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Characteristics of an Accelerated Transverse Mechanical Pulse and Its Implications

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Abstract

In this report, two nontraditional mechanical wave-related issues are addressed. 1) Customary, for practical reasons the characteristics of the sinusoidal pulses progressing at a constant speed only is considered. And 2) the literature search shows that there has been no interest in exploring the characteristics of the pulses in a curvilinear two-dimensional space. By relaxing the first restriction we consider a scenario that which a mechanical pulse progresses with variable speed, specifically at a constant acceleration. We develop its equation of motion conducive to a homogenous linear partial differential equation with variable coefficients then we apply it to a practical problem. To address the 2nd point we depict a circular orbital embodying two Gaussian pulses circulating in opposite directions. Utilizing a Computer Algebra System (CAS), *Mathematica*, we develop animations visually easing the comprehension of the issues on hand.

Keywords

Mechanical Transverse Pulse, Accelerated Pulse, Computer Algebra System, *Mathematica*

1. Introduction

Generally speaking, most physics texts begin addressing the wave-related issues by considering transverse and longitudinal waves. For the former, waves in a string and for the latter sound in a gas are considered popular representative examples. This approach suppresses the fact that waves are composite entities. As such wave issues have the potential to be addressed by focusing on the properties of its constituents *i.e.* pluses. By confining the scope of the investigation to the mentioned special examples one leaves the "wave chapters" having the notion that waves always move at a constant speed. As such formulation developed quantifying their speeds.

In this note, we wish to deviate from the traditional norm. We consider suitable non-harmonic pulses. Specifically the localized Gaussian pulses. Then we consider a practical scenario that which the pulse would propagate with a variable speed, specifically at a constant acceleration. To be transparent, we confine the scope of the investigation to a one-dimensional case and give a practical experiment to its support.

Following the laid objectives, in the second section of the report, we focus on the properties of the mobile signals in a two-dimensional curvilinear space. We choose a circle representing the curvilinear space, other shapes such as an ellipse or other shapes may be considered as well.

This report is composed of three sections. In addition to the Introduction, the second section embodies a detailed description of our analysis. It contains also the needed *Mathematica* codes [1], the interested reader may duplicate the output. This section also includes a 3D display of two mobile circulating pulses. The last section is the Conclusions, explaining what we have learned while developing this report.

2. Procedure

We begin with constructing a traveling pulse from the ground up. Any reasonable one or multi-variable analytic function depending on the interest may be considered to represent a pulse. For one dimensional case replacing the argument of the function, $f(x)$, by $x \to x - x_0$ translates the function without altering its shape along the positive x-axis by x_0. The alternative is to replace $x \to x + x_0$ translating the function along the negative direction. The transformation is discrete as if the function is jumped by one step. Replacing the discrete parameter x_0 with a continuous variable the function will slide along one of the mentioned directions continuously. In other words, the static function $f(x)$ becomes mobile, *i.e.*, the pulse would propagate.

With this fundamental notion, the entire wave industry is funded, [2] [3] [4] [5] [6]. For instance sinusoidal, harmonic transverse waves are given by $y(x;t) = y_0 \sin[k(x \pm vt)]$, or longitudinal acoustic waves are $S(x;t) = s_0 \sin[k(x \pm vt)]$, [2] [3] [4]. In these expressions vt with t being the continuous parameter, time is substituted for the x_0. The linearity of the t parameter signifies the uniform motion of the signal. Phenomenons such as wave interference, resonances, etc. are based on the applications of these waves. As pointed out in the abstract one leaves the "wave chapters" with the notion that wave motions are confined to uniform motions, *i.e.*, constants speeds. One of the objectives of our investigation is to extend the scope of the study to include cases that which the time variable is not linear. Although any power of time may be considered, here we focus only on the second order. This corresponds to a motion at a constant acceleration. By introducing a practical example we justify the reason for our interest.

Consider an accelerated pulse by replacing $x_0 \to -\frac{1}{2}at^2$, *i.e.*, a pulse that is sliding along the positive *x*-axis with constant acceleration a, $f(x;t) = f\left(x - \frac{1}{2}at^2\right)$. To form its equation of motion we calculate its first order partial declivities,

$$\begin{cases} \partial_x f(x;t) = f'(x;t) \\ \partial_t f(x;t) = -atf'(x;t) \end{cases} \tag{1}$$

where the primes are the derivatives w/the argument *i.e.* $x - 1/2at^2$. Dividing the terms of (1) yields,

$$\left(\partial_x + \frac{1}{at}\partial_t\right) f(x;t) = 0. \tag{2}$$

Equation (2) is the equation of motion of a pulse that is progressive with a constant acceleration a along the positive *x*-axis. It is useful for the succinctness if one defines an operator, $\hat{O}$, to referee (2),

$$\hat{O}_+ := \partial_x + \frac{1}{at}\partial_t, \tag{3}$$

so that (2) reads,

$$\hat{O}_+ f\left(x - \frac{1}{2}at^2\right) = 0. \tag{4}$$

For a signal that is sliding along the negative *x*-axis, one may easily show that its equation of motion is subject to an alike (4),

$$\hat{O}_- f\left(x + \frac{1}{2}at^2\right) = 0. \tag{5}$$

And for a practical situation when a pulse is set in simultaneous motion along both directions the corresponding equation of motion becomes,

$$\hat{O}_+ \hat{O}_- f(x;t) = 0, \tag{6}$$

where,

$$\hat{O}_+ \hat{O}_- := \partial_x^2 + \left(\frac{1}{a^2}\frac{1}{t^3}\right)\partial_t - \frac{1}{(at)^2}\partial_t^2, \tag{7}$$

noting, $\left[\hat{O}_+, \hat{O}_-\right] = 0$. The bracket is the commutator, meaning the operators permute.

The general solution to (6) is,

$$f(x;t) = c_1 g\left(x - \frac{1}{2}at^2\right) + C_2 h\left(x + \frac{1}{2}at^2\right); \tag{8}$$

with C_1 and C_2 are constants. The *g* and *h* represent the accelerated mobile signals along the mentioned directions.

Noticing (7) distinguishing itself vs. the "standard" equation of motion. The "standard" refers to the motion with a constant speed, [4]. Note also (6) is a linear asymmetric homogeneous partial differential equation with variable coefficients. It is noted the partial directives are not symmetric. Also note by replacing $at \to v$, (7) becomes closely comparable with the "standard" equation of motion.

Here we consider a practical setting putting the developed equation of motion into action. Consider a rope, a line with no negligible mass that vertically hangs from a ceiling. Assume the line has a length ℓ and mass of m, see **Figure 1**. For sake of simplicity consider a case where the line has a constant mass density. The objective is to look into the kinematics of a transverse pulse that originates at the free end of the line progressing upward towards the pivot. **Figure 1** depicts the setting. It is known [2] [3] [4] as the speed "wave" in a massive line with sustained constant tension, T, is,

$$v = \sqrt{\frac{T}{\mu}}, \tag{9}$$

where $\mu = \frac{m}{\ell}$ is the linear massdensity. Assuming this relationship is just for the scenario on hand we replace the tension with,

$$T(y) = \mu y g. \tag{10}$$

As shown in **Figure 1**, y, is an arbitrary point along the line, and g is the gravity acceleration. Substituting (10) in (9) and replacing $v \to \mathrm{d}y/\mathrm{d}t$ the solution of the corresponding equation yields,

$$y = \frac{1}{2}\left(\frac{1}{2}gt^2\right). \tag{11}$$

I.e. the pulse crawls upward along the line accelerated with a half acceleration of a free fall. Two points: 1) the pulse is not progressing at a constant speed and 2) its speed increasing at a constant rate, its acceleration is constant. We reason this is because, at the bottom of the line where the pulse originates the tension is at minimum, $T = 0$, and at the pivot, the tension is at maximum, $T = mg$. This is

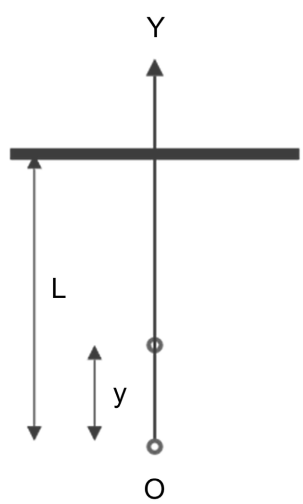

Figure 1. A rope of length L and mass m is hung from a pivot. The origin of the coordinate system is set at the bottom.

exactly the opposite of what an upward project does. It is also interesting to note that theoretically increasing speed ultimately could reach the light speed. This would require a line one-quarter of a light-year long! Note also if the pulse originates at the pivot on the way down the line it will decelerate with the mentioned characters.

Based on mentioned theoretical and experimental development, we craft an animation. Its snapshot is depicted in **Figure 2**. Two scenarios are considered. The Gaussian pulses are set in upwards motion.

Manipulate[Rotate[Plot[0.2e^(-5(x-t)^2),{x,-3,10},PlotRange->{0,0.5}, PlotStyle->Thickness[0.008],GridLines→Automatic,PlotLabel→"Const Speed"],90Degree],{t,0,8,0.01},ControlPlacement->Top]

Manipulate[Rotate[Plot[0.2e^(-5(x-t^2)^2),{x,-3,10},PlotRange->{0,0.5}, PlotStyle->{Red,Thickness[0.008]},GridLines→Automatic,PlotLabel→"Const Acc"],90Degree],{t,0,2.97,0.01},ControlPlacement->Top]

The code includes two Gaussian signals; $e^{-5(x-t)^2}$ in blue and $e^{-5\left(x-t^2\right)^2}$ in red. These two pulses are used because they are localized signals. Harmonic pulses are not localized they are not suitable for our honed objectives. The first one corresponds to a mobile pulse at a constant speed and the second one corresponds

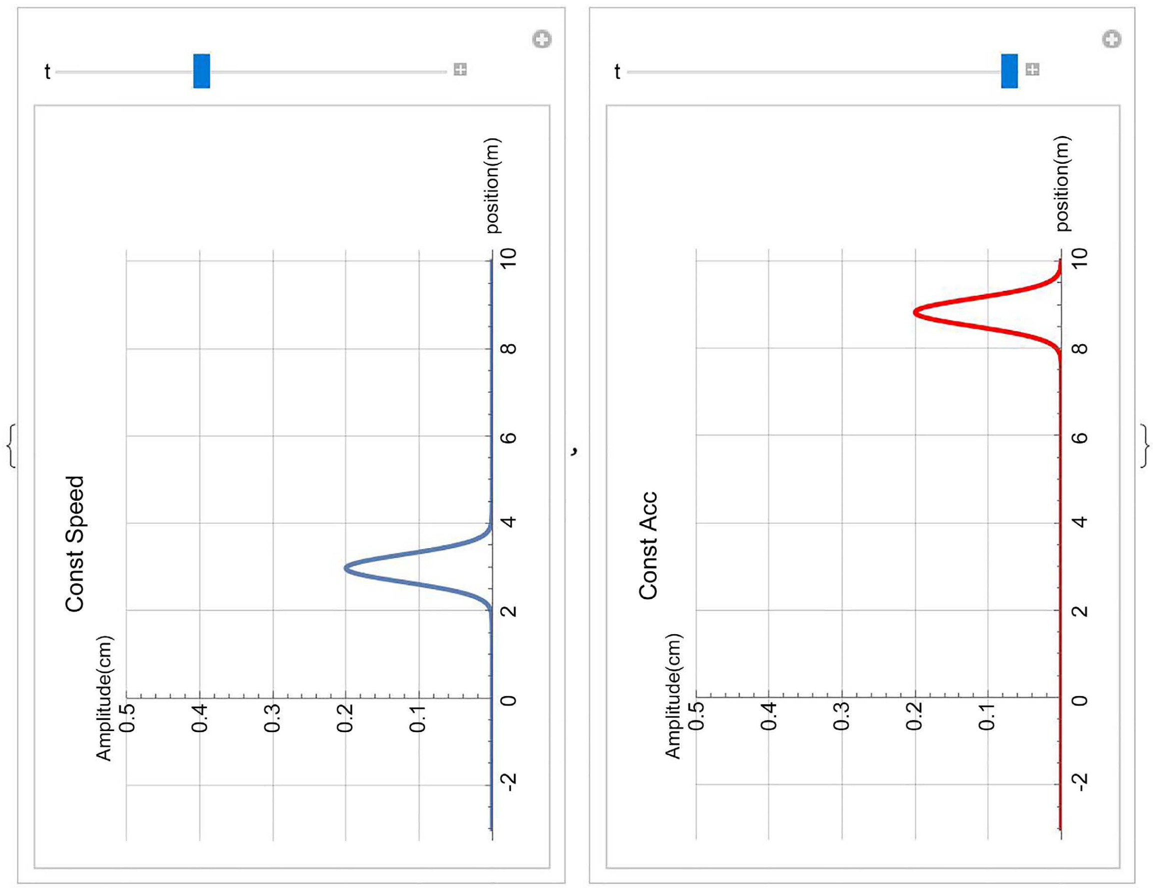

Figure 2. These are two Gaussian localized signals. The left and the right panels depict snap-shots of progressive pulses at a constant speed and a constant acceleration, respectively.

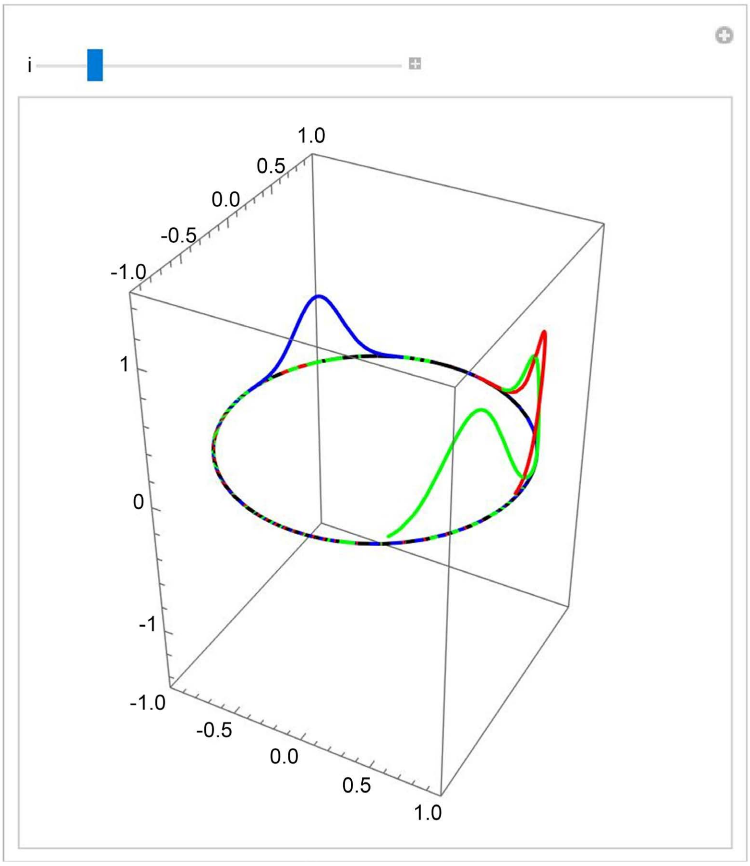

Figure 3. A 3D orbital housing two circulating pulses are the show. The Gaussian pulses are the rotating signals. The blue pulse rotates clockwise while the red one rotates counter-clockwise, respectively. The green signal is the superposed signal.

to the accelerated case, respectfully. As shown these two signals by preserving their original shapes move at different rates distancing different distances. The time duration of these animations is 2.97 s. The left panel in blue is the constant speed, and the right panel in red is the constant acceleration. Clearly, these show their differences.

We conclude this report by considering a pulse-related issue that is skipped attention in the literature [5] [6] [7]. **Figure 3** depicts the case of interest. A circular orbital houses two localized Gaussian pulses circulating in opposite directions; clockwise and counterclockwise, respectively. The kinematics of the motions are optional. Meaning each pulse may rotate uniformly or at constant acceleration. The question of interest would be to analyze the feature of their interference pattern, resonances, etc. **Figure 3** shows two such pulses; the blue pulse moves clockwise while the red rotates counter-clockwise. The green signal is the result of their superposition. The code conducive to the shown scenario and its corresponding analysis is left in the interest of the reader.

3. Conclusions

The main interest of this investigating report is to show that there are numerous unattended issues in the wave-elated phenomena. The most common issues related to the progressive waves are discussed at the introductory level in physics textbooks. As pointed out one leaves the "wave chapters" thinking that waves only travel at a constant speed. And that the waves are only sinusoidal. Yet, if one views the localized pulses as massless marbles one may establish a comprehensive interchangeable feature amongst them. As such we develop a theoretical base describing accelerated progressive pulses and support the theory with a plausible experimental setting. Although we limited the theory to the pulsating motion at constant acceleration our view may be extended to include higher-order time dependencies; *i.e.*, variable acceleration. It is left to the interest of the reader to explore the features such as super-positioning of the traveling pulses by adapting Gaussian signals. The kinematics of the signals might exhibit interesting features.

The graphs in this report are made with *Mathematica*. Interested readers may find [8] [9] resourceful.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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