

A Note on SK, SK₁, SK₂ Indices of Interval Weighted Graphs

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Abstract

In this study, the SK, SK₁ and SK₂ indices are defined on weighted graphs. Then, the SK, SK₁ and SK₂ indices are defined on interval weighted graphs. Their behaviors are investigated under some graph operations by using these definitions.

Keywords

SK Index, SK₁ Index, SK₂ Index, Weighted Graph, Interval Weighted Graph

1. Introduction

A topological index of a chemical compound is an integer, derived following a certain rule, which can be used to characterize the chemical compound and predict certain physiochemical properties like boiling point, molecular weight, density, refractive index, and so forth [1].

Molecules and molecular compounds are often modeled by molecular graph. A molecular graph is a representation of the structural formula of a chemical compound in terms of graph theory, whose vertices correspond to the atoms of the compound and edges correspond to chemical bonds [2].

Let $G = (V, E)$ be a graph with the vertex set $V(G)$ and edge set $E(G)$ and $v_G = |V(G)|$ vertices and $e_G = |E(G)|$ edges. The degree d_v of the vertex $v \in V(G)$ is the number of first neighbors of v . The edge of the graph G , connecting the vertices u and v , will be denoted by $e = uv$. Throughout this paper, the graphs considered are assumed to be connected. A connected graph is a graph such that there is a path between all pairs of vertices, see books [3] [4].

We now recall some graph operations we shall need in this paper.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs. The sum $G_1 + G_2$ of these two graphs is defined as the graph having the vertex set

$V(G_1 + G_2) = V_1 \cup V_2$ and the edge set
 $E(G_1 + G_2) = E_1 \cup E_2 \cup \{(u, v) : u \in V_1, v \in V_2\}$ [5].

The cartesian product $G_1 \times G_2$ is the graph with vertex set $V(G_1 \times G_2) = V_1 \times V_2$; the vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ of $G_1 \times G_2$ are adjacent if and only if $[u_1 = v_1, u_2 v_2 \in E_2]$ or $[u_2 = v_2, u_1 v_1 \in E_1]$ [5].

Definition 1.1. ([1], SK index). The SK index of a graph $G = (V, E)$ is defined as

$$SK(G) = \frac{1}{2} \sum_{uv \in E(G)} d(u) + d(v)$$

where $d(u)$ and $d(v)$ are the degrees of the vertices u and v in G , respectively.

Definition 1.2. ([1], SK₁ index). The SK₁ index of a graph $G = (V, E)$ is defined as

$$SK_1(G) = \frac{1}{2} \sum_{uv \in E(G)} d(u)d(v)$$

where $d(u)$ and $d(v)$ are the degrees of the vertices u and v in G , respectively.

Definition 1.3. ([1], SK₂ index). The SK₂ index of a graph $G = (V, E)$ is defined as

$$SK_2(G) = \frac{1}{4} \sum_{uv \in E(G)} [d(u) + d(v)]^2$$

where $d(u)$ and $d(v)$ are the degrees of the vertices u and v in G , respectively.

2. Graph Operations on the SK, SK₁, SK₂ Indices of Weighted Graphs

In this section, we define the SK, SK₁ and SK₂ indices on weighted graphs. A weighted graph is a graph each edge of which has been assigned to a number called the weight of the edge. All the weight of the edges are assumed to be positive definite [6] [7].

Let G be a weighted graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set E . Denote by w_{ij} the positive definite weight matrix of order p of the edge ij and assume that $w_{ij} = w_{ji}$. We write $i \sim j$ if vertices i and j are adjacent. Let $w_i = \sum_{j: i \sim j} w_{ij}$ be the weight matrix of the vertex i [6] [7].

Definition 2.1. Let $G = (V, E)$ be a connected weighted graph having n vertices. Let each edge of G be weighted with positive real numbers. The weighted SK index $SK(G, w)$ of G is defined as follows:

$$SK(G, w) = \frac{1}{2} \sum_{uv \in E(G)} w(u) + w(v)$$

where $w(u)$ is the sum of the weights on u .

Definition 2.2. Let $G = (V, E)$ be a connected weighted graph having n ver-

tices. Let each edge of G be weighted with positive real numbers. The weighted SK_1 index $SK_1(G, w)$ of G is defined as follows:

$$SK_1(G, w) = \frac{1}{2} \sum_{uv \in E(G)} w(u)w(v)$$

where $w(u)$ is the sum of the weights on u .

Definition 2.3. Let $G = (V, E)$ be a connected weighted graph having n vertices. Let each edge of G be weighted with positive real numbers. The weighted SK_2 index $SK_2(G, w)$ of G is defined as follows:

$$SK_2(G, w) = \frac{1}{4} \sum_{uv \in E(G)} [w(u) + w(v)]^2$$

where $w(u)$ is the sum of the weights on u .

Theorem 2.4. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple, connected graphs. Then the SK , SK_1 and SK_2 indices of the sum of graphs G_1 and G_2 are respectively given by

$$SK(G_1 + G_2) = \frac{1}{2} \left[\sum_{ij \in E_1} (d_i + n_{G_2}) + (d_j + n_{G_2}) + \sum_{ij \in E_2} (d_i + n_{G_1}) + (d_j + n_{G_1}) \right. \\ \left. + \sum_{i \in V_1, j \in V_2} (d_i + n_{G_2}) + (d_j + n_{G_1}) \right]$$

$$SK_1(G_1 + G_2) = \frac{1}{2} \left[\sum_{ij \in E_1} (d_i + n_{G_2})(d_j + n_{G_2}) + \sum_{ij \in E_2} (d_i + n_{G_1})(d_j + n_{G_1}) \right. \\ \left. + \sum_{i \in V_1, j \in V_2} (d_i + n_{G_2})(d_j + n_{G_1}) \right]$$

$$SK_2(G_1 + G_2) = \frac{1}{4} \left[\sum_{ij \in E_1} [(d_i + n_{G_2}) + (d_j + n_{G_2})]^2 \right. \\ \left. + \sum_{ij \in E_2} [(d_i + n_{G_1}) + (d_j + n_{G_1})]^2 \right. \\ \left. + \sum_{i \in V_1, j \in V_2} [(d_i + n_{G_2}) + (d_j + n_{G_1})]^2 \right]$$

Theorem 2.5. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple, connected weighted graphs. Then the weighted SK , SK_1 and SK_2 indices of the sum of graphs G_1 and G_2 are respectively given by

$$SK(G_1 + G_2, w) = \frac{1}{2} \left[\sum_{i, j \in V_1, ij \in E_1} (w_i + \sum_{k \in V_2} w_k) + (w_j + \sum_{k \in V_2} w_k) \right. \\ \left. + \sum_{i, j \in V_2, ij \in E_2} (w_i + \sum_{t \in V_1} w_t) + (w_j + \sum_{t \in V_1} w_t) \right. \\ \left. + \sum_{i \in V_1, j \in V_2} (w_i + \sum_{k \in V_2} w_k) + (w_j + \sum_{t \in V_1} w_t) \right]$$

$$SK_1(G_1 + G_2, w) = \frac{1}{2} \left[\sum_{i, j \in V_1, ij \in E_1} (w_i + \sum_{k \in V_2} w_k)(w_j + \sum_{k \in V_2} w_k) \right. \\ \left. + \sum_{i, j \in V_2, ij \in E_2} (w_i + \sum_{t \in V_1} w_t)(w_j + \sum_{t \in V_1} w_t) \right. \\ \left. + \sum_{i \in V_1, j \in V_2} (w_i + \sum_{k \in V_2} w_k)(w_j + \sum_{t \in V_1} w_t) \right]$$

$$\begin{aligned}
& + \sum_{i,j \in V_2, ij \in E_2} \left(w_i + \sum_{t \in V_1} w_t \right) \left(w_j + \sum_{t \in V_1} w_t \right) \\
& + \sum_{i \in V_1, j \in V_2} \left(w_i + \sum_{k \in V_2} w_k \right) \left(w_j + \sum_{t \in V_1} w_t \right) \Big] \\
\text{SK}_2(G_1 \times G_2, w) &= \frac{1}{4} \left[\sum_{i,j \in V_1, ij \in E_1} \left[\left(w_i + \sum_{k \in V_2} w_k \right) + \left(w_j + \sum_{k \in V_2} w_k \right) \right]^2 \right. \\
& + \sum_{i,j \in V_2, ij \in E_2} \left[\left(w_i + \sum_{t \in V_1} w_t \right) + \left(w_j + \sum_{t \in V_1} w_t \right) \right]^2 \\
& \left. + \sum_{i \in V_1, j \in V_2} \left[\left(w_i + \sum_{k \in V_2} w_k \right) + \left(w_j + \sum_{t \in V_1} w_t \right) \right]^2 \right]
\end{aligned}$$

Theorem 2.6. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple, connected graphs. Then the SK, SK₁ and SK₂ indices of the cartesian product of graphs G_1 and G_2 are respectively given by

$$\begin{aligned}
\text{SK}(G_1 \times G_2) &= \frac{1}{2} \left[\sum_{(u_i, v_j)(u_k, v_l) \in E(G_1 \times G_2)} \left(d(u_i) + d(v_j) \right) + \left(d(u_k) + d(v_l) \right) \right] \\
\text{SK}_1(G_1 \times G_2) &= \frac{1}{2} \left[\sum_{(u_i, v_j)(u_k, v_l) \in E(G_1 \times G_2)} \left(d(u_i) + d(v_j) \right) \left(d(u_k) + d(v_l) \right) \right] \\
\text{SK}_2(G_1 \times G_2) &= \frac{1}{4} \left[\sum_{(u_i, v_j)(u_k, v_l) \in E(G_1 \times G_2)} \left[\left(d(u_i) + d(v_j) \right) + \left(d(u_k) + d(v_l) \right) \right]^2 \right]
\end{aligned}$$

Theorem 2.7. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple, connected weighted graphs. Then the weighted SK, SK₁ and SK₂ indices of the cartesian product of graphs G_1 and G_2 are respectively given by

$$\begin{aligned}
\text{SK}(G_1 \times G_2, w) &= \frac{1}{2} \left[\sum_{(u_i, v_j)(u_k, v_l) \in E(G_1 \times G_2)} \left(w(u_i) + w(v_j) \right) + \left(w(u_k) + w(v_l) \right) \right] \\
\text{SK}_1(G_1 \times G_2, w) &= \frac{1}{2} \left[\sum_{(u_i, v_j)(u_k, v_l) \in E(G_1 \times G_2)} \left(w(u_i) + w(v_j) \right) \left(w(u_k) + w(v_l) \right) \right] \\
\text{SK}_2(G_1 \times G_2, w) &= \frac{1}{4} \left[\sum_{(u_i, v_j)(u_k, v_l) \in E(G_1 \times G_2)} \left[\left(w(u_i) + w(v_j) \right) + \left(w(u_k) + w(v_l) \right) \right]^2 \right]
\end{aligned}$$

3. Graph Operations on the SK, SK₁, SK₂ Indices of Interval Weighted Graphs

In this section, we define the SK, SK₁ and SK₂ indices on interval weighted graphs. An interval weighted graph (interval graph) is a weighted graph in which each edge is assigned an interval or an interval square matrix. All the interval square matrices are assumed to be of the same order and to be positive definite [8].

Let G be an interval graph on n vertices. Denote by \tilde{w}_{ij} the positive definite

interval matrix of order p of the edge ij and assume that $\tilde{w}_{ij} = \tilde{w}_{ji}$. We write $i \sim j$ if vertices i and j are adjacent. Let $\tilde{w}_i = \sum_{j:j \sim i} \tilde{w}_{ij}$ be the weight interval matrix of the vertex i [8].

Definition 3.1. Let $G = (V, E)$ be a connected interval weighted graph having n vertices. Let weight each edge of G be an interval or an interval square matrix. The interval weighted SK index $SK(G, \tilde{w})$ of G is defined as follows:

$$SK(G, \tilde{w}) = \frac{1}{2} \sum_{uv \in E(G)} \tilde{w}(u) + \tilde{w}(v)$$

where $\tilde{w}(u)$ is the sum of the interval weights on u .

Definition 3.2. Let $G = (V, E)$ be a connected interval weighted graph having n vertices. Let weight each edge of G be an interval or an interval square matrix. The interval weighted SK_1 index $SK_1(G, \tilde{w})$ of G is defined as follows:

$$SK_1(G, \tilde{w}) = \frac{1}{2} \sum_{uv \in E(G)} \tilde{w}(u)\tilde{w}(v)$$

where $\tilde{w}(u)$ is the sum of the interval weights on u .

Definition 3.3. Let $G = (V, E)$ be a connected interval weighted graph having n vertices. Let weight each edge of G be an interval or an interval square matrix. The interval weighted SK_2 index $SK_2(G, \tilde{w})$ of G is defined as follows:

$$SK_2(G, \tilde{w}) = \frac{1}{4} \sum_{uv \in E(G)} [\tilde{w}(u) + \tilde{w}(v)]^2$$

where $\tilde{w}(u)$ is the sum of the interval weights on u .

Theorem 3.4. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple, connected interval weighted graphs. Then the interval weighted SK index of the cartesian product of graphs G_1 and G_2 is respectively given by

$$SK(G_1 + G_2, \tilde{w}) = \frac{1}{2} \left[\sum_{i,j \in V_1, ij \in E_1} (\tilde{w}_i + \sum_{k \in V_2} \tilde{w}_k) + (\tilde{w}_j + \sum_{k \in V_2} \tilde{w}_k) + \sum_{i,j \in V_2, ij \in E_2} (\tilde{w}_i + \sum_{t \in V_1} \tilde{w}_t) + (\tilde{w}_j + \sum_{t \in V_1} \tilde{w}_t) + \sum_{i \in V_1, j \in V_2} (\tilde{w}_i + \sum_{k \in V_2} \tilde{w}_k) + (\tilde{w}_j + \sum_{t \in V_1} \tilde{w}_t) \right]$$

Proof. Let $V = V(G_1) \cup V(G_2)$, $E = E(G_1) \cup E(G_2) \cup \{(u_1, u_2) : u_1 \in V(G_1), u_2 \in V(G_2)\}$. We partition the set of pairs of vertices of $G_1 + G_2$ to obtain the following three sums denoted by K_1, K_2, K_3 , respectively.

Firstly, for each sum, we consider \tilde{w}_i as the sum of the weights in each vertex i . In K_1 , we collect all pairs of vertices i and j so that i, j are in $V(G_1)$ and ij is in $E(G_1)$. Hence, i and j are adjacent vertices in $E(G_1)$. For K_1 , we obtain,

$$K_1 = \frac{1}{2} \left[\sum_{i,j \in V_1, ij \in E_1} (\tilde{w}_i + \sum_{k \in V_2} \tilde{w}_k) + (\tilde{w}_j + \sum_{k \in V_2} \tilde{w}_k) \right].$$

For the second sum K_2 , we take the vertices i and j in $V(G_2)$ so that ij is in

$E(G_2)$. Hence,

$$K_2 = \frac{1}{2} \left[\sum_{i,j \in V_2, ij \in E_2} (\tilde{w}_i + \sum_{t \in V_1} \tilde{w}_t) + (\tilde{w}_j + \sum_{t \in V_1} \tilde{w}_t) \right].$$

In the third sum K_3 , i is taken in $V(G_1)$ and j is taken in $V(G_2)$. So,

$$K_3 = \frac{1}{2} \left[\sum_{i \in V_1, j \in V_2} (\tilde{w}_i + \sum_{k \in V_2} \tilde{w}_k) + (\tilde{w}_j + \sum_{t \in V_1} \tilde{w}_t) \right].$$

The result now follows by adding the three contributions and simplifying the resulting expression.

Theorem 3.5. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple, connected interval weighted graphs. Then the interval weighted SK_1 index of the cartesian product of graphs G_1 and G_2 is respectively given by

$$\begin{aligned} SK_1(G_1 + G_2, \tilde{w}) = & \frac{1}{2} \left[\sum_{i,j \in V_1, ij \in E_1} (\tilde{w}_i + \sum_{k \in V_2} \tilde{w}_k) (\tilde{w}_j + \sum_{k \in V_2} \tilde{w}_k) \right. \\ & + \sum_{i,j \in V_2, ij \in E_2} (\tilde{w}_i + \sum_{t \in V_1} \tilde{w}_t) (\tilde{w}_j + \sum_{t \in V_1} \tilde{w}_t) \\ & \left. + \sum_{i \in V_1, j \in V_2} (\tilde{w}_i + \sum_{k \in V_2} \tilde{w}_k) (\tilde{w}_j + \sum_{t \in V_1} \tilde{w}_t) \right] \end{aligned}$$

Proof. The proof is similarly done to the proof of Theorem 3.4.

Theorem 3.6. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple, connected interval weighted graphs. Then the interval weighted SK_2 index of the cartesian product of graphs G_1 and G_2 is respectively given by

$$\begin{aligned} SK_2(G_1 + G_2, \tilde{w}) = & \frac{1}{4} \left[\sum_{i,j \in V_1, ij \in E_1} \left[(\tilde{w}_i + \sum_{k \in V_2} \tilde{w}_k) + (\tilde{w}_j + \sum_{k \in V_2} \tilde{w}_k) \right]^2 \right. \\ & + \sum_{i,j \in V_2, ij \in E_2} \left[(\tilde{w}_i + \sum_{t \in V_1} \tilde{w}_t) + (\tilde{w}_j + \sum_{t \in V_1} \tilde{w}_t) \right]^2 \\ & \left. + \sum_{i \in V_1, j \in V_2} \left[(\tilde{w}_i + \sum_{k \in V_2} \tilde{w}_k) + (\tilde{w}_j + \sum_{t \in V_1} \tilde{w}_t) \right]^2 \right] \end{aligned}$$

Proof. The proof is similarly done to the proof of Theorem 3.4.

Theorem 3.7. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple, connected interval weighted graphs. Then the interval weighted SK index of the cartesian product of graphs G_1 and G_2 is respectively given by

$$SK(G_1 \times G_2, \tilde{w}) = \frac{1}{2} \left[\sum_{(u_i, v_j) (u_k, v_l) \in E(G_1 \times G_2)} (\tilde{w}(u_i) + \tilde{w}(v_j)) + (\tilde{w}(u_k) + \tilde{w}(v_l)) \right]$$

Proof. The set of vertices in the graph $G_1 \times G_2$ is $u = (u_1, u_2)$, $v = (v_1, v_2) \in V_1 \times V_2$ for $u_1, v_1 \in V_1$ and $u_2, v_2 \in V_2$. Also, $\tilde{w}(u)$ is the interval weight of the vertex u . Thus, the interval weight of any vertex $(u_1, u_2) \in V_1 \times V_2$ in the graph $G_1 \times G_2$ is $\tilde{w}(u_1) + \tilde{w}(u_2)$.

The SK index is equal half of the sum of degrees of all adjacent vertex pairs of the graph. Since the degrees in an interval weighted graph will turn into interval

weights, it is obtained

$$SK(G_1 \times G_2, \tilde{w}) = \frac{1}{2} \left[\sum_{(u_i, v_j)(u_k, v_l) \in E(G_1 \times G_2)} (\tilde{w}(u_i) + \tilde{w}(v_j)) + (\tilde{w}(u_k) + \tilde{w}(v_l)) \right].$$

Theorem 3.8. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple, connected interval weighted graphs. Then the interval weighted SK1 index of the cartesian product of graphs G_1 and G_2 is respectively given by

$$SK_1(G_1 \times G_2, \tilde{w}) = \frac{1}{2} \left[\sum_{(u_i, v_j)(u_k, v_l) \in E(G_1 \times G_2)} (\tilde{w}(u_i) + \tilde{w}(v_j))(\tilde{w}(u_k) + \tilde{w}(v_l)) \right]$$

Proof. The proof is similarly done to the proof of Theorem 3.7.

Theorem 3.9. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple, connected interval weighted graphs. Then the interval weighted SK2 index of the cartesian product of graphs G_1 and G_2 is respectively given by

$$SK_2(G_1 \times G_2, \tilde{w}) = \frac{1}{4} \left[\sum_{(u_i, v_j)(u_k, v_l) \in E(G_1 \times G_2)} [(\tilde{w}(u_i) + \tilde{w}(v_j)) + (\tilde{w}(u_k) + \tilde{w}(v_l))]^2 \right]$$

Proof. The proof is similarly done to the proof of Theorem 3.7.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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