Applied Mathematics, 2009, 1, 1-16

Scientific Research

ON COMPLEX VARIABLE METHOD IN FINITE ELASTICITY

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Abstract. We highlight the alternative presentation of the Cauchy-Riemann conditions for the analyticity of a complex variable function and consider plane equilibrium problem for an elastic transversely isotropic layer, in finite deformation. We state the fundamental problems and consider traction boundary value problem, as an example of fundamental problem-one. A simple solution of "Lame's problem" for an infinite layer is obtained. The profile of the deformed contour is given; and this depends on the order of the term used in the power series specification for the complex potential and on the material constants of the medium.

AMS Mathematics Subject Classification: 73G05.
Key words and phrases: Analyticity, complex potentials, fundamental problems, finite deformation, anisotropic parameter.

1. Introduction

In the last three or so decades now, in continuum mechanics, specifically in solid mechanics, it is modern to apply *finite element method* to solving problems. This is particularly so in the cases where analytical solution is not readily attainable. It is true that robust computational methods and corresponding packages abound everywhere today. However, it is no less true that, when analytical method for a given problem exists, then the analytical solution is more informative than any numerical solution. So, sustained efforts is always on-going to unravel and develope suitable analytical method of solution in continuum mechanics. A ready avenue to obtaining analytical solutions in plane problems is to invoke the complex variable method. An example in this direction is the so-called theory of *first* and the *second fundamental problems* [1] in plane problems of elasticity.

In fact, the universal compatibility equation of Levy-Maurice, that provided the theoretical basis and hence opened the floodgate of the photoelastic method of analysis in solid mechanics, has revealed the potency of plane problems in elasticity. In the same vein, the pioneering works of Kolosov, Muskhelishvili and Stevenson [2] have endowed the plane problem of elasticity with the effective tools of complex analysis. Later, this tool of complex analysis has come to be extended to the investigation of phenomena in the theory of finite elasticity [3,4]. Ogden and Isherwood [5] considered some finite plane strain problems using the stress function method. In the present work, a complex potential approach is adopted.

Specifically, if an expression could be shown to satisfy a Laplace equation, hence analytic in the appropriate sense, then it can be expressed in a pertinent convergent series; which often admits some form of analytical solution and other indepth analysis. In fact, on the basis of the Cauchy integral theorem/formular, by the known value of an analytic function on a contour of a region we can establish that function in the whole plane region. The treasure in this for the plane problem of elasticity is that if we can find or specify, for instance, the resultant force vector and the resultant moment vector on the contour of a plane region then, by the Cauchy integral formular, any plane problem could be assumed solved. Consequently, most problems are reduced to finding the two analytic functions in the domain where solutions are sought.

The question then arises, how do we find these analytical functions in the case of finite deformations? Moreover, an explicit and tractable universal expression for the energy function is not often known in finite elasticity. Here, on the basis of the semi-linear material of John [4,6], we give the general relations for the static, plane, finite deformation problem of elasticity for a transversely isotropic body.

Using the parameter of anisotropy, we carry out the anisotropic expansion of pertinent state variables and reduce the initial-boundary value problem to a recurrence system of Laplace and Poisson equations. For the first system we give the complex potentials and the fundamental problems, which are nonlinear due to the consideration of finite deformations. An example traction boundary value problem is then considered for an infinite transversely isotropic layer, on the basis of the fundamental problem-one.

PROBLEM SETTING

Analyticity of a Complex Variable Expression

We recall that the analyticity of a complex variable function $f(z) = u(x,y) + iv(x,y), \quad i \equiv \sqrt{-1}$, in the finite plane $\mathcal{C}$, is guaranteed by the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial u}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1.1}$$

An equivalent expression to (1.1), through the complex variable $z = x + iy$ and its conjugate $\overline{z} = x - iy$, is also given in the literature as

$$\frac{\partial f}{\partial \overline{z}} = 0. \tag{1.2}$$

In this paper, we shall adopt the presentation in (1.2).

Geometry of Deformation

Let Ω be a transverse isotropic medium in three dimensional euclidean space E^3. We look at the equilibrum state of Ω in plane finite deformation. The deformation of Ω is given by specifying the position vector $\vec{X}$ of a particle prior to deformation in the initial (reference) configuration Ω_o with the boundary Σ_o and its orientation normal unit vector $\vec{N}$ and the position vector $\vec{x}$ in the current configuration Ω with the boundary Σ and its orientation normal vector $\vec{n}$:

$$\vec{X} = X^1\vec{e_1} + X^2\vec{e_2} + X^3\vec{e_3}, \tag{1.3}$$

$$\vec{x} = x^1\vec{e_1} + x^2\vec{e_2} + x^3\vec{e_3}, \tag{1.4}$$

such that for plane deformation:

$$x^\alpha = x^\alpha(X^1, X^2), \quad x^3 = lX^3, \quad \alpha = 1, 2, \tag{1.5}$$

where X^m, x^m are the material coordinates in Ω_o and Ω respectively; $\vec{e_m}$ is the orthonormal basis; $m = 1, 2, 3$; l is any real constant.

Let the deformation gradient be the tensor-gradient of the position vector $\vec{x}$ in Ω taken with respect to the initial configuration Ω_o. That is, applying the operator of gradient-vector in the reference configuration, $\overset{o}{\nabla} \equiv \vec{e_m}\frac{\partial}{\partial X^m}$, on the position vector $\vec{x}$ in the current configuration, we obtain the tensor-gradient (or *deformation gradient*):

$$\mathbf{A} \equiv \overset{o}{\nabla}\vec{x} = \vec{e}_\alpha \otimes \vec{e}_\beta \frac{\partial x^\beta}{\partial X^\alpha} + \vec{e}_3 \otimes \vec{e}_3 l. \tag{1.6}$$

We also consider the *rotation tensor* of the medium:

$$\mathbf{R} = \mathbf{V}^{-1} \cdot \mathbf{A} = \mathbf{I}\cos\chi + (1 - \cos\chi)\vec{e}_3 \otimes \vec{e}_3 - \vec{e}_3 \times \mathbf{I}\sin\chi \tag{1.7}$$

where, $\mathbf{V}$, such that $\mathbf{V}^2 = \mathbf{A} \cdot \mathbf{A}^T$, is the symmetric left stretch tensor, arising from the polar decomposition of the deformation gradient, $\mathbf{A} = \mathbf{V} \cdot \mathbf{R}$; $\mathbf{I} = \vec{e}_1 \otimes \vec{e}_1 + \vec{e}_2 \otimes \vec{e}_2 + \vec{e}_3 \otimes \vec{e}_3$ is the unit tensor in E^3, $\vec{e}_3 \times \mathbf{I}$ is the cross product of $\vec{e}_3$ and $\mathbf{I}$ and

$$\cos\chi = \frac{1}{q}\left(\frac{\partial x^1}{\partial X^1} + \frac{\partial x^2}{\partial X^2}\right), \quad \sin\chi = \frac{1}{q}\left(\frac{\partial x^2}{\partial X^1} - \frac{\partial x^1}{\partial X^2}\right),$$

$$q = \sqrt{\left(\frac{\partial x^1}{\partial X^1} + \frac{\partial x^2}{\partial X^2}\right)^2 + \left(\frac{\partial x^2}{\partial X^1} - \frac{\partial x^1}{\partial X^2}\right)^2}. \tag{1.8}$$

For any vectors (or tensors, as may be appropriate) ϕ and ψ, here and elsewhere, we denote their dot product, double dot product and cross product respectively by $\phi \cdot \psi$, $\phi \cdot \cdot \psi$ and $\phi \times \psi$.

Static Equation for Transversely Isotropic Material

We look at the equilibrium of Ω in a state of plane finite deformation. For this, we first introduce the energy function for an isotropic semi-linear material in finite deformation, proposed by F. John [6]:

$$W = \mu S_2 + 1/2\lambda S_1^2, \tag{1.9}$$

where S_1 and S_2 are the invariants $S_1 = \mathbf{I} \cdot \cdot (\mathbf{V} - \mathbf{I})$, $\ S_2 = (\mathbf{V} - \mathbf{I}) \cdot \cdot (\mathbf{V} - \mathbf{I})$. λ and μ are the Lame constants. Then, on the basis of (1.9) we consider an energy function [7,8] for a transversely isotropic semi-linear material:

$$W = \lambda_2 S_2 + 1/2\lambda_1 S_1^2 + \lambda_0 S_0, \tag{1.10}$$

where $S_0 = \vec{c} \cdot \mathbf{V}^2 \cdot \vec{c}$ is an additional invariant of deformation, due to anisotropy. $\vec{c}$ is the unit vector characterizing the direction of anisotropy (i.e., the axis of transverse isotropy). $\lambda_0, \lambda_1, \lambda_2$ are the material constants. In the case of a randomly unidirectional fibre reinforced composite or a lamina composite the material constants represent the effective moduli [8,9,10]

$$\lambda_2 = \langle \mu \rangle, \quad \lambda_1 = \langle \lambda \rangle + \frac{\left\langle \frac{\lambda}{(\lambda + 2\mu)} \right\rangle^2}{\left\langle \frac{1}{(\lambda + 2\mu)} \right\rangle} - \left\langle \frac{\lambda^2}{(\lambda + 2\mu)} \right\rangle, \quad \lambda_3 = \frac{1}{\langle \frac{1}{\mu} \rangle}, \quad \lambda_o = 2(\lambda_3 - \lambda_2). \tag{1.11}$$

We note that in the case of degeneracy into isotropy, the energy function (1.10) automatically reduces to the energy (1.9) and, accordingly, for the effective moduli $\lambda_3, \lambda_2, \lambda_1$, we have

$$\lambda_3 = \lambda_2 = \mu, \quad \lambda_1 = \lambda, \qquad \lambda_o = 0. \tag*{(1.11)$'$}$$

For any finite function $\varphi(\vec{\xi}) \in \Omega$, $\ \langle \varphi \rangle$ denotes its geometric average over Ω $\langle \varphi \rangle = \frac{1}{|\Omega|} \int_\Omega \varphi \, d\,\Omega$, where $|\Omega|$ denotes the volume of Ω.

Now, for hyperelasticity, we take the Frechet derivative [4,11] of the energy with respect to deformation gradient $\mathbf{A}$ and obtain the Piola stress tensor $\mathbf{P}$, to which it is energy conjugate:

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{A}} = 2\lambda_2 \mathbf{A} + (\lambda_1 S_1 - 2\lambda_2)\mathbf{R} + 2\lambda_0 \vec{c} \otimes \vec{c} \cdot \mathbf{A}. \tag{1.12}$$

In the absence of body force, we have the equilibrium equation

$$\overset{o}{\nabla} \cdot \mathbf{P} = 0$$

and the accompanying boundary condition

$$\vec{f} d\Sigma = \vec{N} \cdot \mathbf{P} d\Sigma_o,$$

where $d\Sigma$ is the element of the boundary surface area in the current configuration on which a prescribed force per unit surface area $\vec{f}$ acts while $d\Sigma_o$ is the element of the boundary surface area in the reference configuration, with the outer unit normal vector $\vec{N} = (N_1, N_2)$.

The component form of these relations is:

$$\frac{\partial P^{11}}{\partial X^1} + \frac{\partial P^{21}}{\partial X^2} = 0, \quad \frac{P^{12}}{\partial X^1} + \frac{\partial P^{22}}{\partial X^2} = 0, \quad \frac{\partial P^{33}}{\partial X^3} = 0 \tag{1.13}$$

and

$$f_1 \frac{d\Sigma}{d\Sigma_o} = N_1 P^{11} + N_2 P^{21}, \quad f_2 \frac{d\Sigma}{d\Sigma_o} = N_1 P^{12} + N_2 P^{22}, \tag{1.13$'$}$$

where by the deformation (1.5), we have $d\Sigma = l ds dX^3$ and $d\Sigma_o = dS dX^3$ and the surface area ratio $\frac{d\Sigma}{d\Sigma_o}$ can be expressed through the corresponding arclengths ds and dS as $l\frac{ds}{dS}$.

2. Complex Variable Formulation

Complex Variable Representation of Equilibrium Equation

We now look at Ξ, a two-dimension cross-sectional region of the original medium Ω, as a subspace of the complex plane $\mathcal{C}$. In place of the material coordinates X^1, X^2 and x^1, x^2, in the reference configuration and the current configuration respectively, we introduce the complex variables:

$$Z = X^1 + iX^2 \in \Xi_o \ \ and \ \ z = x^1 + ix^2 \in \Xi, \tag{2.1}$$

where $i = \sqrt{-1}$ is the imaginary unit. Then,

$$\frac{\partial}{\partial Z} = \frac{1}{2}\left(\frac{\partial}{\partial X^1} - i\frac{\partial}{\partial X^2}\right), \quad \frac{\partial}{\partial \overline{Z}} = \frac{1}{2}\left(\frac{\partial}{\partial X^1} + i\frac{\partial}{\partial X^2}\right) \tag{2.2}$$

and

$$2\frac{\partial z}{\partial Z} = \left(\frac{\partial x^1}{\partial X^1} + \frac{\partial x^2}{\partial X^2}\right) + i\left(\frac{\partial x^2}{\partial X^1} - \frac{\partial x^1}{\partial X^2}\right),$$

or, in view of (1.6),

$$\frac{\partial z}{\partial Z} = \frac{1}{2} q \exp{(i\chi)} \quad and \quad \exp{(i\chi)} = \frac{\partial z}{\partial Z} / \left| \frac{\partial z}{\partial Z} \right| . \tag{2.3}$$

The Piola stress tensor (1.12), the equilibrium equations (1.13) and the boundary conditions $(1.13)'$ respectively become

$$P^{11} + iP^{12} = \phi(q)\mathrm{exp}\,(i\chi) + 2i\lambda_2 \frac{\partial z}{\partial X^2} + 2\lambda_o \left(q \exp{(i\chi)} + i \frac{\partial z}{\partial X^2} \right),$$

$$P^{22} - iP^{21} = \phi(q)\mathrm{exp}\,(i\chi) - 2\lambda_2 \frac{\partial z}{\partial X^1}, \tag{2.4}$$

$$P^{33} = P^{33}(X^1, X^2),$$

$$\frac{\partial \Phi(Z, \overline{Z})}{\partial \overline{Z}} = -2\lambda_0 \frac{\partial^2 z}{\partial X^1 \partial X^1} \tag{2.5}$$

and

$$if \frac{ds}{dS} - \frac{dZ}{dS} \Phi(Z, \overline{Z}) + 4\lambda_2 \frac{dz}{dS} = 2\lambda_0 \frac{\partial z}{\partial X^1} iN_1, \tag*{(2.5)$'$}$$

where $N = N_1 + iN_2$; $f = f_1 + if_2$ is the complex form of the boundary force per unit present length; ds and dS are the arc elements in the current and reference configurations respectively, and the yet to be established function

$$\Phi(Z, \overline{Z}) \equiv \phi(q)\mathrm{exp}\,(i\chi), \quad \phi(q) = (\lambda_1 + 2\lambda_2)(q-2) + 2\lambda_2 + \lambda_1(l-1). \tag{2.6}$$

Anisotropic Expansion of State Variables

We note that if Ξ were to be an isotropic medium, then the right-hand side in (2.5) and $(2.5)'$ would vanish. This implies that λ_0 is a true parameter of anisotropy. So, we non-dimensionalize it and expand the dependent variables in terms of

$$\beta = \frac{\lambda_0}{\lambda_1 + 2\lambda_2} < 1. \tag{2.7}$$

That is,

$$z = z_0 + \beta z_1 + \beta^2 z_2 + \beta^3 z_3 + ...,$$

$$\Phi = \Phi_0(Z, \overline{Z}) + \beta\Phi_1(Z, \overline{Z}) + \beta^2\Phi_2(Z, \overline{Z}) + \beta^3\Phi_3(Z, \overline{Z}) + ..., \tag{2.8}$$

$$\vec{f} = \vec{f_0} + \beta\vec{f_1} + \beta^2\vec{f_2} + \beta^3\vec{f_3} + ...$$

Putting (2.8) in (2.5) and $(2.5)'$ we obtain for the equilibrium equation and the boundary condition, respectively,

$$\sum_{m=0}^{\infty} \beta^m \mathcal{F}_m = 0, \tag{2.9}$$

$$\sum_{m=0}^{\infty} \beta^m \mathcal{P}_m = 0, \tag*{(2.9)$'$}$$

where

$$\mathcal{F}_m \equiv \frac{\partial \Phi_m}{\partial \overline{Z}} + 2(\lambda_1 + 2\lambda_2)\frac{\partial^2 z_{m-1}}{\partial X^1 \partial X^1},$$

$$\mathcal{P}_m \equiv i f_m \frac{ds}{dS} - \frac{dZ}{dS}\Phi_m(Z, \overline{Z}) + 4\lambda_2 \frac{dz_m}{dS} - 2\lambda_0 \frac{\partial z_{m-1}}{\partial X^1} i N_1,$$

for $m = 0, 1, 2, ..., z_1 = 0$.

Now, setting every coefficient of the powers of β to zero in equations (2.9) and (2.9)$'$ we obtain a recurrence system of equations, i.e., $\mathcal{F}_m = 0$ and $\mathcal{P}_m = 0; m = 0, 1, 2, \ldots$. The first equation in the recurrence system due to (2.9) is homogeneous (i.e. zero right hand side), while each of the subsequent ones is nonhomogeneous with right hand sides depending recursively on the solution of the previous equation.

Thus, this much is the effect of anisotropy on the medium: and has in no way influenced the fact or exposed the issue of finite deformation. The effect of finite deformation is exposed in what follows.

Complex Potential and Nonlinear Fundamental Problems

We now deduce the *Fundamental Problems*, analogous to the Kolosov - Muskhelishvili first and second fundamental problems of infinitesimal elasticity, given in terms of two complex potentials [1-4,12].

Let

$$z_0 \equiv v = v_1 + iv_2, \quad \Phi_0 \equiv F(Z, \overline{Z}) \quad \textit{and} \quad f_0 \equiv h = h_1 + ih_2. \tag{2.10}$$

Then, the first equation in the recurrence system due to (2.9) is:

$$\frac{\partial F(Z, \overline{Z})}{\partial \overline{Z}} = 0, \tag{2.11}$$

with the corresponding boundary condition from recurrence system (2.9)$'$

$$ih\frac{ds}{dS} = \frac{dZ}{dS}F(Z, \overline{Z}) - 4\lambda_2 \frac{dv}{dS}, \tag*{(2.11)$'$}$$

where $h \equiv f_0$ is the specified force per unit length of the current boundary contour. By the Cauchy-Riemann equations, i.e., (1.2), (2.11) implies that $F(Z, \overline{Z})$ is an analytic function of the variable Z only, in the finite domain. It can then be written as a uniformly convergent series in the form

$$F(Z,\overline{Z}) \equiv F(Z) = \sum_{m=0}^{\infty} \alpha_m Z^m,$$

where α_m are constants.

Now, $F(Z)$ is related to the material positions $v(Z,\overline{Z})$ in the current plane in a pertinent manner. In fact, setting $l = 1$ in (1.6) for convenience and using (2.3), (2.6) and (2.10), we write

$$F(Z) = \phi_0(q_0)\exp(i\chi_0), \quad q_0 = 2\left|\frac{\partial v}{\partial Z}\right|, \quad \exp(i\chi_0) = \frac{\partial v}{\partial Z}\Big/\left|\frac{\partial v}{\partial Z}\right|. \tag{2.12}$$

Then,

$$F(Z) = 2(\lambda_1 + 2\lambda_2)\frac{\partial v}{\partial Z} - 2(\lambda_1 + \lambda_2)\frac{F(Z)}{|F(Z)|}$$

and

$$(\lambda_1 + 2\lambda_2)\frac{\partial v}{\partial Z} = \lambda_2 {\varphi'}^2(Z) + (\lambda_1 + \lambda_2)\frac{\varphi'(Z)}{\overline{\varphi'(Z)}}, \tag{2.13}$$

where

$$F(Z) \equiv 2\lambda_2 {\varphi'}^2(Z). \tag{2.14}$$

Following Goursat or Muskhelishvili [1,2], we integrate (2.13) and differentiate the result with respect to $\overline{Z}$ to give

$$(\lambda_1 + 2\lambda_2)v(Z,\overline{Z}) = \lambda_2 \int {\varphi'}^2(Z)d\,Z - (\lambda_1 + \lambda_2)\left[\frac{\varphi(Z)}{\overline{\varphi'(Z)}} + \overline{\psi(Z)}\right] \tag{2.15}$$

and

$$-\frac{\lambda_1 + 2\lambda_2}{\lambda_1 + \lambda_2}\frac{\partial v}{\partial \overline{Z}} = \frac{\varphi(Z)\overline{\varphi''(Z)}}{\overline{\varphi'(Z)\varphi'(Z)}} - \overline{\psi'(Z)}. \tag{2.16}$$

Here, $\psi(Z)$ is another analytic function that emerges as a result of integration, in the form of $\overline{\psi(Z)}$. Thus, in place of $F(Z)$ we now have two analytic functions, called *complex potentials*, through which we can define any particle position $v(Z,\overline{Z})$ on the current plane Ξ.

Now, on any arc length S on the contour Γ_o of the reference configuration we have:

$$\frac{dv}{dS} = \frac{\partial v}{\partial Z}\frac{dZ}{dS} + \frac{\partial v}{\partial \overline{Z}}\frac{d\overline{Z}}{dS}.$$

Putting (2.15), (2.16) in this, we obtain

$$-i(\lambda_1 + 2\lambda_2)\frac{dv}{dS} = \left[\lambda_2 {\varphi'}^2(Z) + (\lambda_1 + \lambda_2)\frac{\varphi'(Z)}{\overline{\varphi'(Z)}}\right] N$$

$$-(\lambda_1+\lambda_2)\left[\frac{\varphi(Z)\overline{\varphi''(Z)}}{\overline{\varphi'(Z)}\varphi'(Z)}+\overline{\psi'(Z)}\right]\overline{N}, \tag{2.17}$$

where $N = N_1 + iN_2 = -i\frac{dZ}{dS}$; $\overline{N} = N_1 - iN_2 = i\frac{d\overline{Z}}{dS}$.

This now enables us to obtain boundary conditions pertinent for equilibrium. By (1.13)' we have

$$\left[{\varphi'}^2(Z)-\frac{\varphi'(Z)}{\overline{\varphi'(Z)}}\right]N-\left[\frac{\varphi(Z)\overline{\varphi''(Z)}}{\overline{\varphi'(Z)}\varphi'(Z)}+\overline{\psi'(Z)}\right]\overline{N}=\frac{\lambda_1+2\lambda_2}{2\lambda_2(\lambda_1+\lambda_2)}h\frac{ds}{dS}. \tag{2.18}$$

We note that on any contour Γ_o of the cross-section of the body, for any function $\theta(Z,\overline{Z})$:

$$\int_{\Gamma_o}\theta(Z,\overline{Z})\overline{N}dS=\int_{\Gamma_o}(N_1-iN_2)\theta(Z,\overline{Z})dS=\int_{\Xi_o}\left(\frac{\partial}{\partial X^1}-i\frac{\partial}{\partial X^2}\right)\theta(Z,\overline{Z})d\Xi_o$$

$$=2\int_{\Xi_o}\frac{\partial\theta(Z,\overline{Z})}{\partial Z}d\Xi_o$$

and

$$\int_{\Gamma_o}\theta(Z,\overline{Z})NdS=\int_{\Xi_o}\frac{\partial\theta(Z,\overline{Z})}{\partial\overline{Z}}d\Xi_o. \tag{2.19}$$

Then, by (2.13), (2.14), (2.16), (2.19) the resultant force vector $H=\int_\Gamma h(x^1,x^2)ds$ and the resultant moment $M=\int_\Gamma(x^1h_2-x^2h_1)ds=\frac{i}{2}\int_\Gamma(v\overline{h}-\overline{v}h)ds$ will vanish.

Now, let

$$Z=r\,\exp(i\theta),\quad \frac{\partial Z}{\partial\theta}=iZ,\quad \frac{\partial\overline{Z}}{\partial\theta}=-i\overline{Z},$$

$$\int{\varphi'}^2(Z)dZ=I(Z),\quad v'(Z,\overline{Z})=\frac{\partial v}{\partial\theta},\quad h_o(Z,\overline{Z})=\frac{k}{2\lambda_2}\frac{ds}{dS}(h_1+ih_2), \tag{2.20}$$

$$c\equiv\frac{\lambda_2}{\lambda_1+\lambda_2}=1-2\nu_o,\quad k\equiv\frac{\lambda_1+2\lambda_2}{\lambda_1+\lambda_2}=1+c=2(1-\nu_o),\quad \nu_o\equiv\frac{1}{2}\frac{\lambda_1}{\lambda_1+\lambda_2}.$$

Then, (2.18) becomes

$$I(Z)-\frac{\varphi(Z)}{\overline{\varphi'(Z)}}+\overline{\psi(Z)}=\frac{ik}{2\lambda_2}H(Z,\overline{Z}) \tag{2.21}$$

and by (2.17)

$$cI(Z)+\frac{\varphi(Z)}{\overline{\varphi'(Z)}}-\overline{\psi(Z)}=kv(Z,\overline{Z}). \tag{2.22}$$

Thus, we have established two nonlinear fundamental boundary-value problems, given respectively by (2.21) and (2.22), for plane finite deformation. These are analogous to those already known in the literature for the case of linear elasticity [1,2].

(i)Fundamental Problem I: Find the state of elastic equilibrium of a body when the external load applied on the boundary Γ is specified by (2.21), i.e., $H(Z, \overline{Z})$ is given on the contour Γ.

(ii)Fundamental Problem II: find the state of elastic equilibrium for a body whose contour deforms from Γ_o into Γ, given by the position function (2.22), i.e., $v(Z, \overline{Z})$ is given on the contour Γ.

This implies that the solution of plane problems of finite elasticity impinges on knowing $\varphi(Z)$ and $\psi(Z)$. These can be established by their values on the appropriate boundary contours, on the basis of the Cauchy integral theorem, due to their analyticity property.

Physical Components of Stress

It is desirable to express the physical components of stress in the current configuration, since, in practice, it is in this form that they are often used. For this purpose, we make use of the expression connecting the Cauchy stress $\mathbf{T}$ and the Piola stress $\mathbf{P}$, noting that $0 < J \equiv |\frac{\partial x^\alpha}{\partial X^\beta}| = \frac{\partial v}{\partial Z}\frac{\partial \overline{v}}{\partial \overline{Z}} - \frac{\partial v}{\partial \overline{Z}}\frac{\partial \overline{v}}{\partial Z}$:

$$\mathbf{T} = \frac{1}{J}\mathbf{A}^T \cdot \mathbf{P} = \frac{1}{J}\left(\frac{\partial x^\alpha}{\partial X^\beta} P^{\beta\gamma} \vec{e}_\alpha \otimes \vec{e}_\gamma + P^{33} \vec{e}_3 \otimes \vec{e}_3\right); \alpha, \beta, \gamma = 1, 2$$

and

$$t_1 + t_2 = \vec{e}_1 \cdot \mathbf{T} \cdot \vec{e}_1 + \vec{e}_2 \cdot \mathbf{T} \cdot \vec{e}_2 = \frac{1}{J}\frac{\partial x^\alpha}{\partial X^\beta} P^{\alpha\beta},$$

$$t_1 + t_2 + 4\lambda_2 = \frac{1}{J} q_0 \phi(q_0),$$

$$\begin{aligned} t_2 - t_1 - 2it_{12} &= \vec{e}_2 \cdot \mathbf{T} \cdot \vec{e}_2 - \vec{e}_1 \cdot \mathbf{T} \cdot \vec{e}_1 - i(\vec{e}_1 \cdot \mathbf{T} \cdot \vec{e}_2 + \vec{e}_2 \cdot \mathbf{T} \cdot \vec{e}_1) \\ &= \frac{1}{J}\left[\frac{\partial x^2}{\partial X^\alpha} P^{\alpha 2} - \frac{\partial x^1}{\partial X^\alpha} - i(\frac{\partial x^1}{\partial X^\alpha} P^{\alpha 2} + \frac{\partial x^2}{\partial X^\alpha} P^{\alpha 1})\right] \\ &= \frac{1}{J}\phi(q_0)\exp(i\chi_0)\left(\frac{\partial v}{\partial X^1} + i\frac{\partial v}{\partial X^2}\right) = -\frac{4}{J}\frac{\phi(q_0)}{q_0}\frac{\partial v}{\partial Z}\frac{\partial v}{\partial \overline{Z}}. \end{aligned}$$

On the contour Γ we have,

$$t_n + t_{ns} = 2\lambda_2 {\varphi'}^2(Z)\frac{\partial \overline{v}}{\partial s}\frac{\partial Z}{\partial s} - 2\lambda_2 \qquad (2.23)$$

where, t_n, t_{ns} are the normal and tangential physical components of the Cauchy stress, with respect to the contour Γ.

In the cases of simple specification of potentials, it is possible to obtain exact solutions.

3. Example: An Exact Solution

Using the semi-inverse method, we can consider examples on the application of the exposed theory of the complex variable method to large deformation theory of elasticity, with respect to two cases of exact boundary value problems. The first case is the so-called *simple solution*, whereby the two complex potentials, $\varphi(Z)$ and $\psi(Z)$, are carefully specified from the onset. The second case, named *main* (or *mixed*) *solution* is implemented by specifying one of the potentials, say $\varphi(Z)$, while the second, $\psi(Z)$ is deduced from the problem vis-a-vis $\varphi(Z)$. By (2.22) and (2.23) we then obtain the displacement and the physical stress field on every contour Γ of the current configuration, Ξ.

Specifically, we consider here a simple problem, by the fundamental problem-one.

Simple Solution for an Infinite Plane - "Lame's Problem"

Consider the equilibrium of an infinite transversely-isotropic plane; in the current configuration having a unit circular hole, centered at the origin. The axis of transversal-isotropy is in the radial direction. We are then interested in the displacement and stress field developed in the medium as a result of an applied "dead load", σ_1, on the boundary of the hole.

By the semi-inverse method, in the case of an infinite domain, let

$$\varphi(Z) = Z, \quad \psi(Z) = \frac{b}{Z^m}, \tag{3.1}$$

where b is a real number and m is a positive integer.

Now, by putting (3.1) in (2.22), we obtain the displacement field

$$v(Z, \overline{Z}) = Z - k'\overline{Z}^{-m} = r\exp(i\theta) - kr^{-m}\exp(im\theta), \quad Z \equiv r\exp(i\theta), \quad k' \equiv \frac{b}{k}. \tag{3.2}$$

This implies that (3.1) transforms a circle of radius $r =$ const. into a closed curve $R \equiv |v| =$ const. :

$$R^2 = v\overline{v} = r^2 - 2k'r^{1-m}\cos(m-1)\theta + (k'r^{-m})^2. \tag{3.3}$$

With respect to stress, consider the following:

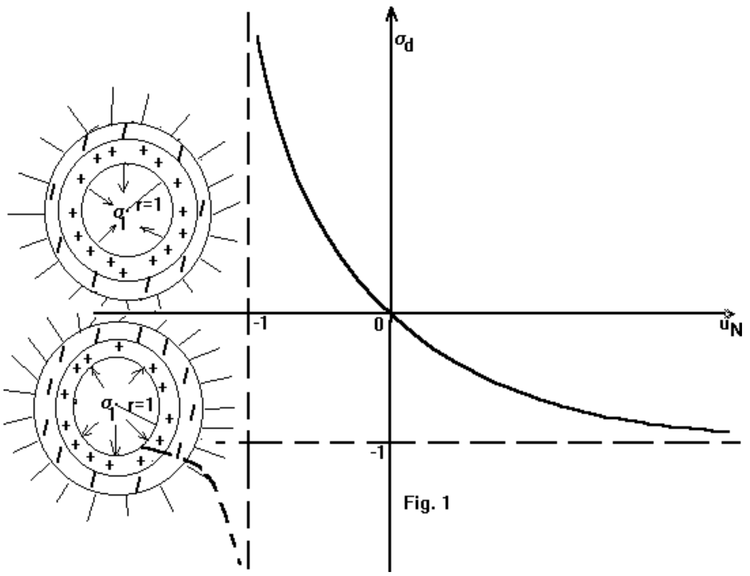

Figure 1. Block partitioning for L

$$\frac{\partial Z}{\partial \theta} = iZ, \quad \frac{\partial v}{\partial \theta} = i(Z - mk'\overline{Z}^{-m}), \quad \frac{\partial \overline{v}}{\partial \theta} = -i(\overline{Z} - mk'Z^{-m}) \quad \textit{and}$$

$$\frac{\partial \overline{v}}{\partial \theta}\frac{\partial Z}{\partial \theta} = r^2 - mk'r^{1-m}\sigma^{1-m},$$

$$R_\theta \equiv \frac{\partial v}{\partial \theta}\frac{\partial \overline{v}}{\partial \theta} = r^2 - 2mk'r^{1-m}\cos(m-1)\theta + (mk'r^{-m})^2. \tag{3.4}$$

By (2.23) we obtain

$$\begin{aligned}
&t_n + it_{ns} \\
&= 2\lambda_2[-1 + R_\theta^{-2}(r^2 - mk'r^{1-m}\sigma^{1-m})] \\
&= 2\lambda_2\{-1 + R_\theta^{-2}\{r^2 - mk'r^{1-m}[\cos(1-m)\theta + i\sin(1-m)\theta]\}\} \\
&= 2mk'\lambda_2 R_\theta^{-2}\{[r^{1-m}\cos(1-m)\theta - mk'r^{-2m}] - ir^{1-m}\sin(1-m)\theta\}.
\end{aligned}$$

This gives

$$t_n = 2mk'\lambda_2 R_\theta^{-2}[r^{1-m}\cos(1-m)\theta - mk'r^{-2m}],$$

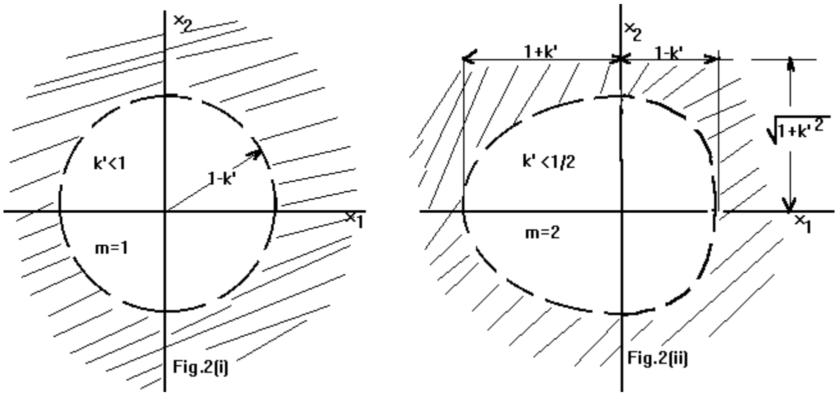

FIGURE 2. Block partitioning for L

$$t_{ns} = -2mk'\lambda_2 R_\theta^{-2} r^{1-m}\sin(1-m)\theta. \tag{3.5}$$

When $m = 1$, from (3.2), (3.3) and (3.5) we obtain

$$v(Z,\overline{Z}) = Z - \frac{k'}{\overline{\overline{Z}}},\;\; R = (r - \frac{k'}{r})^{1/2}, \quad t_n = 2k'\lambda_2(r^2-k')^{-1},\;\; t_{ns} = 0. \tag{3.6}$$

That is, the only non-vanishing components for the displacement and stress are the normal ones. What is more, the expression $(3.6)_1$ for the particle position in the current configuration, where $m = 1$, is similar to the corresponding solutions (6.8), (6.24) obtained in [5]. When $m > 1$ in (3.2), the nonlinear effect becomes obvious. On the other hand, the situation $m = 1$ corresponds to Lame's problem of classical elasticity:

$$u_n(r) = -\frac{k'}{r};\quad t_n(r) = \frac{2k'\lambda_2}{r^2-k'},$$

the only unknown constant k' is found from the loading condition, on the boundary: $t_n(r)_{|r=1} = \sigma_1$. Then, $k' = \frac{\sigma_1}{2\lambda_2+\sigma_1}$, and the displacement will have the expression:

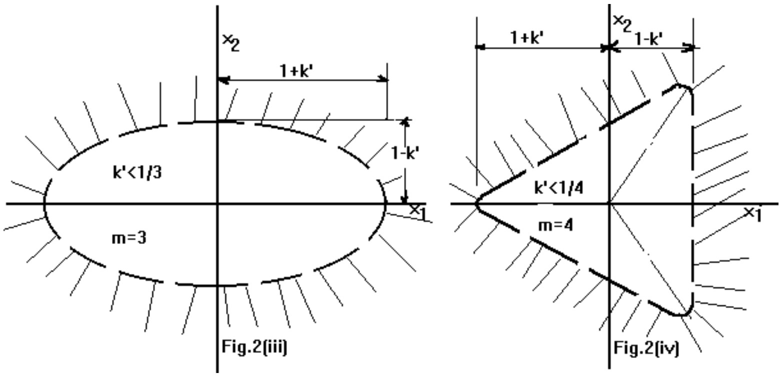

Figure 3. Block partitioning for L

$$u_n = -\frac{\sigma_1}{2\lambda_2 + \sigma_1}\frac{1}{r}.$$

This gives pertinent constraint on the character and magnitude of the admissible boundary load σ_1.

The variation of the radial displacement against the dimensionless boundary stress $\sigma_d \equiv \frac{\sigma_1}{2\lambda_2}$, for a concentric cylindrical composite layer of unit internal radius, is shown in figure 1. The transverse isotropy is in the radial direction; above the horizontal-axis (i.e. the radial displacement axis) the cross-section of the medium is in compression while below, it is under internal pressure. For $\sigma_d > -1$ we obtain a stress-displacement relation similar to the Lame's problem of classical elasticity: the theoretical dashed curve below the asymptotic line $\sigma_d = -1$ is not countenanced by the small deformation theory.

Based on equations (3.1) and (3.2), the specific type of curve described by (3.3) depends on the numerical values of k' and m. So also is the stress field given by (3.5). Consequently, for the particle positions given by (3.3) , we identify the following situations: (i) m = 1 gives a circle, as also obtained in the linear theory, (ii) m = 2 corresponds to a Pascal curve, (iii) m = 3 describes an ellipse, (iv) m

= 4 approximates a triangle etc. Figure 2. shows the shape of these curves in the current configuration for different combinations of values of m and k'.

5. Conclusion

The application of the complex variable method to obtain the fundamental problems one and two for plane equilibrium problem, in the case of anisotropic finite elasticity, has been carried out. The discussion here-in shows that the relevance and potency of the complex variable method in plane problems remain as valid as ever, and as when it was first introduced by Kolosov-Muskhelishvili-Stevensen [2,3] at about the begining of this century; even, in the case of finite deformations and anisotropy. All the relations obtained here, due to finite deformation, also remain true for the case of isotropy/homogeneity, when the effective constants take the corresponding expressions given in (1.11)'. In fact, (2.11) is valid independent of (1.11)'; since material constants do not feature there.

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