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The Stability of the Gauss-Laguerre Rule for Cauchy P.V. Integrals on the Half Line

Maria Rosaria Capobianco[1], Giuliana Criscuolo[2]

[1]Istituto per le Applicazioni del Calcolo "Mauro Picone"—CNR, Napoli, Italy
[2]Dipartimento di Matematica e Applicazioni, Università degli Studi di Napoli "Federico II", Napoli, Italy
Email: mariarosaria.capobianco@cnr.it, giuliana.criscuolo@unina.it

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Abstract

In this paper, the authors give a different and more precise analysis of the stability of the classical Gauss-Laguerre quadrature rule for the Cauchy P.V. integrals on the half line. Moreover, in order to obtain this result they give some new estimates for the distance of the zeros of the Laguerre polynomials that can be useful also in other contests.

Keywords

Gauss Quadrature, Exponential Weights, Cauchy P.V. Integrals, Stability

1. Introduction

A careful analysis of Gauss-Laguerre formulas for ordinary integrals can be found in [1]. The present paper is instead aimed at the use of the Gauss-Laguerre formulas for the approximation of the Cauchy principal value integrals. The main results on the topic can be found in [2] (see also the references therein). The aim of the present paper is to give the upper bound of the stability factor when this kind of quadrature rule is used.

We consider the class of singular principal value integrals

$$\mathcal{H}(fw_\alpha;t)=\int_0^\infty \frac{f(x)}{x-t}w_\alpha(x)\mathrm{d}x=\lim_{\varepsilon\to=0^+}\frac{f(x)}{x-t}w_\alpha(x)\mathrm{d}x,\quad t>0, \tag{1}$$

where f satisfies the smoothness conditions

$$f\in C^0_{\sqrt{w_\alpha}}:=\left\{f\in C^0[0,\infty):\lim_{x\to\infty}f(x)\sqrt{w_\alpha}=0\right\}, \tag{2}$$

and

$$\int_0^1 u^{-1}\omega(f;u)_{\sqrt{w_\alpha},\infty}\mathrm{d}u<\infty, \tag{3}$$

where $\omega(f;.)_{\sqrt{w_\alpha},\infty}$ is the weighted Ditzian-Lubinsky modulus of smoothness [3].

We propose a Gauss-Laguerre type quadrature formula to evaluate the singular principal value integral $\mathcal{H}(fw_\alpha;t)$ defined by (1) assuming that the function *f* has good integration property at the superior limit of integration interval; this assumption is the same that assures the boundedness of $\mathcal{H}(fw_\alpha)$. In this first part we study the stability of the proposed procedure with respect to the distance of the singularity *t* from the quadrature knots. The proposed method to compute (1) is well known in the case of bounded intervals. Even though the fundamental idea is not new, a thorough investigation of this algorithm is of interest since there are significant differences between the case of a bounded interval and the case of an unbounded interval from the point of view of approximation theory.

2. The Stability of Gauss-Laguerre Quadrature Rule

The *GL* type formula to evaluate $\mathcal{H}(fw_\alpha)$ is constructed by interpolating the function *f* on $x_{n,k}^\alpha, k=1,\cdots,n$, and on the singularity *t* assuming that $t \neq x_{n,k}^\alpha, k=1,\cdots,n$. Taking into account that such formula can be written

$$\begin{aligned}\mathcal{H}(fw_\alpha;t) &= \mathcal{H}(w_\alpha;t)f(t)+\sum_{k=1}^n \lambda_{n,k}^\alpha \frac{f(x_{n,k}^\alpha)-f(t)}{x_{n,k}^\alpha-t}+\mathcal{R}_n^{GL}(f;w_\alpha;t)\\ &=\left[\mathcal{H}(w_\alpha;t)-\sum_{k=1}^n\frac{\lambda_{n,k}^\alpha}{x_{n,k}^\alpha-t}\right]f(t)+\sum_{k=1}^n\frac{\lambda_{n,k}^\alpha}{x_{n,k}^\alpha-t}f(x_{n,k}^\alpha)+\mathcal{R}_n^{GL}(f;w_\alpha;t) \quad (4)\\ &=\mathcal{H}_n^{GL}(f;w_\alpha;t)+\mathcal{R}_n^{GL}(f;w_\alpha;t),\end{aligned}$$

we have

$$\mathcal{R}_n^{GL}(f;w_\alpha;t)=R_n^{GL}\left(\frac{f-f(t)}{e_1-t};w_\alpha\right).$$

Therefore, the formula (4) has degree of exactness $2n$, *i.e.* $\mathcal{R}_n^{GL}(f;w_\alpha;t)=0$ whenever *f* is a polynomial of degree $\le 2n$.

Obviously, from a theoretical point of view, this formula turns out to be convergent if the function *f* is sufficiently smooth. Furthermore, it has the advantage of simplicity in the computation of the coefficients, but unfortunately it may exhibit numerical cancellation and generally it cannot converge when a knot $x_{n,k}^\alpha$ is very close to *t*. In order to establish a bound of the amplification factor of (4) which depends on the position of *t* with respect to the points $x_{n,k}^\alpha$, we need some some notations and preliminary lemmas.

If *A* and *B* are two expressions depending on some variables, then we write $A \sim B$ if and only if $\left|AB^{-1}\right|^{\pm 1} \le \text{const}$ uniformly for the variables under consideration.

We bring here some properties of the knots $x_{n,k}^\alpha, k=1,2,\cdots,n$ and of the Christoffel constants $\lambda_{n,k}^\alpha, k=1,2,\cdots,n$ of the *GL* formula (4). These properties can be found in [4] and [5] where are proved for a more general class of weight functions.

Let $x_{n,k}^{\alpha}, k=1,2,\cdots,n$ be the zeros of the *n*-th Laguerre orthogonal polynomial ordered in increasing order. We have

$$\frac{c_1}{n} < x_{n,1}^{\alpha} < x_{n,2}^{\alpha} < \cdots < x_{n,n}^{\alpha} < 4n - 2\alpha + 2 - c_2 (4n)^{\frac{1}{3}}, \tag{5}$$

with some constants c_1 and c_2 independent of $n \ge 1$ and $k \in \{1,2,\cdots,n\}$.

The Christoffel constants $\lambda_{n,k}^{\alpha}, k=1,2,\cdots,n$ admit the following bounds

$$\begin{aligned} \lambda_{n,k}^{\alpha} &\sim \sqrt{\frac{x_{n,k}^{\alpha}}{4n - x_{n,k}^{\alpha}}} w_\alpha\left(x_{n,k}^{\alpha}\right), \quad k \in \{1,2,\cdots,n\}, \\ \lambda_{n,k}^{\alpha} &\sim \Delta x_{n,k}^{\alpha} w_\alpha\left(x_{n,k}^{\alpha}\right), \quad k \in \{2,3,\cdots,n\}, \end{aligned} \tag{6}$$

uniformly for $n \ge 1$ and where $\Delta x_{n,k}^{\alpha} = x_{n,k}^{\alpha} - x_{n,k-1}^{\alpha}$.

Let $r_n = f - t_n$ and $t_n \in \Pi_n$ be such that

$E_n(f)_{\sqrt{w_\alpha},\infty} = \min_{p_n \in \Pi_n} \left\|(f - p_n)\sqrt{w_\alpha}\right\|_\infty = \left\|(f - t_n)\sqrt{w_\alpha}\right\|_\infty$.

For any $t > 0$, $n \in N$ we denote by $x_{n,c}^{\alpha}$ the knot closest to *t*, defined by

$$\left|x_{n,c}^{\alpha} - t\right| = \min_{1 \le k \le n} \left|x_{n,k}^{\alpha} - t\right|.$$

Lemma 1. *We have*

$$\sum_{k=2, k\ne c}^{n} \frac{\Delta x_{n,k}^{\alpha}}{\left|x_{n,k}^{\alpha} - t\right|} = O(\log_e n), \quad t > 0.$$

Proof. It easy seen that

$$\frac{\Delta x_{n,k}^{\alpha}}{\left|x_{n,k}^{\alpha} - t\right|} = O(1), \quad k = 2,\cdots,n, k \ne c, t > 0,$$

and

$$\frac{\Delta x_{n,k}^{\alpha}}{\Delta x_{n,k+1}^{\alpha}} = O(1), \quad k = 2,\cdots,n-1.$$

Thus

$$\begin{aligned} \sum_{k=2, k\ne c}^{n} \frac{\Delta x_{n,k}^{\alpha}}{\left|x_{n,k}^{\alpha} - t\right|} &= O\left(\sum_{k=2}^{c-2} \frac{\Delta x_{n,k+1}^{\alpha}}{\left|x_{n,k}^{\alpha} - t\right|}\right) + \sum_{k=c+2}^{n} \frac{\Delta x_{n,k}^{\alpha}}{\left|x_{n,k}^{\alpha} - t\right|} + O(1) \\ &= O\left(\left(\int_{x_{n,1}^{\alpha}}^{x_{n,c-1}^{\alpha}} + \int_{x_{n,c+1}^{\alpha}}^{x_{n,n}^{\alpha}}\right) \frac{\mathrm{d}x}{|x - t|}\right) + O(1). \end{aligned}$$

Hence performing the integrations we get the lemma. □

Lemma 2. For any $t > 0$ such that $t \ne x_{n,k}^{\alpha}, k = 1,\cdots,n$, and with some constant C independent of *f*, *n* and *t*,

$$\left|\sum_{k=1}^{n} \frac{\lambda_{n,k}^{\alpha}}{x_{n,k}^{\alpha} - t} r_n\left(x_{n,k}^{\alpha}\right)\right| \le C E_n(f)_{\sqrt{w_\alpha},\infty} \left[\log_e n + \frac{\Delta x_{n,c}^{\alpha}}{\left|x_{n,c}^{\alpha} - t\right|} \sqrt{w_\alpha\left(x_{n,c}^{\alpha}\right)}\right],$$

where $x_{n,0}^{\alpha} = 2x_{n,1}^{\alpha} - x_{n,2}^{\alpha}$.

Proof. We write

$$\left|\sum_{k=1}^{n}\frac{\lambda_{n,k}^{\alpha}}{x_{n,k}^{\alpha}-t}r_n\left(x_{n,k}^{\alpha}\right)\right|=\left|\sum_{k=1}^{n}\frac{\lambda_{n,k}^{\alpha}w_{\alpha}^{-1/2}\left(x_{n,k}^{\alpha}\right)}{x_{n,k}^{\alpha}-t}r_n\left(x_{n,k}^{\alpha}\right)\sqrt{w_{\alpha}\left(x_{n,k}^{\alpha}\right)}\right|$$
$$\le E_n\left(f\right)_{\sqrt{w_{\alpha}},\infty}\left[\sum_{k=1,k\neq c}^{n}\frac{\lambda_{n,k}^{\alpha}w_{\alpha}^{-1/2}\left(x_{n,k}^{\alpha}\right)}{\left|x_{n,k}^{\alpha}-t\right|}+\frac{\lambda_{n,c}^{\alpha}w_{\alpha}^{-1/2}\left(x_{n,c}^{\alpha}\right)}{\left|x_{n,c}^{\alpha}-t\right|}\right]. \tag{7}$$

First, we obtain from (6)

$$\sum_{k=2,k\neq c}^{n}\frac{\lambda_{n,k}^{\alpha}w_{\alpha}^{-1/2}\left(x_{n,k}^{\alpha}\right)}{\left|x_{n,k}^{\alpha}-t\right|}\le C\sum_{k=2,k\neq c}^{n}\frac{\Delta x_{n,k}^{\alpha}}{\left|x_{n,k}^{\alpha}-t\right|}\le C\log_e n. \tag{8}$$

Here we have used Lemma 1.

Now, if $c=1$ then the last sum of (7) is equal to the first sum in (8). On the other hand, if $c\neq 1$, we have $\left|t-x_{n,1}^{\alpha}\right|\ge \Delta x_{n,2}^{\alpha}/2$; then

$$\frac{\lambda_{n,1}^{\alpha}w_{\alpha}^{-1/2}\left(x_{n,1}^{\alpha}\right)}{\left|x_{n,1}^{\alpha}-t\right|}\le C.$$

So, in all cases,

$$\sum_{k=1,k\neq c}^{n}\frac{\lambda_{n,k}^{\alpha}w_{\alpha}^{-1/2}\left(x_{n,k}^{\alpha}\right)}{\left|x_{n,k}^{\alpha}-t\right|}\le C\log_e n,\quad t>0. \tag{9}$$

Finally, again by using (6),

$$\frac{\lambda_{n,c}^{\alpha}}{\left|x_{n,c}^{\alpha}-t\right|}w_{\alpha}^{-1/2}\left(x_{n,c}^{\alpha}\right)\sim\frac{\Delta x_{n,c}^{\alpha}}{\left|x_{n,c}^{\alpha}-t\right|}\sqrt{w_{\alpha}\left(x_{n,c}^{\alpha}\right)},\quad t>0, \tag{10}$$

taking into account the definition of $x_{n,0}^{\alpha}$. Combining (7), (9) and (10) the assertion follows. □

Lemma 3. For any $t>0$ such that $t\neq x_{n,k}^{\alpha}, k=1,\cdots,n$, and with some constant C independent of *f*, *n* and *t*,

$$\left|\left[\mathcal{H}\left(w_{\alpha};t\right)-\sum_{k=1}^{n}\frac{\lambda_{n,k}^{\alpha}}{x_{n,k}^{\alpha}-t}\right]r_n\left(t\right)\right|$$
$$\le CE_n\left(f\right)_{\sqrt{w_{\alpha}},\infty}\left[\frac{t}{x_{n,c}^{\alpha}}+\frac{\Delta x_{n,c}^{\alpha}}{\left|x_{n,c}^{\alpha}-t\right|}\right]w_{\alpha}^{-1/2}\left(t\right)w_{\alpha}\left(x_{n,c}^{\alpha}\right),$$

where $x_{n,0}^{\alpha}=2x_{n,1}^{\alpha}-x_{n,2}^{\alpha}$.

Proof. The lemma has been proved [2] with respect to the Hermite weight. Following the same steps of the proof of the Lemma 3.3 in [2], it is possible to derive the assertion. Here we omit the details. □

In order to estimate the stability of the *GL* formula (4), we define

$$\mathcal{K}_n^{GL}\left(w_{\alpha};t\right)=\left|\mathcal{H}\left(w_{\alpha};t\right)-\sum_{k=1}^{n}\frac{\lambda_{n,k}^{\alpha}}{x_{n,k}^{\alpha}-t}\right|+\sum_{k=1}^{n}\frac{\lambda_{n,k}^{\alpha}}{\left|x_{n,k}^{\alpha}-t\right|},\quad t>0,n\in N.$$

Theorem 4. For any $t>0$ such that $t\neq x_{n,k}^{\alpha}, k=1,\cdots,n$, and with some constant C independent of *n* and *t*,

$$\mathcal{K}_n^{GL}\left(w_\alpha;t\right) \le C\sqrt{w_\alpha\left(x_{n,c}^\alpha\right)}\left\{\log_e n + \left[\frac{t}{x_{n,c}^\alpha} + \frac{\Delta x_{n,c}^\alpha}{\left|x_{n,c}^\alpha - t\right|}\right]\sqrt{w_\alpha\left(x_{n,c}^\alpha\right)}\right\},$$

where $x_{n,0}^\alpha = 2x_{n,1}^\alpha - x_{n,2}^\alpha$.

Proof. Following the same steps of the proofs of the Lemmas 2 and 3, we have

$$\sum_{k=1}^{n} \frac{\lambda_{n,k}^\alpha}{\left|x_{n,k}^\alpha - t\right|} \le C\sqrt{w_\alpha\left(x_{n,c}^\alpha\right)}\left\{\log_e n + \frac{\Delta x_{n,c}^\alpha}{\left|x_{n,c}^\alpha - t\right|}\right\}, \quad t > 0, \tag{11}$$

$$\left|\mathcal{H}\left(w_\alpha;t\right) - \sum_{k=1}^{n} \frac{\lambda_{n,k}^\alpha}{x_{n,k}^\alpha - t}\right| \le C w_\alpha\left(x_{n,c}^\alpha\right)\left[\frac{t}{x_{n,c}^\alpha} + \frac{\Delta x_{n,c}^\alpha}{\left|x_{n,c}^\alpha - t\right|}\right], \quad t > 0, \tag{12}$$

respectively. Then, the assertion follows by (11) and (12) taking into account the definition of $\mathcal{K}_n^{GL}\left(w_\alpha;t\right)$. □

Corollary 1. Assume that $f \in C^0_{\sqrt{w_\alpha}}$, where the space $C^0_{\sqrt{w_\alpha}}$ is defined in (2) and *f* satisfies the condition (3). If $0 < t \sim n$ and *n* are such that $t \ge 2x_{n,n}^\alpha - x_{n,n-1}$, then

$$\mathcal{K}_n^{GL}\left(w_\alpha;t\right) \le C\log_e n,$$

with some constant *C* independent of *n* and *t*.

Proof. Taking into account that the assumptions on *t* and *n* give $x_{n,c}^\alpha = x_{n,n}^\alpha$, we have $t/x_{n,c}^\alpha \sim 1$ and $\Delta x_{n,c}^\alpha / \left|x_{n,c}^\alpha - t\right| \le 1$. Thus, the assertion follows from Theorem 4. □

Now assume that $t > 0$ is fixed. In order to specify the bound of the amplification factor $\mathcal{K}_n^{GL}\left(w_\alpha;t\right)$, we need to use a suitable subsequence of $\left\{\mathcal{H}_n\left(f;w_\alpha;t\right)\right\}_{n=1}^\infty$. To derive it, the following lemma turns out helpful.

Lemma 5. Let $w_\alpha\left(x\right) = x^\alpha e^{-x}, \alpha > -1$. There exists n_0 such that uniformly for $n \ge n_0$, $2 \le k \le n$,

$$x_{n+1,k+1}^\alpha - x_{n,k}^\alpha \ge C_1 a_n^{-1}\Delta x_{n,k}^\alpha\left[x_{n,k}\left(a_n - x_{n,k}^\alpha\right)\right]^{1/2}, \tag{13}$$

$$x_{n,k}^\alpha - x_{n+1,k}^\alpha \ge C_2 a_n^{-1}\Delta x_{n,k}^\alpha\left[x_{n,k}\left(a_n - x_{n,k}^\alpha\right)\right]^{1/2}, \tag{14}$$

where $a_n \sim n$ is the Mhaskar-Rakhmanov-Saff number [6], and C_1, C_2 are positive constants.

Proof. At first we remark that $x_{n+1,1}^\alpha < x_{n,1}^\alpha < x_{n+1,2}^\alpha < x_{n,2}^\alpha < \cdots < x_{n+1,k}^\alpha < x_{n,k}^\alpha < x_{n+1,k+1}^\alpha < x_{n,k+1}^\alpha < \cdots < x_{n,n-1}^\alpha < x_{n+1,n}^\alpha < x_{n,n}^\alpha < x_{n+1,n+1}^\alpha$, (cfr. [5]).

Let us denote

$$\ell_{n,k}\left(w_\alpha;x\right) = \frac{L_n\left(w_\alpha;x\right)}{L_n'\left(w_\alpha;x_{n,k}^\alpha\right)\left(x - x_{n,k}^\alpha\right)}, \quad k = 1,\cdots,n, \tag{15}$$

the fundamental Lagrange polynomials with respect to the points $x_{n,j}^\alpha, j = 1,\cdots,n$, zeros of the *n*-th Laguerre polynomial $L_n\left(w_\alpha\right)$.

By using the relation

$$\max_{x \ge \frac{a_n}{n^2}}\left|\frac{\sqrt{w_\alpha\left(x\right)}\ell_{n,k}\left(x\right)}{\sqrt{w_\alpha\left(x_{n,k}^\alpha\right)}}\right| \sim 1,$$

(see Theorem 1.3(c) in [7]), we obtain from (15)

$$x_{n+1,k+1}^{\alpha} - x_{n,k}^{\alpha} \ge C \frac{\sqrt{w_\alpha\left(x_{n,k}^{\alpha}\right)}}{\sqrt{w_\alpha\left(x_{n+1,k+1}^{\alpha}\right)}} \frac{\left|L_n\left(w_\alpha; x_{n+1,k+1}^{\alpha}\right)\right|}{\left|L_n'\left(w_\alpha; x_{n,k}^{\alpha}\right)\right|}, \quad k = 1, \cdots, n. \tag{16}$$

By using

$$\left|L_n'\left(w_\alpha; x_{n,k}^{\alpha}\right)\right| \sim \left(\Delta x_{n,k}^{\alpha}\right)^{-1} \left[x_{n,k}\left(a_n - x_{n,k}^{\alpha}\right)\right]^{-1/4} \sqrt{w_\alpha\left(x_{n,k}^{\alpha}\right)}, \quad k = 2, \cdots, n,$$

$$\left|L_n\left(w_\alpha; x_{n+1,k+1}^{\alpha}\right)\right| \sim a_n^{-1} \left[x_{n+1,k}\left(a_n + 1 - x_{n+1,k+1}^{\alpha}\right)\right]^{1/4} \sqrt{w_\alpha\left(x_{n+1,k+1}^{\alpha}\right)}, \quad k = 1, \cdots, n,$$

(see Theorem 1.3(a), (b) and Theorem 1.4 in [7]) and taking into account that $x_{n,k}^{\alpha} \sim x_{n+1,k+1}^{\alpha}$ and $a_n - x_{n,k}^{\alpha} \sim a_{n+1} - x_{n+1,k+1}^{\alpha}$, we obtain (13) from (16). Similarly we can prove (14). □

Assuming that $t > 0$ is fixed, we define the set

$$N_t = \left\{ n \in N : \left|x_{n,c}^{\alpha} - t\right| \ge C \frac{\Delta x_{n,c}^{\alpha}}{\sqrt{n}} \right\},$$

where the parameter $C > 0$ is chosen a priori. Being t fixed we have that $x_{n,c}^{\alpha}\left(a_n - x_{n,c}^{\alpha}\right) \sim a_n$. Thus, in view of Lemma 5, we deduce that N_t is an infinite set. Indeed, if $n \notin N_t$, then $n+1 \in N_t$.

Finally, we remark that to construct the formula we have assumed $t \ne x_{n,k}^{\alpha}, k = 1, \cdots, n$. To derive a rule and the related error bound for all t, it would be of interest to investigate the limit case $x_{n,c}^{\alpha} \to t$. Of course, this would require additional assumptions on the function f. However, the restriction $t \ne x_{n,k}^{\alpha}, k = 1, \cdots, n$, does not influence the effective approximation of $\mathcal{H}\left(fw_\alpha; t\right)$. Indeed, for any fixed $t > 0$ it is possible to construct the subsequence $\left\{\mathcal{H}_n^{GL}\left(f; w_\alpha; t\right)\right\}_{n \in N_t}$ and the related error can be derived.

Corollary 2. If $t > 0$ is fixed, then

$$\mathcal{K}_n^{GL}\left(w_\alpha; t\right) \le C\sqrt{n},$$

with some constant C independent of $n \in N_t$.

Proof. The assertion follows from Theorem 4 taking into account the definition of the set N_t. □

3. Conclusion

Theorem 4 provides the upper bound of the amplification factor of the Gauss-Laguerre quadrature formula in the case of Cauchy principal value integrals. Corollary 2 instead provides a useful definition of a subsequence of the Gauss-Laguerre formula with respect to which the amplification factor is particularly useful in applications.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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