Transitioning from Discrete to Continuous Distribution *Mathematica* vs. Excel

—An Example

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**Abstract**

Frequencies of the repeated integers of the first \( n \) digits of e.g. \( \pi \) utilizing commercial software are listed. The discrete distribution is utilized to evaluate its statistical moments. The distribution is fitted with a polynomial generating a continuous replica of the former. Its statistical moments are evaluated and compared to the former. The procedure clarifies the mechanism transiting from discrete to a continuous domain. Applying *Mathematica* the fitted polynomial is replaced with an interpolated function with controlled smoothing factor refining the quality of the fit and its corresponding moments. Knowledge learned assists in the understanding of the standard procedure calculating moments of e.g. Maxwell-Boltzmann continuous distribution in kinetic Theory of gases.

**Keywords**

Digits of \( \pi \), Discrete Distribution, Continuous Distribution, Moments of Distribution, Excel, *Mathematica*

1. Introduction

Tabulating the statistical information such as distribution moments for either the discrete and/or continuous ensembles is the quantities of paramount interest when working with either abstract mathematical or collected data in natural sciences. It is somewhat trivial to evaluate the moments for a discrete data set, it is not obvious how to systematically transit from the discrete to the continuous situation.

One of the objectives of this report is that by way of example first to show how the moments are evaluated for a set of the discrete mathematical ensemble, and
then by applying the same conceptual method to extend the procedure for the continuous case.

To achieve this goal, we select one of the ~32,000 known constants in science e.g., the value of \( \pi \). The shown procedure identically may be applied to any of the chosen constants. For an instant e, the Euler constant \( \gamma \), the golden ratio \( \varphi \), etc. Here in this report, we have chosen the \( \pi \). We form an ensemble comprised of \( n \) digits of \( \pi \) naturally, this is a set of discrete integers. We then show how the statistical moments of the set are evaluated. Taking advantage of the commercially available software, e.g., Excel [1] we tally the data conducive to the needed distribution function. Having the distribution function on hand we evaluate the moments, such as the first, second, third, etc. Excel is an excellent numeric-based program with certain limitations. For instance, because it is a single-precision compiler it displays the digits of \( \pi \) up to 16 significant figures. As such it limits the number of elements of \( \pi \text{List} \). To circle this, one may use a commercially available scientific software e.g., Mathematica [2]. This allows extending the number of the digits of \( \pi \text{List} \) literally to “infinite.” To transit from discrete to continuous and hence to evaluate the moments we form the extended \( \pi \text{List} \) say with 50 elements. This list then is imported to Excel and is used as a basis to form the continuous distribution function by fitting it using a polynomial. Here again, Excel is limited to a maximum 6th order polynomial. For the sake of consistency when we utilize the Mathematica, we apply the same polynomial power; this results in the identical result. However, Mathematica has a useful option smoothening the quality of the fit. Utilizing this option, we perfect the fit. We include tables embodying the values of the calculated moments for all the scenarios.

This report is comprised of four sections. In addition to Section 1, introduction that outlines the motivation and goals, Section 2 is procedure; a description that embodies Mathematica codes, charts, tables as well as selected Excel’s charts. The interested reader may easily duplicate the steps and modify the codes adjusting to the need, for information c.f. [3] [4]. Section 3 is the conclusions and comments on what we learned.

2. Procedure

For the sake of efficiency, we begin with Mathematica, as such first we form the \( \pi \text{List} \), a list of digits of \( \pi \). \( N_{\text{max}} \) defines the number of desired significant digits, e.g., 50. Shown program is crafted such that with this input parameter one single keystroke runs the entire program with the needed output.

\[
N_{\text{max}}=50;
\]

\[
\pi=\text{First}[\text{RealDigits}[\text{N}[\pi,N_{\text{max}}]]];
\]

Next, we tabulate the tallied digits, (see Table 1)

\[
\text{table} = \text{TableForm}[\text{Tally}[\pi]/.\{p_\_,q_\_\}->\{q,p\},\text{TableHeadings} \rightarrow \{\text{Automatic,}
\text{"Frequency","digit/Event"}\}]
\]

By defining a few auxiliary components, we display the Frequency vs. the Range. This is shown in Figure 1.
Table 1. Frequencies of the first 50 digits of Pi vs. the digits.

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Digits</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 1. Display of frequencies vs. ranges i.e., the event.

```
lengthtallypi = Length[Tally[pi]]; data50 = Transpose[Table[{Table[[1,n,2]],Table[[1,n,1]],{n,1,lengthtallypi]}}]; listplotdata50 = ListPlot[data50,AxesOrigin -> {0,0},PlotStyle -> Blue,AxesLabel -> {"event","Frequency"},GridLines -> Automatic] To check the integrity of the normalized distribution we form, Frequencies = Table[Table[[1,n,1]],{n,1,lengthtallypi}]; events = Table[Table[[1,n,2]],{n,1,lengthtallypi}]; frequencies = N[1/Nmax Frequencies]; Apply[Plus,frequencies] 1. This shows the output of this step is a 1. Assuring the distribution is normalized to unity.
```
Note also, Apply is a standard Mathematica command. In this particular-code, it applies the Plus function adding the elements of the listed frequencies.

Next, we evaluate the mean and the RMS of the distribution and compare them to the output of Mathematica’s library functions,

\[
\text{discreteaverage,mean} = \{\text{Apply[Plus,frequencies events]}, \text{N[Mean[pi]]}]\};
\]

\[
\text{discreteRMS,RMS} = \{\text{sqrt[Apply[Plus, frequencies events^2]}], \text{N[RootMeanSquare[pi]]}]\};
\]

\[\{4.939,4.94\}\]

\[\{5.662,5.662\}\]

As shown, these are identical. These steps ensure the accuracy of our program conducive to laying the bases for transiting to the continuous scenario. We also evaluate the 3rd moment making the point that evaluating the n\text{th} order moment is no challenge.

\[\text{discretethirdmoment} = \text{cuberoot[Apply[Plus,frequencies events^3]]};\]

The summary of the calculation is shown in Table 2.

To transit to the continuous domain and be compatible with the capabilities of Excel we consider a 5th order polynomial for the model. Note that by trial and error the 5th and the 6th order proven to be indistinguishable.

\[\text{model} = a + bx + cx^2 + dx^3 + ex^4 + fx^5;\]

\[
\text{Flatten[data50,\{2\}]}[[1]];
\]

\[
\text{fit1} = \text{FindFit[Flatten[data50,\{2\}]}[[1]], \text{model}, \{a,b,c,d,e,f\},x];
\]

\[
\text{fit1x} = \text{model/.fit1}
\]

\[
\text{Show[\{listplotdata50,ListPlot[Table[\{x,fit1x\},\{x,0,9\}],PlotStyle->Red\}]}]
\]

\[
1.03357 + 4.47364 \times - 0.845862 \times^2 - 0.0632867 \times^3 + 0.02162 \times^4 - 0.00102564 \times^5
\]

Noting the numeric coefficients of the fitted polynomials utilized Mathematica and Excel are the same.

Figure 2 and Figure 3 show the fitted polynomials have the correct trend, however, they aren’t to satisfaction. Taking these polynomials as their face value we evaluate their corresponding statistical moments.

To do so and fulfill one of our aimed objectives i.e., transiting from discrete to continuous we take the following steps: since normalized discrete distribution is subject to

\[
\frac{1}{N_{\text{max}}} \sum_{i=1}^{i} F_i , \text{ where } F_i \text{ is the number of events we replace } \frac{1}{N_{\text{max}}} \rightarrow \frac{\Delta x}{9}
\]

and \[F_i \rightarrow F(x)\] where \(H(x)\) is the fitted polynomial. With these substitutions, the normalization condition reads,

\[
\frac{1}{N_{\text{max}}} \sum_{i=1}^{i} F_i \rightarrow \frac{1}{9} \int_{0}^{\alpha} F(x) dx ,
\]

And because the fitted polynomial or any fitted modeled function is not normalized by multiplying the model function with a constant we enforce the normalization so,

\[
\text{normalized Distribution} = \alpha = 9 \left(\int_{0}^{\alpha} \text{fit1}x \text{dx} \right)^{-1} ;
\]
Table 2. First three moments associated with the first 50 digits of Pi.

<table>
<thead>
<tr>
<th>Moment</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average</td>
<td>4.94</td>
</tr>
<tr>
<td>RMS</td>
<td>5.66</td>
</tr>
<tr>
<td>3\text{rd} moment</td>
<td>6.17</td>
</tr>
</tbody>
</table>

Figure 2. The blue dots are the data shown in Figure 1, the red dots are the values of the Frequencies applied to the polynomial, i.e., our model.

Figure 3. Excel fitted curve with the explicitly employed polynomial.

Now by applying the normalized distribution we evaluate the moments,

\[
\text{continuous Average} = \frac{1}{9} \int_{0}^{\alpha} x (\alpha fit_1 x) \, dx ;
\]

\[
\text{rms Continuous} = \sqrt{\int_{0}^{\alpha} x^2 \frac{1}{9} (\alpha fit_1 x) \, dx} ;
\]

\[
\text{third Moment} = \sqrt{\int_{0}^{\alpha} x^3 \frac{1}{9} (\alpha fit_1 x) \, dx} ;
\]
The summary of the output is tabulated in Table 3. The tabulated values of Table 2 and Table 3 reveal the differences between the discrete and its corresponding continuous distributions.

As we pointed out Figure 2 and/or Figure 3 are not quite to our satisfaction. If we were using Excel as an ultimate tool this would have been our best fit with its accompanied evaluated moments given in Table 3. However, *Mathematica* offers a procedure improving the fit quality. With *Mathematica* the discrete limited data are shown with the blue dots in Figure 1 may be interpolated generating unlimited implicit data making the fit procedure much satisfactory. After trial and error, we noted Interpolation Order of degree 4 works the best. Figure 4 is the result.

As shown interpolated function fits exactly the data, dots shown on the right plate are doubly overlapped dots. Meaning the smoothened data is exactly overlaps with discrete ones. The green continuous curve is the $5^{th}$ order polynomial that is smoothened by *Mathematica*. Having such a perfect fit i.e., continuous distribution, we calculate its moments. Beforehand the green curve ought to be normalized,

$$\beta = N \left[ 9 \int_0^9 \text{interpolate}[x] \, dx \right]^{-1};$$

And the moments are,

- average$\text{Interpolated}[x] = \frac{1}{9} N \text{Integrate}[x \cdot (\beta \text{interpolate}[x]), \{x, 0, 9\}]$;
- $\text{rmsInterpolated}[x] = \sqrt{\frac{1}{9} N \text{Integrate}[x^2 \cdot (\beta \text{interpolate}[x]), \{x, 0, 9\}]}$;
- $\text{thirdmomentInterpolated}[x] = \text{cuberooot}[\frac{1}{9} N \text{Integrate}[x^3 \cdot (\beta \text{interpolate}[x]), \{x, 0, 9\}]];$

<table>
<thead>
<tr>
<th>Table 3. Average, RMS and the third moment of the continuous distribution of the first 50 digits of $\pi$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average</td>
</tr>
<tr>
<td>RMS</td>
</tr>
<tr>
<td>$3^{rd}$ moment</td>
</tr>
</tbody>
</table>

Figure 4. The left plate is the same as Figure 2, the right plate is the Interpolated fit including the smoothening factor.
Table 4. Summary of the first three moments associated with the three scenarios. Description of each case is embedded in the text.

<table>
<thead>
<tr>
<th></th>
<th>Average</th>
<th>RMS</th>
<th>3rd moment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discrete</td>
<td>4.94</td>
<td>5.66</td>
<td>6.17</td>
</tr>
<tr>
<td>Polynomial</td>
<td>4.56</td>
<td>5.23</td>
<td>5.72</td>
</tr>
<tr>
<td>Interpolated</td>
<td>4.47</td>
<td>5.16</td>
<td>5.66</td>
</tr>
</tbody>
</table>

Table 4 is the calculated moments for three scenarios in this report.

Table 4 embodies the values of the body of the work presented in this report. It compares the values of various moments of discrete, continuous, and improved continuous distributions. To form an opinion about the quality of these moments and the associated quality of distribution functions one needs to have Figure 4 in mind.

3. Conclusions

We set two aims crafting this investigation. For practical purposes by a way of example we show how the common knowledge evaluating statistical moments of a discrete distribution is extended by evaluating similar moments for a continuous distribution. Steps shown in this report transiting between these two sets of distributions fill the missing gap that is overlooked in the literature. Our approach identified the shortcoming of a useful program such as Excel justifying looking for a replacement such as the powerful scientific program Mathematica. For a manageable list of a discrete list of integer numbers, we set the list length to 50, as mentioned Mathematica contrary to Excel can extend the list length literally to infinite. Specific of the given example is applied for π, identical steps may be taken utilizing e.g., the e, Euler gamma γ, etc. and combinations amongst e.g., π, eπ, so there is no limitation replicating examples.

The lesson learned is that with a solid understanding of steps needed to calculate the moments of a continuous distribution, calculation conducive to the moments for distributions encounter in physical science, e.g., speed distribution given by Maxwell-Boltzmann [5], or probability distribution for quantum system easily may be duplicated as well.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.
References


