Some Convexificators-Based Optimality Conditions for Nonsmooth Mathematical Program with Vanishing Constraints

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Abstract

In this paper, by using the notion of convexificator, we introduce the generalized standard Abadie constraint qualification and the generalized MPVC Abadie constraint qualification, and define the generalized stationary conditions for the nonsmooth mathematical program with vanishing constraints (MPVC for short). We show that the generalized strong stationary is the first order necessary optimality condition for nonsmooth MPVC under the generalized standard Abadie constraint qualification. Sufficient conditions for global or local optimality for nonsmooth MPVC are also derived under some generalized convexity assumptions.

Keywords

Mathematical Program with Vanishing Constraints, Optimality Conditions, Convexificator

1. Introduction

We consider the following mathematical program with vanishing constraints

$$
\begin{aligned}
\min \quad & f(x) \\
\text{s.t.} \quad & g_i(x) \le 0, i = 1, 2, \cdots, m; \\
& h_j(x) = 0, \ j = 1, 2, \cdots, p; \\
& H_i(x) \ge 0, i = 1, 2, \cdots, l; \\
& G_i(x) H_i(x) \le 0, i = 1, 2, \cdots, l,
\end{aligned}
\tag{1.1}
$$

where $f : R^n \to R, g : R^n \to R^m, h : R^n \to R^p$ and $G, H : R^n \to R^l$ are the

given functions.

The MPVC problem was firstly introduced by Achtziger and Kanzow [1], and it originated from the optimization topology design problems in mechanical structures [1]. The current researches show that the robot motion planning problem [2], the economic dispatch problem [3] and the nonlinear integer optimal control [4] [5] can all be transformed into the MPVC problem. As pointed out in [1], the major difficulty in solving problem (1) is that it does not satisfy most of the standard constraint qualifications, including linearly independent constraint qualification (LICQ for short) and Mangasarian-Fromovitz constraint qualification (MFCQ for short) at any interesting feasible point so that the standard optimization methods are likely to fail for this problem. The MPVC can be formulated as a mathematical program with equilibrium constraints (MPEC for short) and vice versa. However, the formulation has certain disadvantages like the introduction of additional solutions, the large dimension and so on. In [6], the MPVC can be formulated as a nonsmooth MPEC, but the reformulation violates the MPEC type constraint qualifications, which causes some trouble when solving the MPEC formulation by the suitable algorithms. These observations motivate us to consider it as an independent class of interesting optimization problems. The MPVC has attracted much attention in recent years. Several theoretical properties and different numerical approaches for MPVC can be found in [1]-[25].

It is well known that convexifactor is one of the important tools of nonsmooth analysis; the concept of convexificator was firstly introduced by Demyanov [26] in 1994 as a generalization of the notation of upper convex and lower concave approximation. It can be viewed as a weaker version of the notion of subdifferential. Indeed, the convexificator is in general a closed set unlike the well-known subdifferentials which are convex and compact sets. Moreover, for a locally Lipschitz function, most known subdifferentials are convexificators and these known subdifferentials may contain the convex hull of a convexificator [27]. Therefore, from the viewpoint of optimization and applications, the optimality conditions using convexificators are sharper than those using Clarke, Michel-Penot subdifferentials, etc. Convexificators were further studied by Demyanov and Jeyakumar [28], Jeyakumar and Luc [27], Dutta and Chandra [29] [30], etc. Recently, the notion of convexificators has been used to extend various results in nonsmooth analysis; see, e.g., [31] [32] [33] [34]. For nonsmooth optimization problems, various convexificators-based results with respect to the Fritz-John type and the Karush-Kuhn-Tucker type necessary optimality conditions have been developed in [32] [33] [34] [35] [36]. Very recently, Ansari, Movahedian and Nobakhtian [37] deal with constraint qualifications, stationary concepts and optimality conditions for a nonsmooth mathematical program with equilibrium constraints by using the notion of convexificators. However, the corresponding results about the nonsmooth mathematical program with vanishing constraints can be very few. Until recently, based on the Clarke subdifferential, Kazemi and Kanzi [21] study a broad class of mathematical programming with

non-differentiable vanishing constraints. They firstly propose some various qualification conditions for this problem. Then, these constraint qualifications are applied to obtain, under different conditions, several stationary conditions of Karush-Kuhn-Tucker type.

In this paper, different from [21], by utilizing the concept of convexificators which is weaker than the Clarke subdifferential, we introduce the generalized standard Abadie constraint qualification and the generalized MPVC Abadie constraint qualification, and define the generalized stationary conditions for the nonsmooth MPVC. We derive the necessary and sufficient optimality conditions for nonsmooth MPVC under the generalized standard Abadie constraint qualification and some generalized convexity assumptions.

The rest of the paper is organized as follows. Section 2 contains the preliminaries and basic definitions which are used in the sequel. In Section 3, some necessary and sufficient optimality conditions are derived for nonsmooth MPVC based on the notion of convexificators. We close with some final remarks in the end.

2. Prelimilaries

In this section, we will give some basic definitions, which will be used in the sequel.

Let $S \subseteq R^n$ be a nonempty subset contains the origin. The convex hull of *S*, the closure of *S* and the convex cone generated by *S* is denoted by coS, clS and $coneS$, respectively. The negative polar cone is defined by

$$S^- = \left\{ v \in R^n : \langle x, v \rangle \le 0, \forall x \in S \right\}.$$

Let $x \in clS$, the contingent cone $T(x, S)$ to *S* at *x* is defined by

$$T(x, S) = \left\{ v \in R^n \mid \exists t_n \downarrow 0, \exists v_n \to v, \text{s.t. } x + t_n v_n \in S \right\}.$$

Let $f : R^n \to R \cup \{+\infty\}$ be an extended real valued function. The lower and upper Dini directional derivatives of *f* at *x* in the direction *v* are defined, respectively, by

$$f^-(x, v) = \liminf_{t \to 0^+} \frac{f(x + tv) - f(x)}{t}$$

and

$$f^+(x, v) = \limsup_{t \to 0^+} \frac{f(x + tv) - f(x)}{t}.$$

Definition 2.1 [27] *A function* $f : R^n \to R \cup \{+\infty\}$ *is said to admit an upper convexificator,* $\partial^* f(x)$ *at* $x \in R^n$ *if* $\partial^* f(x)$ *is a closed set and for every* $v \in R^n$,

$$f^-(x, v) \le \sup_{\xi \in \partial^* f(x)} \langle \xi, v \rangle.$$

Definition 2.2 [27] *A function* $f : R^n \to R \cup \{+\infty\}$ *is said to admit a lower convexificator,* $\partial_* f(x)$ *at* $x \in R^n$ *if* $\partial_* f(x)$ *is a closed set and for every*

$v \in R^n$,

$$f^+(x,v) \geq \inf_{\xi \in \partial_* f(x)} \langle \xi, v \rangle.$$

A closed set $\partial^* f(x) \subset R^n$ is said to be a convexificator of f at x if it is both upper and lower convexificator of f at x.

Definition 2.3 [30] A *function* $f : R^n \to R \cup \{+\infty\}$ *is said to admit an upper semi-regular convexificator,* $\partial^* f(x)$ *at* $x \in R^n$ *if* $\partial^* f(x)$ *is a closed set and for every* $v \in R^n$,

$$f^+(x,v) \leq \sup_{\xi \in \partial^* f(x)} \langle \xi, v \rangle.$$

If the equality holds in the above inequality, then $\partial^* f(x)$ is called as an upper regular convexificator of f at x.

Definition 2.4 [30] *Let* $f : R^n \to R \cup \{+\infty\}$ *be an extended real valued function that has an upper semi-regular convexificator at* $x \in R^n$. *Then f is said to be*

(i) ∂^* convex at $\overline{x}$ if for every $x \in R^n$,
$f(x) \geq f(\overline{x}) + \langle \xi, x - \overline{x} \rangle, \forall \xi \in \partial^* f(\overline{x})$.

(ii) ∂^* pseudoconvex at $\overline{x}$ if for every $x \in R^n$,
$f(x) < f(\overline{x}) \Rightarrow \langle \xi, x - \overline{x} \rangle < 0, \forall \xi \in \partial^* f(\overline{x})$.

(iii) ∂^* quasiconvex at $\overline{x}$ if for every $x \in R^n$,
$f(x) \leq f(\overline{x}) \Rightarrow \langle \xi, x - \overline{x} \rangle \leq 0, \forall \xi \in \partial^* f(\overline{x})$.

3. Optimality Conditions for Nonsmooth Mathematical Program with Vanishing Constraints

In this section, we will develop several optimality conditions for nonsmooth MPVC in terms of the concept of convexificator. It is worth mentioning that since the upper convexificator is not necessary unique, all the new definitions given in this section depend on the the choice of the convexificator.

First, we introduce some notations. For the problem (1), we denote the feasible region by *X*, that is,

$$X = \left\{ x \in R^n \mid g(x) \leq 0, h(x) = 0, H_i(x) \geq 0, G_i(x) H_i(x) \leq 0, i = 1, 2, \cdots, l \right\}.$$

For $\overline{x} \in X$, we define the following index sets:

$$I_g = \left\{ i \mid g_i(\overline{x}) = 0 \right\},$$

$$I_{+0} = \left\{ i \mid H_i(\overline{x}) > 0, G_i(\overline{x}) = 0 \right\},$$

$$I_{+-} = \left\{ i \mid H_i(\overline{x}) > 0, G_i(\overline{x}) < 0 \right\},$$

$$I_{0+} = \left\{ i \mid H_i(\overline{x}) = 0, G_i(\overline{x}) > 0 \right\},$$

$$I_{0-} = \left\{ i \mid H_i(\overline{x}) = 0, G_i(\overline{x}) < 0 \right\},$$

$$I_{00} = \left\{ i \mid H_i(\overline{x}) = 0, G_i(\overline{x}) = 0 \right\}.$$

Now, we assume that all the functions have an upper convexificator at $\overline{x}$. For

the above index sets, we introduce the following notations:

$$g = \bigcup_{i \in I_g} co\partial^* g_i(\overline{x}),$$

$$h = \bigcup_{i=1}^{p} co\partial^* h_i(\overline{x}) \cup co\partial^*(-h_i)(\overline{x}),$$

$$G_{I_{+0}} = \bigcup_{i \in I_{+0}} co\partial^* G_i(\overline{x}),$$

$$H_{I_{0+}} = \bigcup_{i \in I_{0+}} co\partial^* H_i(\overline{x}) \cup co\partial^*(-H_i)(\overline{x}),$$

$$H_{I_{0-}} = \bigcup_{i \in I_{0-}} co\partial^*(-H_i)(\overline{x}),$$

$$H_{I_{00}} = \bigcup_{i \in I_{00}} co\partial^*(-H_i)(\overline{x}),$$

$$(GH)_{I_{00}} = \bigcup_{i \in I_{00}} co\partial^* G_i(\overline{x}) \cup co\partial^* H_i(\overline{x}),$$

$$\Gamma(\overline{x}) = g^- \cap h^- \cap G_{I_{+0}}^- \cap H_{I_{0+}}^- \cap H_{I_{0-}}^- \cap H_{I_{00}}^-,$$

$$\Lambda(\overline{x}) = g^- \cap h^- \cap G_{I_{+0}}^- \cap H_{I_{0+}}^- \cap H_{I_{0-}}^- \cap H_{I_{00}}^- \cap (GH)_{I_{00}}^-.$$

Utilizing the above notations, motivated by [37], we are ready to introduce the Abadie type constraint qualification in the form of convexificator which is very important to establish the optimality conditions.

Definition 3.1. *Let $\overline{x} \in X$, and assume that all of the functions have an upper convexificator at $\overline{x}$. We say that the generalized standard Abadie constraint qualification (GS ACQ for short) holds at $\overline{x}$ if at least one of the dual sets used in the definition of $\Gamma(\overline{x})$ is nonzero and $\Gamma(\overline{x}) \subset T(X, \overline{x})$.*

Definition 3.2. *Let $\overline{x} \in X$, and assume that all of the functions have an upper convexificator at $\overline{x}$. We say that the generalized MPVC Abadie constraint qualification (GMPVC ACQ for short) holds at $\overline{x}$ if at least one of the dual sets used in the definition of $\Lambda(\overline{x})$ is nonzero and $\Lambda(\overline{x}) \subset T(X, \overline{x})$.*

Remark 3.1. *Since $\Lambda(\overline{x}) \subset \Gamma(\overline{x})$, the GS ACQ implies the GMPVC ACQ.*

Following the procedure in this section, we will formulate several extended version of stationary concepts for MPVC in the context of convexificator.

Definition 3.3. *A feasible point $\overline{x}$ of MPVC is called as a generalized weakly stationary point (GW stationary point) if there are vectors $\lambda = (\lambda^g, \lambda^h, \lambda^H) \in R^{m+p+2l}$ and $\mu = (\mu^h, \mu^G, \mu^H) \in R^{p+2l}$ such that the following conditions hold true:*

$$0 \in co\partial^* f(\overline{x}) + \sum_{i \in I_g} \lambda_i^g co\partial^* g_i(\overline{x}) + \sum_{j=1}^{p} \left[\lambda_j^h co\partial^* h_j(\overline{x}) + \mu_j^h co\partial^*(-h_j)(\overline{x}) \right]$$

$$+ \sum_{i=1}^{l} \lambda_i^H co\partial^*(-H_i)(\overline{x}) + \sum_{i=1}^{l} \left[\mu_i^G co\partial^* G_i(\overline{x}) + \mu_i^H co\partial^* H_i(\overline{x}) \right], \tag{3.1}$$

$$\lambda_{I_g}^g \geq 0, \lambda_j^h, \mu_j^h \geq 0, j = 1, 2, 3, \cdots, p, \lambda_i^H, \mu_i^G, \mu_i^H \geq 0, i = 1, 2, 3, \cdots, l. \tag{3.2}$$

$$\lambda_{I_{+0} \cup I_{+-}}^H = 0, \mu_{I_{+0} \cup I_{+-}}^H = 0, \mu_{I_{0+} \cup I_{+-} \cup I_{0-}}^G = 0, \lambda_i^H - \mu_i^H \geq 0, i \in I_{0-}. \tag{3.3}$$

Definition 3.4. *A feasible point* $\overline{x}$ *of MPVC is called as a generalized T sta-tionary point* (*GT stationary point for short*) *if there are vectors* $\lambda = \left(\lambda^g, \lambda^h, \lambda^H \right) \in R^{m+p+2l}$ *and* $\mu = \left(\mu^h, \mu^G, \mu^H \right) \in R^{p+2l}$ *such that* (3.1)-(3.3) *and the following conditions hold true:*

$$\forall i \in I_{00}, \mu_i^G \left(\lambda_i^H - \mu_i^H \right) \leq 0.$$

Definition 3.5. *A feasible point* $\overline{x}$ *of MPVC is called as a generalized M sta-tionary point* (*GM stationary point for short*) *if there are vectors* $\lambda = \left(\lambda^g, \lambda^h, \lambda^H \right) \in R^{m+p+2l}$ *and* $\mu = \left(\mu^h, \mu^G, \mu^H \right) \in R^{p+2l}$ *such that* (3.1)-(3.3) *and the following conditions hold true:*

$$\forall i \in I_{00}, \mu_i^G \left(\lambda_i^H - \mu_i^H \right) = 0.$$

Definition 3.6. *A feasible point* $\overline{x}$ *of MPVC is called as a generalized S sta-tionary point* (*GS stationary point for short*) *if there are vectors* $\lambda = \left(\lambda^g, \lambda^h, \lambda^H \right) \in R^{m+p+2l}$ *and* $\mu = \left(\mu^h, \mu^G, \mu^H \right) \in R^{p+2l}$ *such that* (3.1)-(3.3) *and the following conditions hold true:*

$$\forall i \in I_{00}, \mu_i^G = 0, \left(\lambda_i^H - \mu_i^H \right) \geq 0.$$

Remark 3.2. *If all the functions are differentiable, then these notions reduce to the stationary concepts defined in* [25]. *Directly from the definitions, we get the following relationships between these stationary concepts.*

$$GS\text{-}staionary \Rightarrow GM\text{-}staionary \Rightarrow GT\text{-}staionary \Rightarrow GW\text{-}staionary.$$

On the other hand, it is obviously that the above stationary concepts include the ones of Karush-Kuhn?C Tucker type which are proposed by Kazemi et al in [21] *as a special case.*

In the rest of this section, we will focus our attention to the necessary and suf-ficient optimality conditions for the nonsmooth MPVC under the framework of convexificator. The following theorem is the first main result of this paper. We will see that this result is proved under very weak assumptions. Only the objec-tive function is assumed to be Lipschitz, while the other functions do not satisfy any type of continuity.

Theorem 3.1. *Let* $\overline{x}$ *be a local optimal solution of MPVC* (1.1). *Suppose that f is a locally Lipschitz at* $\overline{x}$ *which admits a bounded upper semi-regular convexificator* $\partial^* f \left(\overline{x} \right)$. *Assume that GS-ACQ holds at* $\overline{x}$ *and the cone*

$$K = cone\ co\ g + cone\ co\ h + cone\ co\ H_{I_{0+}} + cone\ co\ H_{I_{0-}}$$
$$+ cone\ co\ G_{I_{+0}} + cone\ co\ H_{I_{00}} \tag{3.4}$$

is closed. Then $\overline{x}$ *is a GS stationary point.*

Proof. Firstly, we will prove that

$$0 \in co\ \partial^* f \left(\overline{x} \right) + K. \tag{3.5}$$

Assume that (3.5) does not hold, one has $co\ \partial^* f \left(\overline{x} \right) \bigcap -K = \varnothing$. In view of $co\ \partial^* f \left(\overline{x} \right)$ being a bounded upper semi-regular convexificator, we know that $co\ \partial^* f \left(\overline{x} \right)$ is compact and convex. Since K is a closed convex set, thus utilizing the convex separation theorem, there exists a nonzero vector $v \in R^n$ and a real

number $\rho \in R$ satisfying

$$\sup \langle \xi, v \rangle < \rho < \inf \langle \eta, v \rangle, \forall \xi \in co\, \partial^* f(\overline{x}), \forall \eta \in -K. \tag{*}$$

Notice that $-K$ is a cone, this implies that $\rho = 0$ and

$$\sup \langle \xi, v \rangle < 0, \forall \xi \in co\, \partial^* f(\overline{x}). \tag{3.6}$$

By the definition of upper semi-regular convexificator and (3.6), one gets $f^+(x; v) < 0$. Hence, there is a $\delta > 0$ such that

$$f(\overline{x} + tv) < f(\overline{x}), \forall t \in (0, \delta). \tag{3.7}$$

On the other hand, using the relationships (*) and $\rho = 0$, we obtain

$$\langle \zeta, v \rangle \le 0, \forall \zeta \in K.$$

This implies that

$$\begin{aligned}
&\langle \zeta_i^1, v \rangle \le 0, \forall \zeta_i^1 \in co\, \partial^* g_i(\overline{x}), \forall i \in I_g, \\
&\langle \zeta_i^2, v \rangle \le 0, \forall \zeta_i^2 \in co\, \partial^* h_i(\overline{x}) \cup co\, \partial^*(-h_i)(\overline{x}), \forall i = 1,2,3,\cdots,p, \\
&\langle \zeta_i^3, v \rangle \le 0, \forall \zeta_i^3 \in co\, \partial^* H_i(\overline{x}) \cup co\, \partial^*(-H_i)(\overline{x}), \forall i \in I_{0+}, \\
&\langle \zeta_i^4, v \rangle \le 0, \forall \zeta_i^4 \in co\, \partial^*(-H_i)(\overline{x}), \forall i \in I_{0-}, \\
&\langle \zeta_i^5, v \rangle \le 0, \forall \zeta_i^5 \in co\, \partial^* G_i(\overline{x}), \forall i \in I_{+0}, \\
&\langle \zeta_i^6, v \rangle \le 0, \forall \zeta_i^6 \in co\, \partial^*(-H_i)(\overline{x}), \forall i \in I_{00}.
\end{aligned} \tag{3.8}$$

This is to say

$$v \in g^- \cap h^- \cap G_{I_{+0}}^- \cap H_{I_{0+}}^- \cap H_{I_{0-}}^- \cap H_{I_{00}}^- = \Gamma(\overline{x}).$$

Taking into account GS-ACQ at $\overline{x}$, we obtain $v \in T(X, \overline{x})$. Thus, there exist the sequences $t_k \to 0$ and $v_k \to v$ such that $\overline{x} + t_k v_k \in X, \forall k \in N$, where N denotes the natural number set. On the other hand, since f is Lipschitz near $\overline{x}$ with the modulus $L > 0$, we have for all sufficiently large k,

$$\begin{aligned}
f(\overline{x} + t_k v_k) &\le f(\overline{x} + t_k v) + L t_k \|v_k - v\| \\
&\le f(\overline{x}) + L t_k \|v_k - v\|.
\end{aligned} \tag{3.9}$$

Combining (3.7) and (3.9), we get for all sufficiently large k,

$$f(\overline{x} + t_k v_k) < f(\overline{x}),$$

which contradicts the local optimality of $\overline{x}$. Thus, (3.5) is true. This implies that there exist the nonnegative multipliers $\lambda_i^g, i \in I_g$, $\lambda_j^h, \mu_j^h, j = 1,2,3,\cdots,p$, $\mu_i^H, i \in I_{0+}$, $\lambda_i^H, i \in I_{0+} \cup I_{0-} \cup I_{00}$, $\mu_i^G, i \in I_{+0}$ such that

$$\begin{aligned}
0 \in co\partial^* f(\overline{x}) &+ \sum_{i \in I_g} \lambda_i^g co\partial^* g_i(\overline{x}) + \sum_{j=1}^p \left[\lambda_j^h co\partial^* h_j(\overline{x}) + \mu_j^h co\partial^*(-h_j)(\overline{x}) \right] \\
&+ \sum_{i \in I_{0+} \cup I_{0-} \cup I_{00}} \lambda_i^H co\partial^*(-H_i)(\overline{x}) + \sum_{i \in I_{0+}} \mu_i^H co\partial^* H_i(\overline{x}) + \sum_{i \in I_{+0}} \mu_i^G co\partial^* G_i(\overline{x}).
\end{aligned} \tag{3.10}$$

Let $\lambda_{I_{+0} \cup I_{+-}}^H = 0$, $\mu_{I_{00} \cup I_{0-} \cup I_{+-} \cup I_{+0}}^H = 0$, $\mu_{I_{+-} \cup I_{0+} \cup I_{0-} \cup I_{00}}^G = 0$, we obtain from (3.10),

$$0 \in co\partial^* f\left(\overline{x}\right) + \sum_{i \in I_g} \lambda_i^g co\partial^* g_i\left(\overline{x}\right) + \sum_{j=1}^{p}\left[\lambda_j^h co\partial^* h_j\left(\overline{x}\right) + \mu_j^h co\partial^*\left(-h_j\right)\left(\overline{x}\right)\right]$$

$$+ \sum_{i=1}^{l} \lambda_i^H co\partial^*\left(-H_i\right)\left(\overline{x}\right) + \sum_{i=1}^{l}\left[\mu_i^G co\partial^* G_i\left(\overline{x}\right) + \mu_i^H co\partial^* H_i\left(\overline{x}\right)\right],$$

$$\lambda_{I_g}^g \geq 0, \lambda_j^h, \mu_j^h \geq 0, j = 1,2,3,\cdots, p, \lambda_i^H, \mu_i^G, \mu_i^H \geq 0, i = 1,2,3,\cdots, l.$$

$$\lambda_{I_{+0} \cup I_{+-}}^H = 0, \mu_{I_{+0} \cup I_{+-} \cup I_{0-} \cup I_{00}}^H = 0, \mu_{I_{0+} \cup I_{+-} \cup I_{0-}}^G = 0,$$

$$\forall i \in I_{00}, \mu_i^G = 0, \lambda_i^H - \mu_i^H = \lambda_i^H \geq 0.$$

This shows that $\overline{x}$ is a GS stationary point and the proof is complete.

Since the constraint functions admitting a bounded upper semi-regular convexificator assure that the set K in (3.4) is closed, we immediately obtain the following corollary of Theorem 3.1.

Corollary 3.1. *Let $\overline{x}$ be a local optimal solution of MPVC* (1.1). *Suppose that f is a locally Lipschitz function at $\overline{x}$. Assume also that f and the constraint functions admit a bounded upper semi-regular convexificator. If the GS-ACQ holds at $\overline{x}$, then $\overline{x}$ is a GS stationary point.*

Now, we provide the following example to illustrate Theorem 3.1, this example is a modified version of Example 4.7 in [37].

Example 3.1. *Consider the following two-dimension nonsmooth MPVC problem*:

$$\begin{aligned} \min \quad & f(x) = |x_1| - |x_2| \\ \text{s.t.} \quad & g(x) = |x_2| \leq 0; \\ & H(x) = x_2 \geq 0; \\ & G(x)H(x) = x_1 x_2 \leq 0. \end{aligned}$$

Obviously, 0 is the global optimal solution of the above problem and we have

$$f^+(0;v) = |v_1| - |v_2|, g^+(0;v) = |v_2|, (-H)^+(0;v) = -v_2,$$

$$G^+(0;v) = v_1, H^+(0;v) = v_2.$$

Moreover, we obtain the following bounded upper semi-regular convexificators for these functions

$$\partial^* f(0) = \left\{(1,-1)^{\mathrm{T}}, (-1,1)^{\mathrm{T}}\right\}, \partial^* g(0) = \left\{(0,-1)^{\mathrm{T}}, (0,1)^{\mathrm{T}}\right\},$$

$$\partial^*(-H)(0) = \left\{(0,-1)^{\mathrm{T}}\right\}, \partial^* G(0) = \left\{(1,0)^{\mathrm{T}}\right\}, \partial^* H(0) = \left\{(0,1)^{\mathrm{T}}\right\}.$$

Hence, we get

$$g^- = \left\{v \mid v_2 = 0\right\}, H_{I_{00}}^- = \left\{v \mid v_2 \geq 0\right\}.$$

This implies that the GS-ACQ is satisfied at 0 and K is closed. By taking $\lambda_g = \lambda^H = \mu^G = \mu^H = 0$, one gets

$$0 \in co\partial^* f(0) + \lambda_g co\partial^* g(0) + \lambda^H co\partial^*(-H)(0) + \mu^G co\partial^* G(0) + \mu^H co\partial^* H(0).$$

This shows that the GS-stationarity of 0.

The next result shows that the GM-stationarity is a necessary optimality condition for MPVC if GMPVC ACQ is satisfied at a local optimal solution of

MPVC.

Theorem 3.2. *Let* $\overline{x}$ *be a local optimal solution of MPVC* (1.1). *Suppose that f is a locally Lipschitz at* $\overline{x}$, *and assume that f and the constraint functions admit a bounded upper semi-regular convexificator. If the GMPVC-ACQ holds at* $\overline{x}$, *then* $\overline{x}$ *is a GM stationary point.*

Proof. Firstly, we claim that

$$0 \in co\, \partial^* f\left(\overline{x}\right) + K + cone \bigcup\nolimits_{i \in I_{00}} co\partial^* G_i\left(\overline{x}\right) \cup co\partial^* H_i\left(\overline{x}\right). \tag{3.11}$$

Suppose that (3.11) does not hold. Since $co\, \partial^* f\left(\overline{x}\right)$ is compact and convex, and $K + cone \bigcup\nolimits_{i \in I_{00}} co\partial^* G_i\left(\overline{x}\right) \cup co\partial^* H_i\left(\overline{x}\right)$ is closed and convex, similar to the proof of Theorem 3.1, we can find a nonzero vector $v \in R^n$ and the sequences $t_k \downarrow 0$ and $v_k \to v$ such that for all sufficiently large k,

$$f\left(\overline{x} + t_k v_k\right) < f\left(\overline{x}\right),$$

which contradicts the local optimality of $\overline{x}$. Thus (11) holds true. This implies that there exist the nonnegative multipliers $\lambda_i^g, i \in I_g$, $\lambda_j^h, \mu_j^h, j = 1, 2, 3, \cdots, p$, $\mu_i^H, i \in I_{0+}$, $\lambda_i^H, i \in I_{0+} \bigcup I_{0-} \bigcup I_{00}$, $\mu_i^G, i \in I_{+0}$, $\mu_i^H, i \in I_{00}$ such that

$$\begin{aligned} 0 \in co\partial^* f\left(\overline{x}\right) + \sum_{i \in I_g} \lambda_i^g co\partial^* g_i\left(\overline{x}\right) + \sum_{j=1}^p \left[\lambda_j^h co\partial^* h_j\left(\overline{x}\right) + \mu_j^h co\partial^*\left(-h_j\right)\left(\overline{x}\right)\right] \\ + \sum_{i \in I_{0+} \bigcup I_{0-} \bigcup I_{00}} \lambda_i^H co\partial^*\left(-H_i\right)\left(\overline{x}\right) + \sum_{i \in I_{0+} \bigcup I_{00}} \mu_i^H co\partial^* H_i\left(\overline{x}\right) + \sum_{i \in I_{+0}} \mu_i^G co\partial^* G_i\left(\overline{x}\right). \end{aligned} \tag{3.12}$$

Let $\lambda_{I_{+0} \bigcup I_{+-}}^H = 0, \mu_{I_{0-} \bigcup I_{+-} \bigcup I_{+0}}^H = 0, \mu_{I_{+-} \bigcup I_{0+} \bigcup I_{0-} \bigcup I_{00}}^G = 0$, we obtain from (3.12),

$$\begin{aligned} 0 \in co\partial^* f\left(\overline{x}\right) + \sum_{i \in I_g} \lambda_i^g co\partial^* g_i\left(\overline{x}\right) + \sum_{j=1}^p \left[\lambda_j^h co\partial^* h_j\left(\overline{x}\right) + \mu_j^h co\partial^*\left(-h_j\right)\left(\overline{x}\right)\right] \\ + \sum_{i=1}^l \lambda_i^H co\partial^*\left(-H_i\right)\left(\overline{x}\right) + \sum_{i=1}^l \left[\mu_i^G co\partial^* G_i\left(\overline{x}\right) + \mu_i^H co\partial^* H_i\left(\overline{x}\right)\right], \end{aligned}$$

$$\lambda_{I_g}^g \geq 0, \lambda_j^h, \mu_j^h \geq 0, j = 1, 2, 3 \cdots, p, \lambda_i^H, \mu_i^G, \mu_i^H \geq 0, i = 1, 2, 3, \cdots, l,$$

$$\lambda_{I_{+0} \bigcup I_{+-}}^H = 0, \mu_{I_{+0} \bigcup I_{+-}}^H = 0, \mu_{I_{0+} \bigcup I_{+-} \bigcup I_{0-}}^G = 0, \lambda_i^H - \mu_i^H \geq 0, i \in I_{0-},$$

$$\forall i \in I_{00}, \mu_i^G = 0, \mu_i^G\left(\lambda_i^H - \mu_i^H\right) = 0.$$

This shows that $\overline{x}$ is a GM stationary point and the proof is complete.

Next, we will show that the GW stationarity is a global or local sufficient optimality condition under certain generalized convexity assumptions.

Theorem 3.3. *Let* $\overline{x}$ *be a feasible GW stationary point of MPVC* (1.1) *and define the following index sets*:

$$I_{00}^G = \left\{i \in I_{00} \mid \mu_i^G > 0\right\}, I_{00}^H = \left\{i \in I_{00} \mid \mu_i^H > 0\right\}, I_{0-}^H = \left\{i \in I_{0-} \mid \mu_i^H > 0\right\},$$

$$I_{0+}^H = \left\{i \in I_{0+} \mid \mu_i^H > 0\right\}, I_{+0}^G = \left\{i \in I_{+0} \mid \mu_i^G > 0\right\}.$$

Assume that f is ∂^* pseudoconvex and $g_i\left(i \in I_g\right)$, $\pm h_i\left(i = 1, 2, 3, \cdots, p\right)$ and $-H_i\left(i \in I_{0+} \bigcup I_{0-} \bigcup I_{00}\right)$ are ∂^* quasiconvex at $\overline{x}$. Then the following assertions hold true:

(i) If $I_{00}^G \bigcup I_{00}^H \bigcup I_{0+}^H \bigcup I_{0-}^H \bigcup I_{+0}^G = \varnothing$, then $\overline{x}$ is a global optimal solution of MPVC.

(ii) If $G_i \left(i \in I_{+0}^G \right)$ and $H_i \left(i \in I_{0+}^H \right)$ are continuous and ∂^* quasiconvex at $\overline{x}$, and $I_{00}^G \bigcup I_{00}^H \bigcup I_{0-}^H = \varnothing$, then $\overline{x}$ is a local optimal solution of MPVC.

(iii)If $G_i \left(i \in I_{+0}^G \right)$ and $H_i \left(i \in I_{0+}^H \bigcup I_{0-}^H \right)$ are continuous and ∂^* quasiconvex at $\overline{x}$, and $\overline{x}$ is an interior point relative to the set

$X \bigcap \left\{ x \mid G_i(x) = 0, H_i(x) = 0, i \in I_{00}^G \bigcup I_{00}^H \bigcup I_{0-}^H \right\}$, then $\overline{x}$ is a local solution of MPVC.

Proof. Let $\overline{x}$ be an arbitrary feasible point of (1.1). Since $g_i(x) \le 0 = g_i(\overline{x})$ for $i \in I_g$, by ∂^* quasiconvexity of g_i at $\overline{x}$, we get

$$\langle \zeta_i, x - \overline{x} \rangle \le 0, \forall \zeta_i \in \partial^* g_i(\overline{x}), \forall i \in I_g. \tag{3.13}$$

Similarly, we get

$$\langle \eta_i, x - \overline{x} \rangle \le 0, \forall \eta_i \in \partial^* h_i(\overline{x}), \forall i = 1,2,3,\cdots,p.$$
$$\langle v_i, x - \overline{x} \rangle \le 0, \forall v_i \in \partial^* (-h_i)(\overline{x}), \forall i = 1,2,3,\cdots,p. \tag{3.14}$$
$$\langle \xi_i, x - \overline{x} \rangle \le 0, \forall \xi_i \in \partial^* (-H_i)(\overline{x}), \forall i \in I_{0+} \bigcup I_{0-} \bigcup I_{00}.$$

(i) We multiply each inequality in (3.14) by $\lambda_i^g, i \in I_g$, $\lambda_j^h, \mu_j^h, j = 1,2,3,\cdots,p$, $\lambda_i^H, i \in I_{0+} \bigcup I_{0-} \bigcup I_{00}$, respectively, and adding, we get

$$\left\langle \sum_{i \in I_g} \lambda_i^g \zeta_i + \sum_{j=1}^p \left[\lambda_j^h \eta_i + \mu_j^h v_i \right] + \sum_{i \in I_{0+} \bigcup I_{0-} \bigcup I_{00}} \lambda_i^H \xi_i, x - \overline{x} \right\rangle \le 0.$$

Since $I_{00}^G \bigcup I_{00}^H \bigcup I_{0+}^H \bigcup I_{0-}^H \bigcup I_{+0}^G = \varnothing$, taking into account the GW stationarity of $\overline{x}$, one gets

$$\left\langle \sum_{i \in I_g} \lambda_i^g \zeta_i + \sum_{j=1}^p \left[\lambda_j^h \eta_i + \mu_j^h v_i \right] + \sum_{i=1}^l \lambda_i^H \xi_i + \sum_{i=1}^l \left[\mu_i^G \tau_i + \mu_i^H \delta_i \right], x - \overline{x} \right\rangle \le 0,$$

$$\forall \tau_i \in \partial^* G_i(\overline{x}), \forall \delta_i \in \partial^* H_i(\overline{x}).$$

This implies that there exists $\vartheta \in co\, \partial^* f(\overline{x})$ such that $\langle \vartheta, x - \overline{x} \rangle \ge 0$. The ∂^* pseudoconvexity of f at $\overline{x}$ shows that $f(x) \ge f(\overline{x})$ for all $x \in X$. Hence, $\overline{x}$ is a global optimal solution of MPVC.

(ii) For any $i \in I_{+0}$, since $H_i(\overline{x}) > 0$, the continuity of H_i implies that $H_i(x) > 0$ for all feasible points x which is sufficiently close to $\overline{x}$. This shows that $G_i(x) \le 0$ for such x. Hence, for x which is sufficiently close to $\overline{x}$, one has

$$G_i(x) \le 0 = G_i(\overline{x}), \forall i \in I_{+0}.$$

Utilizing the ∂^* quasiconvexity of $G_i \left(i \in I_{+0}^G \right)$ at $\overline{x}$, we deduce that for the feasible point x which is sufficiently close to $\overline{x}$,

$$\langle \tau_i, x - \overline{x} \rangle \le 0, \forall \tau_i \in \partial^* G_i(\overline{x}), \forall i \in I_{+0}^G. \tag{3.15}$$

Similarly, one gets, for the feasible point x which is sufficiently close to $\overline{x}$,

$$\langle \delta_i, x - \overline{x} \rangle \le 0, \forall \delta_i \in \partial^* H_i(\overline{x}), \forall i \in I_{0+}^H. \tag{3.16}$$

Similar to the proof of case (i), we can find $\vartheta \in co\, \partial^* f(\overline{x})$ such that $\langle \vartheta, x - \overline{x} \rangle \ge 0$. The ∂^* pseudoconvexity of f at $\overline{x}$ shows that $f(x) \ge f(\overline{x})$ for all feasible point x which is sufficiently close to $\overline{x}$. Hence, $\overline{x}$ is a local op-

timal solution of MPVC.

(iii) Taking into account that $\overline{x}$ is an interior point relative to the set $X \bigcap \{x \mid G_i(x) = 0, H_i(x) = 0, i \in I_{00}^G \bigcup I_{00}^H \bigcup I_{0-}^H\}$, we know that, for the feasible point x which is sufficiently close to $\overline{x}$,

$$G_i(x) = 0 = G_i(\overline{x}), H_i(x) = 0 = H_i(\overline{x}), i \in I_{00}^G \bigcup I_{00}^H \bigcup I_{0-}^H.$$

By the ∂^* quasiconvexity of the above functions at $\overline{x}$, we have

$$\begin{aligned}
\langle \tau_i, x - \overline{x} \rangle \le 0, \forall \tau_i \in \partial^* G_i(\overline{x}), \forall i \in I_{00}^G, \\
\langle \delta_i, x - \overline{x} \rangle \le 0, \forall \delta_i \in \partial^* H_i(\overline{x}), \forall i \in I_{00}^H \bigcup I_{0-}^H.
\end{aligned} \tag{3.17}$$

Multiplying (3.13)-(3.17) by $\lambda_i^g, i \in I_g$, $\lambda_j^h, \mu_j^h, j = 1, 2, 3, \cdots, p$, $\lambda_i^H, i \in I_{0+} \bigcup I_{0-} \bigcup I_{00}$, $\mu_i^G, i \in I_{00}^G \bigcup I_{+0}^G$, $\mu_i^H, i \in I_{00}^H \bigcup I_{0+}^H \bigcup I_{0-}^H$, respectively, and adding, we get

$$\left\langle \sum_{i \in I_g} \lambda_i^g \zeta_i + \sum_{j=1}^p \left[\lambda_j^h \eta_i + \mu_j^h \nu_i \right] + \sum_{i=1}^l \lambda_i^H \xi_i + \sum_{i=1}^l \left[\mu_i^G \tau_i + \mu_i^H \delta_i \right], x - \overline{x} \right\rangle \le 0.$$

This implies that there exists $\vartheta \in co\, \partial^* f(\overline{x})$ such that $\langle \vartheta, x - \overline{x} \rangle \ge 0$. The ∂^* pseudoconvexity of f at $\overline{x}$ shows that $f(x) \ge f(\overline{x})$ for all feasible points x which is sufficiently close to $\overline{x}$. Hence, $\overline{x}$ is a local optimal solution of MPVC.

4. Concluding Remarks

In this paper, under the framework of convexificator, by introducing two generalized MPVC type constraint qualifications and the stationary concepts, we derive the necessary and sufficient optimality conditions for the nonsmooth MPVC using the notion of convexificators. As the future work, some other generalized MPVC type constraint qualifications under the framework of convexificator will be investigated.

Founding

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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