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On the Coupled of NBEM and CFEM for an Anisotropic Quasilinear Problem in an Unbounded Domain with a Concave Angle

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Abstract

In this paper, based on the Kirchhoff transformation and the natural boundary element method, a coupled natural boundary element and curved edge finite element is applied to solve the anisotropic quasi-linear problem in an unbounded domain with a concave angle. By using the principle of the natural boundary reduction, we obtain the natural integral equation on the artificial boundary of circular arc boundary, and get the coupled variational problem and its numerical method. Then the error and convergence of coupling solution are analyzed. Finally, some numerical examples are verified to show the feasibility of our method.

Keywords

Anisotropic Quasilinear Problem, Circular Arc Boundary, Natural Boundary Reduction, Error Estimates

1. Introduction

In this paper, we propose the method of coupling curved edge finite element (CEFE) and natural boundary element (NBE) to solve an anisotropic quasilinear problem in an unbounded domain. The CEFE-NBE method is based on the artificial boundary method [1] [2], namely the coupling of edge finite element (EFE) and natural boundary element (NBE) method [3] [4]. The EFE-NBE method has been used to solve many linear problems [5] [6] and is generalized to solve quasilinear problems [7] [8] [9] [10] [11]. We find the CEFE-NBE method also has been used to solve some linear problems [12] [13] [14] [15], so we try to generalize it to quasilinear boundary value problem.

The standard procedure of the method of coupling curved edge finite element and natural boundary element can be described as follows. We introduce an artificial boundary to divide the original domain into two subregions, a bounded inner region and an unbounded one with special boundary, which are solved by the curved edge finite element and the boundary element method respectively.

Suppose Ω is an infinite domain with a concave angle ω, and $0<\omega\le 2\pi$; the boundaries of Ω are disintegrated into three disjoint parts: $\partial\Omega=\Gamma\cup\Gamma_1\cup\Gamma_2$, the boundary Γ is a simple smooth curve part, Γ_1 and Γ_2 are two half lines. We have

$$\Omega=\{(r,\theta)\mid r>R,0<\theta<\omega\},\ \Gamma=\{(r,\theta)\mid r=R,0<\theta<\omega\},$$

$$\Gamma_1=\{(r,\theta)\mid r>R,\theta=0\},\ \Gamma_2=\{(r,\theta)\mid r>R,\theta=\omega\}.$$

We consider the following quasilinear problem

$$\begin{cases} -\left(\dfrac{\partial}{\partial x}\left(\alpha a(x,u)\dfrac{\partial u}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(\beta a(x,u)\dfrac{\partial u}{\partial y}\right)\right)=f, & \text{in } \Omega_1, \\ \dfrac{\partial u}{\partial n}=0, & \text{on } \Gamma_1\cup\Gamma_2, \\ u=0, & \text{on } \Gamma, \\ u(x) \text{ is bounded}, & \text{as } |x|\to\infty. \end{cases}$$

Suppose that the given function $a(\cdot,\cdot)$ satisfies:

$0<C_0\le a(x,u)\le C_1, \forall u\in R$, and for almost all

$$x\in\Omega, \tag{1}$$

with two constants $C_0, C_1\in R$;

$|a(x,u)-a(x,v)|\le C_L|u-v|, \forall u,v\in R$, and for almost all

$$x\in\Omega, \tag{2}$$

with a constant $C_L>0$.

This problem has many physical applications in the field of continuum mechanics, we also assume that $\dfrac{\partial a}{\partial s},\dfrac{\partial^2 a}{\partial^2 s}$ are continuous and suppose that the given function $f\in L^2(\Omega)$ has compact support, *i.e.*, there exists a constant $R_0>0$, such that

$$supp\, f\subset\Omega_{R_0}=\{x\in R^2\mid |x|\le R_0\}. \tag{3}$$

Furthermore, we assume that

$$a(x,u)\equiv a_0(u),\ \text{when}\ |x|\ge R_0. \tag{4}$$

The rest of the paper is organized as follows. In section 2, we obtain the exact quasilinear elliptical arc artificial boundary condition. In section 3, we give the equivalent variational problems and the finite element approximations and prove the well-posedness and the convergence results of the reduced problems. In section 4, we give some numerical examples to show the efficiency and feasibility of this method.

2. Natural Boundary Reduction

Now, we introduce an artificial boundary

$$\Gamma_0 = \{(R,\theta) \mid \theta \in (0,\omega)\}, \tag{5}$$

with $R \geq R_0$.

Let us introduce the artificial boundary Γ_0 which divides Ω into two regions (see Figure 1), a bounded domain Ω_1 and an unbounded domain Ω_2. Then the problem (1) can be rewritten in the coupled form:

$$\begin{cases} -\left(\dfrac{\partial}{\partial x}\left(\alpha a(x,u)\dfrac{\partial u}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(\beta a(x,u)\dfrac{\partial u}{\partial y}\right)\right) = f, & \text{in } \Omega_1, \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \Gamma_{11} \cup \Gamma_{21}, \\ u = 0, & \text{on } \Gamma, \end{cases} \tag{6}$$

$$\begin{cases} -\left(\dfrac{\partial}{\partial x}\left(\alpha a(x,u)\dfrac{\partial u}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(\beta a(x,u)\dfrac{\partial u}{\partial y}\right)\right) = 0, & \text{in } \Omega_2, \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \Gamma_{12} \cup \Gamma_{22}, \\ u(x) \text{ is bounded}, & \text{as } |x| \to \infty, \end{cases} \tag{7}$$

$u(x)$ and $\alpha a_0(u) n_x \dfrac{\partial u}{\partial x} + \beta a_0(u) n_y \dfrac{\partial u}{\partial y}$ are continuous on Γ_0, where $\Gamma_{11} = \Omega_1 \cap \Gamma_1$, $\Gamma_{21} = \Omega_1 \cap \Gamma_2$, $\Gamma_{12} = \Omega_2 \cap \Gamma_1$ and $\Gamma_{22} = \Omega_2 \cap \Gamma_2$, and $n = (n_x, n_y)$ is the unit exterior normal vector on Γ_0.

Particularly, $a(x,u) \equiv a$ is independent of x and u when $|x| \geq R_0 > 0$, the problem (7) is simplified to the linear exterior elliptic problem [8].

We introduce the so-called Kirchhoff transformation: [8]

$$w(x) = \int_0^{u(x)} a_0(\xi)\,\mathrm{d}\xi, \quad x \in \Omega_2, \tag{8}$$

which gives

$$\nabla w = a_0(u)\nabla u, \tag{9}$$

and

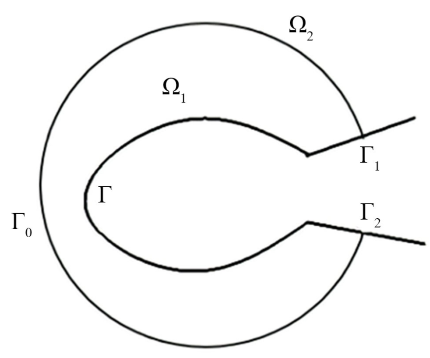

Figure 1. Artificial boundary of area Ω.

$$\left(\alpha\frac{\partial w}{\partial x},\beta\frac{\partial w}{\partial y}\right)=\left(\alpha a_0(u)n_x\frac{\partial u}{\partial x},\beta a_0(u)n_y\frac{\partial u}{\partial y}\right). \tag{10}$$

From (7), that w satisfies the following problem:

$$\begin{cases}-\left(\alpha\frac{\partial^2 w}{\partial x^2}+\beta\frac{\partial^2 w}{\partial y^2}\right)=0, & \text{in } \Omega_2,\\ \frac{\partial u}{\partial n}=0, & \text{on } \Gamma_{12}\cup\Gamma_{22},\\ w(x) \text{ is bounded}, & \text{as } |x|\to\infty.\end{cases} \tag{11}$$

Assume that $w(x)$ is the solution of the problem (13), and the value $w|_{|u|=u_1}$ is given

$$w|_{\Gamma_0}=w(R,\theta),$$

we introduce $x=\sqrt{\alpha}\xi, y=\sqrt{\beta}\eta$, the boundary Γ_0 is changed to the elliptic arc boundary $\overline{\Gamma_0}=\{(\xi,\eta)\,|\,\alpha\xi^2+\beta\eta^2=R^2\}$, the unit exterior normal vector on $\overline{\Gamma_0}$ is

$$\boldsymbol{v}=-\frac{1}{\sqrt{\alpha x^2+\beta y^2}}\left(\sqrt{\alpha}\cos\varphi,\sqrt{\beta}\sin\varphi\right),$$

The next, we need to discuss the relationship between elliptic coordinates (u,φ) and cartesian coordinates (x,y), the relationship can be expressed as below:

$$\begin{cases}x=f_0\cosh u\cos\varphi,\\ y=f_0\sinh u\sin\varphi,\end{cases} \tag{12}$$

where

$$f_0=\sqrt{\frac{\beta-\alpha}{\alpha\beta}}R,\ u_0=\ln\frac{\sqrt{\alpha}+\sqrt{\beta}}{\sqrt{\beta-\alpha}},$$

$$\bar{\Omega}=\{(u,\varphi)\,|\,u>u_0,0<\varphi<\omega\},\bar{\Gamma}=\{(u,\varphi)\,|\,u=u_0,0<\varphi<\omega\},$$

$$\bar{\Gamma}_1=\{(u,\varphi)\,|\,u>u_0,\varphi=0\},\bar{\Gamma}_2=\{(u,\varphi)\,|\,u>u_0,\varphi=\omega\},$$

$$J(u,\varphi)=\begin{vmatrix}\frac{\partial\xi}{\partial u} & \frac{\partial\xi}{\partial\varphi}\\ \frac{\partial\eta}{\partial u} & \frac{\partial\eta}{\partial\varphi}\end{vmatrix}=f_0^2\left(\sinh^2 u\cos^2\varphi+\cosh^2 u\sin^2\varphi\right).$$

The problem (11) is transformed into

$$\begin{cases}-\left(\frac{\partial^2 w}{\partial\xi^2}+\frac{\partial^2 w}{\partial\eta^2}\right)=0, & \text{in } \bar{\Omega}_2,\\ \frac{\partial u}{\partial n}=0, & \text{on } \bar{\Gamma}_{12}\cup\bar{\Gamma}_{22},\\ w(x) \text{ is bounded}, & \text{as } |x|\to\infty.\end{cases} \tag{13}$$

The based on the natural boundary reduction [1] [2], there are the Poisson integral formulas [9].

$$w(r,\theta)=-\frac{1}{2\omega}\left(R^{\frac{2\pi}{\omega}}-r^{\frac{2\pi}{\omega}}\right)\int_0^{\omega}\left[\frac{1}{r^{\frac{2\pi}{\omega}}+R^{\frac{2\pi}{\omega}}-2(Rr)^{\frac{2\pi}{\omega}}\cos\frac{\pi}{\omega}(\theta-\theta')}\right.$$
$$\left.+\frac{1}{r^{\frac{2\pi}{\omega}}+R^{\frac{2\pi}{\omega}}-2(Rr)^{\frac{2\pi}{\omega}}\cos\frac{\pi}{\omega}(\theta+\theta')}\right]w(R,\theta')\mathrm{d}\theta',\ 0<R<r, \tag{14}$$

and the natural integral equation

$$\frac{\partial w}{\partial n}=-\frac{\pi}{4\omega^2R}\int_0^{\omega}\left(\frac{1}{\sin^2\frac{\theta-\theta'}{2\omega}\pi}+\frac{1}{\sin^2\frac{\theta+\theta'}{2\omega}\pi}\right)w(R,\theta')\mathrm{d}\theta'. \tag{15}$$

From (9), we obtain

$$\frac{\partial w}{\partial n}=a_0(u)\frac{\partial u}{\partial n}. \tag{16}$$

Combining (8), (9) and (15), we obtain the exact artificial boundary condition of u on Γ_0,

$$\begin{aligned}&\left|\alpha a_0(u)n_x\frac{\partial u}{\partial x}+\beta a_0(u)n_y\frac{\partial u}{\partial y}\right|_{r=R}\\&=-\frac{\sqrt{\alpha\beta}}{R\pi}\int_0^{\omega}\sum_{n=1}^{\infty}w_0\left(\int_0^{u(R,\theta')}a_0(y)\mathrm{d}y\right)n\cos n(\theta'-\theta)\mathrm{d}\theta'\\&\triangleq\kappa_1(u(u_1,\varphi)).\end{aligned} \tag{17}$$

By the exact quasilinear artificial boundary condition (17), the original problem confines in Ω_1 can be defined as follows

$$\begin{cases}-\left(\frac{\partial}{\partial x}\left(\alpha a(x,u)\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\beta a(x,u)\frac{\partial u}{\partial y}\right)\right)=f, & \text{in } \Omega_1,\\ \frac{\partial u}{\partial n}=0, & \text{on } \Gamma_{11}\cup\Gamma_{21},\\ u=0, & \text{on } \Gamma,\\ \alpha a_0(u)n_x\frac{\partial u}{\partial x}+\beta a_0(u)n_y\frac{\partial u}{\partial y}=\kappa_1(u(u_1,\varphi)), & \text{on } \Gamma_0.\end{cases} \tag{18}$$

3. Variational Problem and Finite Element Approximation

3.1. The Equivalent Variational Problems

Now, we consider the problem (18). First, we will use $W^{m,p}(\Omega)$ denoting the standard Sobolev spaces, $\|\cdot\|_{m.p,\Omega}$ and $|\cdot|_{m,p,\Omega}$ denoting the corresponding norms and semi-norms. In particular, we denote $H^m(\Omega)=W^{m,2}(\Omega)$, $\|\cdot\|_{m,\Omega}=\|\cdot\|_{m.2,\Omega}$ and $|\cdot|_{m,,\Omega}=|\cdot|_{m,2,\Omega}$.

Let us introduce the space

$$V=\left\{v\in H^1(\Omega_1)\,|\,v|_{\Gamma_0}=0\right\}, \tag{19}$$

and the corresponding norms

$$\|v\|_{0,\Omega_1} = \sqrt{\int_{\Omega_1} |v|^2 \,\mathrm{d}x},\ \|v\|_{1,\Omega_1} = \sqrt{\int_{\Omega_1} \left(|v|^2 + |\nabla v|^2\right) \mathrm{d}x}.$$

The boundary value problem (18) is equivalent to the following variational problem

$$\begin{cases} \text{Find } u \in V, \text{ such that} \\ A(u;u,v) + B(u;u,v) = F(v) \quad \forall v \in V, \end{cases} \tag{20}$$

with

$$A(w;u,v) = \int_{\Omega_1} a(x,w)\left(\alpha \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \beta \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\mathrm{d}x, \tag{21}$$

$$B(w;u,v) = \sum_{n=1}^{\infty} \frac{\sqrt{\alpha\beta}}{n\pi} \int_0^{\omega}\int_0^{\omega} a_0\left(w(R,\theta')\right) \frac{\partial u(R,\theta')}{\partial \theta'} \frac{\partial v(R,\theta)}{\partial \theta} \cos n(\theta'-\theta)\,\mathrm{d}\theta'\mathrm{d}\theta \tag{22}$$

$$F(v) = \int_{\Omega_1} f(x) v(x). \tag{23}$$

Lemma 1 [4] There exists $C_0, C_1 > 0$, such that

$$\left|B(w;u,v)\right| \le C_0 \|u\|_{1,\Omega_1} \|v\|_{1,\Omega_1},\ B(u;u,v) \ge C_0 \|u\|_{1,\Omega_1}^2,\ \forall u,v,w \in V.$$

In practice, we need to truncate the series in (17) for some nonnegative integer N, that is

$$\left|\alpha a_0(u) n_x \frac{\partial u}{\partial x} + \beta a_0(u) n_y \frac{\partial u}{\partial y}\right|_{r=R} = \kappa_1^N\left(u(u_1,\varphi)\right), \tag{24}$$

with

$$\kappa_1^N\left(u(u_1,\varphi)\right) = -\frac{\sqrt{\alpha\beta}}{R\pi}\int_0^{\omega} \sum_{n=1}^{N} w_0\left(\int_0^{u(R,\theta')} a_0(y)\,\mathrm{d}y\right) n \cos n(\theta'-\theta)\,\mathrm{d}\theta'. \tag{25}$$

Then, we consider the following approximate problem

$$\begin{cases} -\left(\dfrac{\partial}{\partial x}\left(\alpha a(x,u^N)\dfrac{\partial u^N}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(\beta a(x,u^N)\dfrac{\partial u^N}{\partial y}\right)\right) = f, & \text{in } \Omega_1, \\ \dfrac{\partial u^N}{\partial n} = 0, & \text{on } \Gamma_{11} \cup \Gamma_{21}, \\ u^N = 0, & \text{on } \Gamma, \\ \alpha a_0(u^N) n_x \dfrac{\partial u^N}{\partial x} + \beta a_0(u^N) n_y \dfrac{\partial u^N}{\partial y} = \kappa_1^N\left(u^N(u_1,\varphi)\right), & \text{on } \Gamma_0. \end{cases} \tag{26}$$

The problem (26) is equivalent to the following variational problem

$$\begin{cases} \text{Find } u^N \in V, \text{ such that} \\ A(u^N;u^N,v) + B_N(u^N;u^N,v) = F(v), \quad \forall v \in V, \end{cases} \tag{27}$$

where

$$\begin{aligned} B(w;u,v) &= \sum_{n=1}^{N} \frac{\sqrt{\alpha\beta}}{n\pi} \int_0^{\omega}\int_0^{\omega} a_0\left(w(R,\theta')\right) \frac{\partial u(R,\theta')}{\partial \theta'} \frac{\partial v(R,\theta)}{\partial \theta} \\ &= \cos(\theta'-\theta)\,\mathrm{d}\theta'\mathrm{d}\theta. \end{aligned} \tag{28}$$

Similar with lemma 1, we have:

Lemma 2 There exists a constant $C > 0$, such that

$$\left|B_N(w;u,v)\right| \le C_0 \|u\|_{1,\Omega_1} \|v\|_{1,\Omega_1}, \quad B_N(u;u,v) \ge C_0 \|u\|_{1,\Omega_1}^2, \quad \forall u,v,w \in V.$$

3.2. Finite Element Approximation

Firstly, we divide the arc Γ_0 into M parts and take an edge finite element subdivision τ in Ω_1, so that their nodes on Γ_0 are the same. The curved edge element subdivision $\bar{\tau}$. means that we make a regular T_h on Ω_1, and then we replace the standard triangles T whose two vertices lie on the boundary Γ_0 with a curved triangles $\bar{T}$ whose one edge coincides with the boundary.

Secondly, we let $\bar{T} \in \bar{\tau}$, and use $P_i = (x_i, y_i)(i = 1,2,3)$ to denote the vertex of a curved triangles $\bar{T}$, and P_1P_3 to denote the curved edge. We use area coordinates $(\lambda_1, \lambda_2, \lambda_3)$ to represent a standard triangle T corresponding to the curved triangle $\bar{T}$,

$$x = \sum_{i=1}^{3} x_i \lambda_i, \quad y = \sum_{i=1}^{3} y_i \lambda_i, \tag{29}$$

where $\lambda_1 = 1 - \xi - \eta, \lambda_2 = \xi, \lambda_3 = \eta$, then, based on the linear transformation of the T, we make the following nonlinear transformation, so that the curved triangle $\bar{T}$ can be mapped to the reference element $\tilde{T}$ one-by-one,

$$\begin{cases} x = x_1 + \overline{x_2}\xi + \overline{x_3}\eta + (1 - \xi - \eta)\Phi(\eta), \\ y = y_1 + \overline{y_2}\xi + \overline{y_3}\eta + (1 - \xi - \eta)\Psi(\eta), \end{cases} \tag{30}$$

where

$$\Phi(\eta) = \frac{1}{1-\eta}\left(\varphi\left(s_1 + \overline{s_1}\eta\right) - x_1 - \overline{x_3}\eta\right), \tag{31}$$

$$\Psi(\eta) = \frac{1}{1-\eta}\left(\psi\left(s_1 + \overline{s_3}\eta\right) - y_1 - \overline{y_3}\eta\right), \tag{32}$$

and $\overline{s_3} = s_3 - s_1$, $x = \varphi(s)$, $y = \psi(s)$, $\overline{x_2} = x_2 - x_1$, $\overline{y_2} = y_2 - y_1$, $\overline{x_3} = x_3 - x_1$, $\overline{y_3} = y_3 - y_1$.

In the end, we get a curved edge element subdivision by the above method in $\overline{\Omega_1}$, and it maps a standard triangle in the plane (ξ, η) to a curved triangle one-by-one.

Next, we construct the coordination element, *i.e.* approaching space $S_h \subset H_{\Gamma_0}^1(\Omega_1)$. Triples $\left(\bar{T}, P_1(\bar{T}), \Sigma T\right)$ are used to define the finite element, and use $P_1(\bar{T})$ to denote the space of first degree polynomials and the space of zero degree polynomials on $\tilde{T}$. The following mappings exist

$$P_1(\bar{T}) = \left\{P : \bar{T} \to R; P = \hat{P} \circ F_T^{-1}, \hat{P} \in P_1(\tilde{T})\right\}, \tag{33}$$

and use $\circ$ to denote a composite mapping, and $\Sigma T = \{N_i : i = 1,2,3\}$ is a set of linear functions, we denote $N_i(\varphi) = \varphi(p_i), \varphi = p_i(\bar{T}), i = 1,2,3$ [12] (supplement: see reference [12] for meaning of F_T^{-1}).

According to the reference [12], we propose:

Lemma 3. If $\overline{\tau}$ and the edge finite element are regular, the difference error of each curved triangular element is estimated as follows

$$\left\|v-\Pi_{\bar{T}}v\right\|_{1,\bar{T}} \le Ch_{\bar{T}}\left\|v\right\|_{2,\bar{T}} \quad \forall v\in H^2\left(\overline{T}\right), \tag{34}$$

where C is a constant, Π is interpolation operator, $\Pi_{\bar{T}}v\in P_1\left(\overline{T}\right)$, $\Pi_{\bar{T}}v\left(P_i\right)=v\left(P_i\right)\left(i=1,2,3\right)$ is well-determined.

Proof: we assume τ_h is regular, from reference [12], when $h\to 0$, we have

$$\left\|J\left(F_T\left(\cdot\right)\right)\right\|_{0,\infty,\tilde{T}}=o\left(h_T^2\right),\ \left\|F_T\right\|_{1,\infty,\tilde{T}}=o\left(h_T^l\right),\left(l=1,2\right),\ \left\|F_T^{-1}\right\|_{1,\infty,\tilde{T}}=o\left(h_T^{-1}\right).$$

In the affine condition, according to various properties of F_T, we obtain the difference error of each curved triangular element,

$$\left\|v-\Pi_{\bar{T}}v\right\|_{1,\bar{T}} \le Ch_{\bar{T}}\left\|v\right\|_{2,\bar{T}} \quad \forall v\in H^2\left(\overline{T}\right).$$

Let $S_h=\left\{v_h\in C\left(\Omega_1\right)\cap H^1_{\Gamma_0}\left(\Omega_1\right)\mid v_h|_{\bar{T}}\in P_1\left(\overline{T}\right)\right\}$, combined with (29), we obtain

$$\inf_{v_h\in S_h}\left\|u-v_h\right\|_{1,\Omega_1}\le Ch\left\|u\right\|_{2,\Omega_2},\quad \forall u\in S_h\left(\overline{\Omega}_1\right). \tag{35}$$

The approximate problem of (27) can be written as

$$\begin{cases}\text{Find } u_h^N\in S_h\left(\Omega_1\right),\ \text{such that}\\ A\left(u_h^N;u_h^N,v_h\right)+B_N\left(u_h^N;u_h^N,v_h\right)=F\left(v_h\right),\ \forall v_h\in S_h\left(\Omega_1\right).\end{cases} \tag{36}$$

Similar with existence and uniqueness in [4], we have:

Lemma 4. The variational problems (20), (27) and (36) are uniquely solvable.

From now on, let $u,u^N\in H^2\left(\Omega_1\right)$, and $u_h^N\in S_h$ be the solution of problems (20), (27) and (36) respectively. We also assume that

$$S_h\subset V\cap W^{1,2+\varepsilon}\left(\Omega_1\right),\ \text{for some } \varepsilon\in\left(0,1\right), \tag{37}$$

and we require that $\left\{S_h\right\}_{h\to 0}$ is a family of finite-dimensional subspaces of $V\cap C\left(\Omega_1\right)$, such that

$$\forall v\in V\cap C\left(\Omega_1\right),\exists\left\{v_h\right\}_{h\to 0}:v_h\in S_h,\left\|v-v_h\right\|_{1,\Omega_1}\to 0,\text{ as } h\to 0, \tag{38}$$

$$\left\|v_h\right\|_{1,2,\Omega_1}\le C\left(v\right),\ \forall h, \tag{39}$$

where $C\left(v\right)>0$ is independent of *h*.

The continuous piecewise polynomial spaces, such as S_h, satisfy the condition (37), and if let $v_h=\Pi_h v$, where $\Pi_h:V\to S_h$ is the interpolation operator, we have

$$\left\|v_h\right\|_{1,2+\varepsilon,\Omega_1}\le\left\|\Pi_h v-v\right\|_{1,2+\varepsilon,\Omega_1}+\left\|v\right\|_{1,2+\varepsilon,\Omega_1}\le C\left(v\right),$$

and we can obtain the following lemma.

Theorem 1 [5] Suppose $u,u^N\in H^2\left(\Omega_1\right)$, and $u_h^N\in S_h$ be the solution of problems (20), (27) and (36) respectively, $S_h\subset V\cap W^{1,2+\varepsilon}\left(\Omega_1\right)$. According to the references [4], we know

$$\left\|u-u_h^N\right\|_{1,\Omega_1}\le C\left[h^\sigma+\mathrm{e}^{\left(u_0-u_1\right)\frac{\left(N+1\right)\pi}{\omega}}\left\|u\right\|_{1,\Omega_1}\right]. \tag{40}$$

Proof: For $v \in V$, we assume that

$$w(u,\varphi)=\frac{b_0}{2}+\sum_{n=1}^{+\infty} b_n \mathrm{e}^{(u_0-u)\frac{n\pi}{\omega}} \cos\frac{n\pi\varphi}{\omega}, \ \forall u \geq u_0,$$

$$v(u_1,\theta)=\frac{c_0}{2}+\sum_{n=1}^{+\infty} c_n \cos\frac{n\pi\theta}{\omega}.$$

Then, we have the following result

$$\begin{aligned}
&\left|B(u;u,v)-B_N(u;u,v)\right| \\
&=\left|\frac{2}{n\pi}\int_0^{\omega}\int_0^{\omega}\frac{\partial w(u_1,\varphi)}{\partial\varphi}\frac{\partial v(u_1,\theta)}{\partial\theta}\sum_{n=N+1}^{+\infty}\sin\frac{n\pi\varphi}{\omega}\sin\frac{n\pi\theta}{\omega}\mathrm{d}\varphi\mathrm{d}\theta\right| \\
&=\left|\sum_{n=N+1}^{+\infty}\frac{n\pi}{2}\mathrm{e}^{(u_0-u_1)\frac{n\pi}{\omega}} b_n c_n\right| \\
&\leq C\frac{\mathrm{e}^{(u_0-u_1)\frac{(N+1)\pi}{\omega}}}{(N+1)^{k-1}}\|u\|_{k-\frac{1}{2},\tau_{u_0}}\|v\|_{1,\Omega_1}.
\end{aligned}$$

From references [5] [8] [9], we know

$$\left\|u-u^N\right\|_{1,\Omega_1} \leq C\frac{\left|B(u;u,v)-B_N(u;u,v)\right|}{\|v\|_{1,\Omega_1}},$$

then, we obtain

$$\begin{aligned}
\left\|u-u^N\right\|_{1,\Omega_1} &\leq C\frac{\mathrm{e}^{(u_0-u_1)\frac{(N+1)\pi}{\omega}}\|u\|_{k-\frac{1}{2},\tau_{u_0}}\|v\|_{1,\Omega_1}}{\|v\|_{1,\Omega_1}} \\
&= C\mathrm{e}^{(u_0-u_1)\frac{(N+1)\pi}{\omega}}\|u\|_{k-\frac{1}{2},\tau_{u_0}}.
\end{aligned}$$

Like [8], we introduce operators: $P_h : V \to V_h$, we have

$$\|v-P_h v\|_{1,\Omega_1} \leq C_0 \inf_{v_h\in V_h}\|v-P_h v\|_{1,\Omega_1} \leq C_0 h^{\sigma}, \ 0<\sigma<1, \ \forall v\in V,$$

for $u^N \in V$, we have

$$\left\|u^N-u_h^N\right\|_{1,\Omega_1} \leq \left\|u^N-P_h u^N\right\|_{1,\Omega_1}+\left\|P_h u^N-u_h^N\right\|_{1,\Omega_1} \leq Ch^{\sigma}.$$

Therefore,

$$\left\|u-u_h^N\right\|_{1,\Omega_1} \leq \left\|u-u^N\right\|_{1,\Omega_1}+\left\|u^N-u_h^N\right\|_{1,\Omega_1} = C\left[h^{\sigma}+\mathrm{e}^{(u_0-u_1)\frac{(N+1)\pi}{\omega}}\|u\|_{1,\Omega_1}\right].$$

4. Numerical Example

Example 1. We assume

$$\Omega=\{(r,\theta)\,|\,r>R, 0<\theta<\omega\}, \Gamma=\{(r,\theta)\,|\,r=R, 0<\theta<\omega\},$$

$$\Gamma_1=\{(r,\theta)\,|\,r>R, \theta=0\}, \Omega=\{(r,\theta)\,|\,r>R, \theta=\omega\},$$

with $\omega=\frac{3}{2}\pi, R=0.8, r=1.5, \varepsilon=\frac{\alpha}{\beta}=0.5$. According to references [8] [9] [10], we make $\varepsilon = 0.5$ for easy analysis. We introduce an elliptical boundary

$$\Gamma_0 = \{(R_0,\theta) \mid \theta \in (0,\omega)\},\ R_0 = 1.5.$$

We take our numerical results for problem (1), with

$$a(x,u) = \begin{cases} 16 - r^2 + \dfrac{1}{1+u^2}, & 0.8 \le r \le 1.5, \\ \dfrac{1}{1+u^2}, & r > 1.5, \end{cases}$$

$$u = \tan\left(\frac{y}{x^2+y^2}\right).$$

Furthermore, we let $\Delta r = \frac{1}{m}, \Delta\theta = \frac{3\pi}{2M}$ for some integer $m, M \in N$. The exact solution "u" is solved with $N = 10, \Delta r = \frac{1}{16}, \Delta\theta = \frac{\omega}{64}$.

The numerical results are given in Table 1 and Table 2 (Figures 2-4).

Table 1. Error value when straight triangular element is used for finite element.

mesh	$\|u-u_h\|_{L^2(\Omega_1)}$	ratio $(L^2(\Omega_1))$	$\|u-u_h\|_{L^\infty(\Omega_1)}$	ratio $(L^\infty(\Omega_1))$
h	3.58174E-1		8.76106E-1	
$h/2$	1.30051E-1	2.75411	3.75232E-1	2.33484
$h/4$	4.44990E-2	2.92254	1.50271E-1	2.49704
$h/8$	1.46775E-2	3.03180	4.12880E-2	3.63954

Table 2. Error value of curved edge element in finite element.

mesh	$\|u-u_h\|_{L^2(\Omega_1)}$	ratio $(L^2(\Omega_1))$	$\|u-u_h\|_{L^\infty(\Omega_1)}$	ratio $(L^\infty(\Omega_1))$
h	2.44592E-1		7.47451E-1	
$h/2$	9.79710E-2	2.49660	3.45129E-1	2.165713
$h/4$	3.55530E-2	2.75559	1.48746E-1	2.320254
$h/8$	1.18910E-2	2.98991	4.12630E-2	3.604833

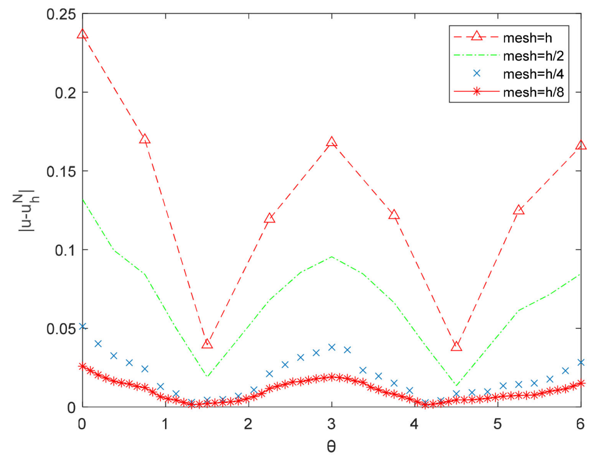

Figure 2. Error value of straight triangular element in finite element with $N = 10$, $\varepsilon = 0.5$.

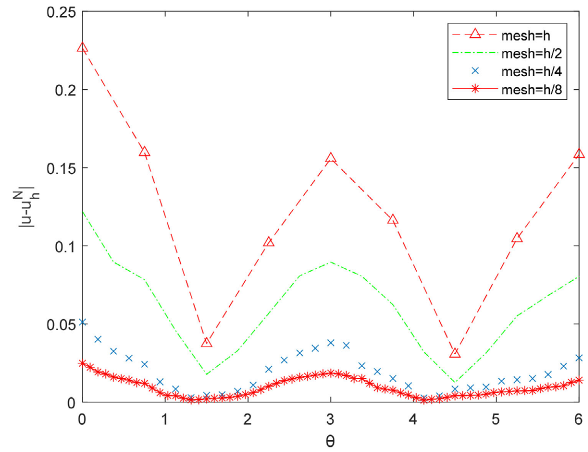

Figure 3. Error value of curved edge element in finite element with $N = 10$, $\varepsilon = 0.5$.

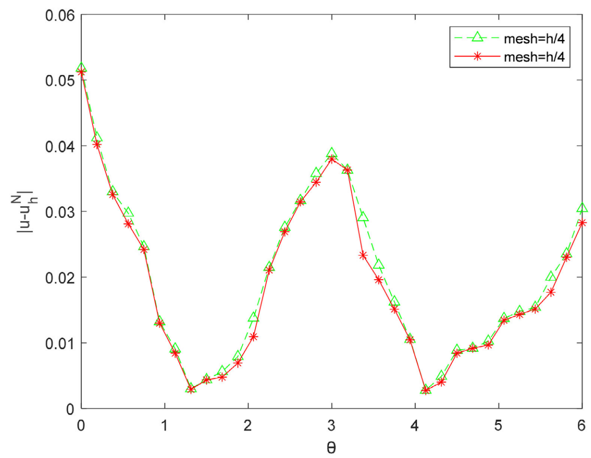

Figure 4. Comparison of error value in **Figure 2** and **Figure 3** with mesh = $h/4$.

These are the three error analysis diagrams of Example 1.

Example 2. We assume

$$\Omega = \{(r,\theta) \mid r > R, 0 < \theta < \omega\}, \Gamma = \{(r,\theta) \mid r = R, 0 < \theta < \omega\},$$

$$\Gamma_1 = \{(r,\theta) \mid r > R, \theta = 0\}, \Omega = \{(r,\theta) \mid r > R, \theta = \omega\},$$

with $\omega = \frac{15}{8}\pi, R = 1, r = 1.5, \varepsilon = \frac{\alpha}{\beta} = 0.5$. Similar with example 1, $\varepsilon = 0.5$. We introduce an elliptical boundary

$$\Gamma_0 = \{(R_0,\theta) \mid \theta \in (0,\omega)\}, R_0 = 1.5.$$

We take our numerical results for problem (1), with

$$a(x,u) = \begin{cases} 4 - r^2 + \dfrac{1}{\sqrt{1+u^2}}, & 1 \le r \le 1.5, \\ \dfrac{1}{1+u^2}, & r > 1.5, \end{cases}$$

$$u = \sin\left(\frac{y}{x^2 + y^2}\right).$$

Furthermore, we let $\Delta r = \frac{1}{m}, \Delta\theta = \frac{15\pi}{8M}$ for some integer $m, M \in N$. The exact solution "u" is solved with $N = 10, \Delta r = \frac{1}{16}, \Delta\theta = \frac{\omega}{64}$.

The numerical results are given in Table 3 and Table 4 (Figures 5-7).

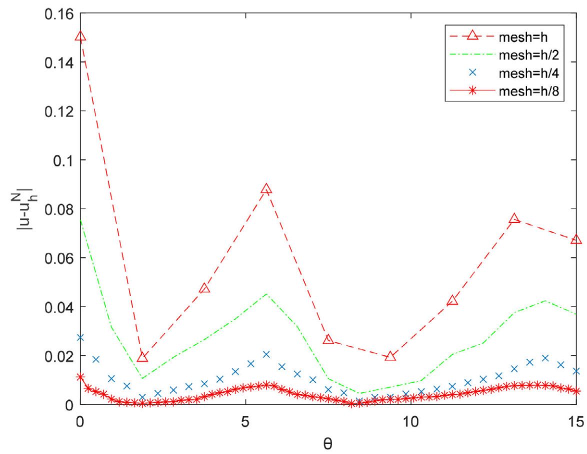

Figure 5. Error value of straight triangular element in finite element with N = 10, $\varepsilon = 0.5$.

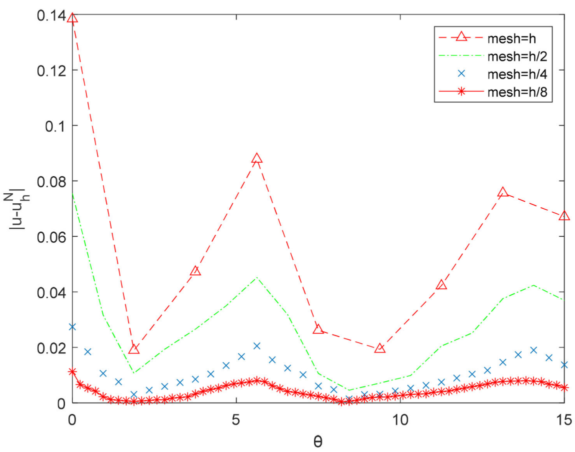

Figure 6. Error value of curved edge element in finite element with N = 10, $\varepsilon = 0.5$.

Table 3. Error value when straight triangular element is used for finite element.

mesh	$\|u-u_h\|_{L^2(\Omega_1)}$	ratio $(L^2(\Omega_1))$	$\|u-u_h\|_{L^\infty(\Omega_1)}$	ratio $(L^\infty(\Omega_1))$
h	5.93620E-1		6.20480E-1	
$h/2$	2.45574E-1	2.41727	2.58075E-1	2.40426
$h/4$	8.78220E-2	2.79628	9.23020E-2	2.79597
$h/8$	2.26990E-2	3.86890	2.60230E-2	3.54698

Table 4. Error value of curved edge element in finite element.

mesh	$\|u-u_h\|_{L^2(\Omega_1)}$	ratio $\left(L^2(\Omega_1)\right)$	$\|u-u_h\|_{L^\infty(\Omega_1)}$	ratio $\left(L^\infty(\Omega_1)\right)$
h	5.20189E-1		5.65939E-1	
$h/2$	2.26124E-1	2.30046	2.46121E-1	2.29943
$h/4$	8.49490E-2	2.66189	9.24893E-2	2.66108
$h/8$	2.26970E-2	3.74275	2.47190E-2	3.74160

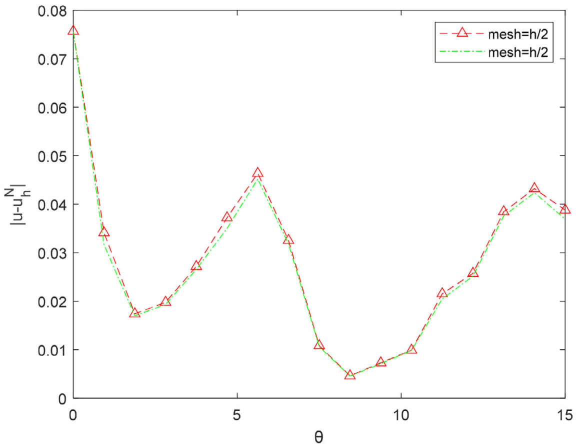

Figure 7. Comparison of error value in **Figure 5** and **Figure 6** with mesh = $h/2$.

These are the three error analysis diagrams of Example 2.

The numerical examples show that the errors can be affected by the location of the artificial boundary and the order of the artificial boundary condition, and it can be reduced by refining the mesh and optimizing finite element subdivision methods. The numerical results are coincident with the theoretical analysis and show the efficiency of the coupling method.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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