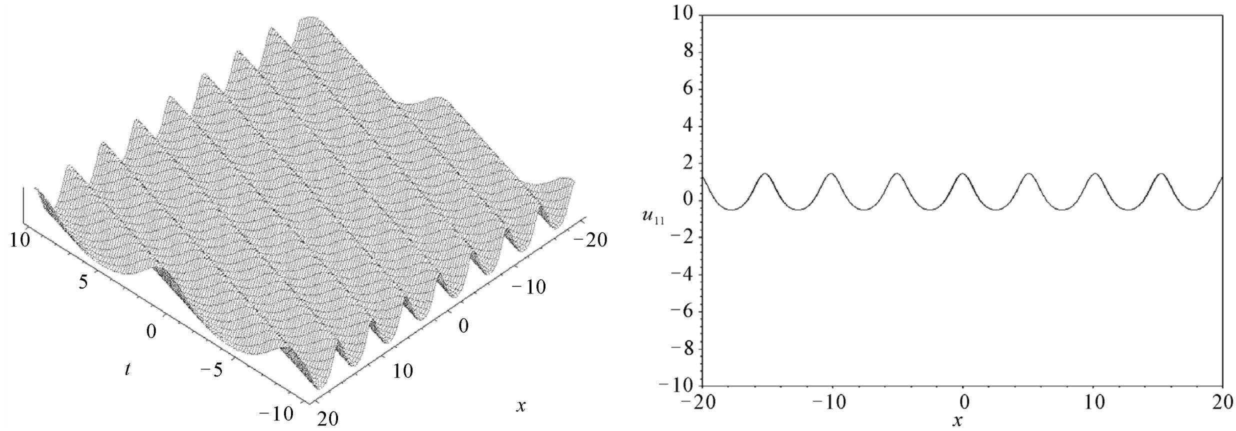


**Figure 2. The plot of the periodic traveling wave solution $u_{11}$ to (1.1) with $c_2$ = 0.5, $k_1$ = 0.9 and the initial status of $u_{11}$.**

**Case 8.** For $c_0 = \frac{c_2 k_3^2}{c_4 p_3^2}, c_1 = c_3 = 0, c_2 < 0, c_4 > 0,$ (2.5) admits two kinds of Jacobian elliptic doubly periodic wave solutions

$$\phi(\xi) = \pm\sqrt{-\frac{c_2 k_3^2}{c_4 p_3}}\,\mathrm{sn}\left(\sqrt{-\frac{c_2}{p_3}}\xi, k_3\right) \tag{3.26}$$

where $p_3 = 1 + k_3^2$ and $k_3 \in (0,1)$ is an arbitrary constant.

This in turn gives two corresponding periodic traveling wave solutions of (1.1)

$$u_{15}(x,t) = -c_2 + \frac{3c_2 k_3^2}{p_3}\mathrm{sn}^2\left[\sqrt{-\frac{c_2}{p_3}}\left(x + \frac{q_3 c_2^2}{p_3^2}t\right), k_3\right] \tag{3.27}$$

and

$$u_{16}(x,t) = -8c_2 + \frac{24 c_2 k_3^2}{p_3}\mathrm{sn}^2\left[\sqrt{-\frac{c_2}{p_3}}\left(x + \frac{176 q_3 c_2^2}{p_3^2}t\right), k_3\right], \tag{3.28}$$

where $q_3 = k_3^4 - k_3^2 + 1$.

## 4. Conclusions and Summary

In this paper, the Fan sub-equation method has been successfully applied to obtain many traveling wave solutions of the (1 + 1) dimensional Kaup-Kupershmidt equation. These rich results show that this method is effective and simple and a lot of solutions can be obtained in the same time. It is also a promising method to solve other nonlinear equations.

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