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On a Dual to the Properties of Hurwitz Polynomials I

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Abstract

In this work we develop necessary and sufficient conditions for describing the family of anti-Hurwitz polynomials, introduced by Vergara-Hermosilla *et al.* in [1]. Specifically, we studied a dual version of the Theorem of Routh-Hurwitz and present explicit criteria for polynomials of low order and derivatives. Another contribution of this work is establishing a dual version of the Hermite-Biehler Theorem. To this aim, we give extensions of the boundary crossing Theorems and a zero exclusion principle for anti-Hurwitz polynomials.

Keywords

Hurwitz Polynomials, Anti-Hurwitz Polynomials, Hermite-Biehler Theorem, Exclusion Principle

1. Introduction

In this paper we present the first part, of a series of three works, on a new approach about the classification of the roots of real polynomials in one variable in the right half complex plane. This new idea arises from the need to obtain simple explicit criteria for the area of the complex plane not covered by the theory of Hurwitz polynomials (also known as stable polynomials). In fact, our results are natural extensions of the classical Theorems of Routh-Hurwitz and Hermite-Biehler for the complement zone; $\mathbb{C}_+ = \{z \in \mathbb{C} : \text{Re}(z) > 0\}$.

In the literature we highlight as main references for the study of roots of real polynomials on the left half complex plane and its applications to system theory in a general framework the books of Gantmacher [2] [3], and the book of Iooss and Joseph [4]. Chappellat, Mansour and Brattacharyya present classic stability criteria with elementary demonstrations in their article [5] while new and interesting ideas about the demonstration of these results have been developed by Holtz in [6]. For a generalization to real polynomials in several variables we mention the work of Fettweis in [7]. The approach introduced in this work consist of a systematic use of the linear transformation $z \mapsto -z$, on the properties that define the Hurwitz polynomials, which leads us to use and explore the notion originally introduced by Vergara-Hermosilla *et al.* in [1] about anti-Hurwitz polynomials. This notion can be recast as a dual result to the main necessary and sufficient conditions on stable polynomials. What is more, our Theorems and Propositions also depend on the coefficients of the polynomial in question which makes it more manipulable for applications in science and engineering. To this end, in Section 5 we will apply our results to a family of polynomials associated with a system of PDE's that describe interactions fluid-structure, for details see [1] or Vergara-Hermosilla [8]. With this preamble,, we are in a position of establish our first main result, which read as:

Proposition. Let $f(X) = a_0X^n + a_1X^{n-1} + \cdots + a_{n-1}X + a_n \in \mathbb{R}[X]$ of degree ≥ 3. Then $f(X)$ is an anti-Hurwitz polynomial, if and only if it satisfies the conditions:

1) $(-1)^i a_i > 0$, for all $i \in \{0, \cdots, n\}$.
2) $(-1)^{\left\lfloor \frac{i+1}{2} \right\rfloor} \Delta_i > 0$, for all $i \in \{1, \cdots, n\}$.

As the practical use of of the Routh-Hurwitz criterion is usually limited, in the context of direct computations, to polynomials of low degrees (3rd, 4th, or 5th), we develop an alternatives result, which is more versatile and, as in the previous case, this extends to the dual zone the classical Boundary Crossing Theorems. More precisely, we will prove our second main result which is dual version of the Zero Exclusion Principle. Our second main result reads as:

Theorem. Let $\mathcal{AH}$ the set of the real anti-Hurwitz polynomials. Suppose $\{f(\lambda, X)\}_{\lambda \in \Omega}$ is a family of real polinomials in the variable X wich depends continuously on the $\lambda \in \Omega \subset \mathbb{R}$, with Ω pathwise connected. Suppose moreover that the family $f(\lambda, X)$ is of degree constant and there is at least one anti-Hurwitz polynomial. Then, the family $\{f(\lambda, X)\}_{\lambda \in \Omega} \subset \mathcal{AH}$, if and only if $p(\lambda, iw) \neq 0$, for all $w \in \mathbb{R}$ and $\lambda \in \Omega$.

With this result, and by defining an appropriate property of anti-alternancy, we will demonstrate the third main result in this paper, which is a dual version of the Theorem of Hermite-Biehler. This third main result reads as:

Theorem. A real polinomial $p(X)$ is Anti-Hurwitz, if and only if satisfies the anti-alternancy property.

The outline of the paper is organized as follow. In Section 2 we state the main definitions and properties that describe the Hurwitz polynomials emphasizing the Hurwitz matrix and the Theorem of Routh-Hurwitz. In Section 3 we define the anti-Hurwitz polynomials, demonstrate our first main result, and establish explicit criteria for real polynomials of less than or equal order 4 and derivatives. In Section 4 we introduce the dual versions of the classical Boundary Crossing results and we proof our second and third main results. Finally, in Section 5 we apply our results for obtaining information about the behavior of the roots of the family of *viscous polynomials* defined in [1].

2. Hurwitz Polynomials

For $n \in \mathbb{N}$ we denote by $\mathcal{P}_n$ the set of all degree *n* polynomials with real coefficients.

Definition 2.1. A polynomial $f(X) \in \mathbb{R}[X]$ is Hurwitz if the real part of all its complex roots is negative *i.e.*, $\mathrm{Re}(u) < 0$ for any $u \in \mathbb{C}$ satisfying $f(u) = 0$.

Let $\mathcal{H}$ denote the set of all Hurwitz polynomials, and we set $\mathcal{H}_n = \mathcal{H} \cap \mathcal{P}_n$. The set of all Hurwitz polynomials in $\mathcal{H}_n$ with positive coefficients is denoted by $\mathcal{H}_n^+$.

Theorem 1 (Stodola condition). *If a polynomial* $f(X) \in \mathbb{R}[X]$ *is Hurwitz, then all its coefficients are of the same sign.*

Proof. The roots of a real polynomial are symmetric with respect to the real line. For $f(X)$, we can write

$$f(X) = a_0 \prod_k (X - s_k) \prod_j (X - \alpha_j - i\beta_j) \prod_j (X - \alpha_j + i\beta_j), \tag{2.1}$$

where each s_k are real roots, and $\alpha_j \pm i\beta_j$ are complex roots of $f(X)$ with nonzero imaginary part. Note that s_j, α_j are negative. Since the expressions $(X - s_k)$ and $X^2 - 2\alpha_j X + (\alpha_j^2 + \beta_j^2)$ have positive coefficients, their product has the same property.

Let $f(X) = a_0 X^n + \cdots + a_{n-1}X + a_n \in \mathcal{P}_n$ be a polynomial. The *Hurwitz matrix* of a polynomial, denoted as $H(f(X))$, is the square matrix of size *n* defined as follows:

$$\begin{pmatrix} a_1 & a_3 & a_5 & \cdots & 0 & 0 & 0 \\ a_0 & a_2 & a_4 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & a_3 & \cdots & 0 & 0 & 0 \\ 0 & a_0 & a_2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a_1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a_0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-2} & a_n & 0 \\ 0 & 0 & 0 & \cdots & a_{n-3} & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & a_{n-4} & a_{n-2} & a_n \end{pmatrix} \tag{2.2}$$

For every $k \in \{1, \cdots, n\}$, let $H_k(f(X))$ denote the square matrix of size k obtained from the first *k* rows and columns of $H(f(X))$, and we set:

$$\Delta_k = \det\left(H_k(f(X))\right), \tag{2.3}$$

where $\det(H_k)$ denotes the determinant of the square matrix H_k.

Theorem 2 (Routh-Hurwitz). A polynomial $f(X) = a_0 X^n + \cdots + a_{n-1}X + a_n \in \mathcal{P}_n$ with $a_0 > 0$ is Hurwitz if and only if $\Delta_k > 0$ for all $k \in \{1, \cdots, n\}$.

For a proof of this result see for instance [2] [4] or [6].

3. Anti-Hurwitz Polynomials

In this section we establish the definition of anti-Hurwitz polynomials and a dual criterion to the Theorem of Routh-Hurwitz. To this end, we introduce the following definition.

Definition 3.1. A polynomial $f(X) \in \mathcal{P}_n$ is said to be anti-Hurwitz if the real part of all its complex roots is positive, *i.e.*, $\mathrm{Re}(u) > 0$ for all $u \in \mathbb{C}$ satisfying $f(u) = 0$.

Lemma 3. A polynomial $f(X) \in \mathcal{P}_n$ is anti-Hurwitz if and only if $f(-X)$ is Hurwitz.

Proof. Let $f(X)$ be an anti-Hurwitz polynomial and u a complex root of $f(-X)$. Then $f(-u) = 0$ and $\mathrm{Re}(-u) > 0$, *i.e.*, $\mathrm{Re}(u) < 0$. Therefore $f(-X)$ is Hurwitz. On the other hand, if $f(-X)$ is a Hurwitz polynomial and u a complex root of $f(X)$, then $f(u) = f(-(-u)) = 0$. In this case, $\mathrm{Re}(-u) < 0$, *i.e.*, $\mathrm{Re}(u) > 0$. Hence, $f(X)$ is anti-Hurwitz.

Lemma 4. Let $f(X) \in \mathbb{R}[X]$ be a polynomial of degree n and Δ_i the determinant of the Hurwitz submatrix $H_i(f(X))$, for $1 \le i \le n$. Then we have

$$\Delta_i^- = (-1)^{\left\lfloor \frac{i+1}{2} \right\rfloor} \Delta_i, \tag{3.1}$$

where Δ_i^- is the determinant of *i*-th Hurwitz submatrix $H_i(f(-X))$.

Proof. The matrix for $H_i(f(-X))$ is written as

$$\begin{pmatrix} -a_1 & -a_3 & -a_5 & \cdots & 0 & 0 & 0 \\ a_0 & a_2 & a_4 & \cdots & 0 & 0 & 0 \\ 0 & -a_1 & -a_3 & \cdots & 0 & 0 & 0 \\ 0 & a_0 & a_2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -a_1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a_0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-2} & a_n & 0 \\ 0 & 0 & 0 & \cdots & -a_{n-3} & -a_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & a_{n-4} & a_{n-2} & a_n \end{pmatrix}. \tag{3.2}$$

Comparing it with the matrix of $H_i(f(X))$, we immediately see that

$$\Delta_i^- = (-1)^{\left\lfloor \frac{i+1}{2} \right\rfloor} \Delta_i. \tag{3.3}$$

Proposition 5. Let $f(X) = a_0 X^n + a_1 X^{n-1} + \cdots + a_{n-1} X + a_n \in \mathbb{R}[X]$ of degree ≥ 3. Then $f(X)$ is an anti-Hurwitz polynomial, if and only if it satisfies the conditions:

1) $(-1)^i a_i > 0$, for all $i \in \{0, \cdots, n\}$.

2) $(-1)^{\left\lfloor \frac{i+1}{2} \right\rfloor} \Delta_i > 0$, for all $i \in \{1, \cdots, n\}$.

Proof. By lemma (3), we know that $f(X)$ is an anti-Hurwitz polynomial if and only if $f(-X)$ is a Hurwitz polynomial. In this case, the coefficient of X^i

in $f(-X)$ is $(-1)^i a_{n-i}$. Without loss of generality, we may suppose that $a_0 > 0$. Now the Stodola Condition (1) and Theorem (2), gives us that
$(-1)^i a_{n-i} > 0$, for $i \in \{0,1,\cdots,n\}$ and $\Delta_i^- > 0$. Hence, we conclude by Lemma (4).

In the following we establish simple criteria on the property of anti-Hurwitz, applicable to real polynomials of less than or equal order 4 and derivatives of polynomials. To this end we consider a polynomial $p(X) \in \mathbb{R}[X]$ and the necessary and sufficient conditions developed in the Proposition 5. The criteria read as:

- The polynomial $p(X) = X^2 + a_1 X + a_2$ is an anti-Hurwitz polynomial, if and only if

$$-a_1, a_2 > 0. \tag{3.4}$$

- The polynomial $p(X) = X^3 + a_1 X^2 + a_2 X + a_3$ is an anti-Hurwitz polynomial, if and only if

$$-a_1, a_2, -a_3 > 0 \text{ and } a_2 - a_1 a_2 > 0. \tag{3.5}$$

- The polynomial $p(X) = X^4 + a_1 X^3 + a_2 X^2 + a_3 X + a_4$ is an anti-Hurwitz polynomial, if and only if

$$-a_1, a_2, -a_3, a_4 > 0 \text{ and } a_1 a_2 a_3 - a_3^2 - a_1^2 a_4 > 0. \tag{3.6}$$

- Let $p(X)$ be an anti-Hurwitz polynomial of degree n and let $P'(X)$ denote the first-order derivative of $p(X)$ with respect to *X*. Then $-p'(X)$ is again an anti-Hurwitz polynomial.

4. A Dual Version of the Theorem of Hermite-Biehler

In this Section we establish a dual version of the Theorem of Hermite-Biehler for anti-Hurwitz polynomials. To this end, we need to introduce dual versions of Boundary Crossing Theorems. We begin the Section with the following definition.

Definition 4.1. Let $p(X) \in \mathbb{R}[X]$ and $w \in \mathbb{R}$. The argument of $p(iw)$ is called the phase of $p(iw)$.

Lemma 6. Let $p(X) = a_X^n + \cdots + a_1 X + a_0$ be an anti-Hurwitz polynomial of degree n. Then, $\arg p(iw)$ is a strictly decreasing function. Moreover, the net change in the phase from $-\infty$ to $+\infty$ is

$$\lim_{w \to +\infty} p(iw) - \lim_{w \to -\infty} p(iw) = n\pi. \tag{4.1}$$

Proof. By the fundamental theorem of algebra, we can write $p(X)$ as a product of its roots

$$p(X) = a_n (X - \alpha_1 - i\beta_1)(X - \alpha_2 - i\beta_2)\cdots(X - \alpha_n - i\beta_n).$$

Plugging $X = iw$, we get

$$p(iw) = a_n(-\alpha_1 + i(w - \beta_1))(-\alpha_2 + i(w - \beta_2))\cdots(-\alpha_n + i(w - \beta_n)),$$

and so, we obtain

$$\arg p(iw) = \arg(a_n) + \arctan\left(\frac{w-\beta_1}{-\alpha_1}\right) + \cdots + \arctan\left(\frac{w-\beta_n}{-\alpha_n}\right). \quad (4.2)$$

Differentiating the above expression with respect to w, we get

$$\frac{\mathrm{d}}{\mathrm{d}w}\arg p(iw) = \frac{1}{1+\left(\frac{w-\beta_1}{-\alpha_1}\right)^2}\left(-\frac{1}{\alpha_1}\right) + \cdots + \frac{1}{1+\left(\frac{w-\beta_1}{-\alpha_n}\right)^2}\left(-\frac{1}{\alpha_n}\right).$$

Since $p(X)$ is an anti-Hurwitz polynomial, we have that $\alpha_k > 0$ for $k \in \{1,\cdots,n\}$. Therefore, $\arg p(iw)$ is decreasing is a decreasing function in w. Now, from Equation (4.2), we have

$$\lim_{w\to+\infty} \arg p(iw) = \arg(\alpha_n) - \frac{n\pi}{2};$$

$$\lim_{w\to-\infty} \arg p(iw) = \arg(\alpha_n) + \frac{n\pi}{2}.$$

The claim now follows.

In the following we enunciate two classic results on stability, whose demonstrations can be consulted in the article of Chappellat *et al.* [5].

Proposition 7. Let $a_n \neq 0$, $p(X) = a_n X^n + \cdots + a_1 X + a_0 = a_n \prod_{j=1}^{m} (X-\omega_j)^{t_j}$, and $q(X) = (a_n+\varepsilon_n)X^n + \cdots + (a_1+\varepsilon_1)X + (a_0+\varepsilon_0)$. Consider the circle C_k of radius r_k centered at ω_k. Let r_k be fixed such that $0 \le r_k < \min|\omega_k - \omega_j|$, for $j \in \{1,2,\cdots,k-1,k+1,\cdots,m\}$. Then, there exists an $\varepsilon > 0$ such that for all $|\varepsilon_1|,|\varepsilon_2|,\cdots,|\varepsilon_n| < \varepsilon$, $q(X)$ has precisely t_k zeros inside the circle C_k.

Corollary 1. Fix m circles $C_1,\cdots,C_m$ that are pairwise disjoint and centered at $\omega_1,\omega_2,\cdots,\omega_m$ respectively. Then, by repeatedly applying Theorem (7), it is always possible to find an $\varepsilon > 0$ such that for any $|\varepsilon_1|,|\varepsilon_2|,\cdots,|\varepsilon_n| < \varepsilon$, $q(X)$ has precisely t_k zeros inside each of the circles C_k.

Remark. In the previous Corollary, we note that $q(X)$ always has $t_1 + t_2 + \cdots + t_m = n$ zeros and must therefore remain of degree n, so necessarily we have $\varepsilon < |a_n|$.

In the following we denote the set of anti-Hurwitz polynomials of degree n by $\mathcal{AH}_n$.

Remark. By Proposition (7), Corollary (1) and Remark (4), we see that if $p(X) = a_n X^n + \cdots + a_1 X + a_0 \in \mathcal{AH}_n$, then there exists an $\varepsilon > 0$ such that for all $|\varepsilon_1|,|\varepsilon_2|,\cdots,|\varepsilon_n| < \varepsilon$, the polynomial $q(X) = (a_n+\varepsilon_n)X^n + \cdots + (a_1+\varepsilon_1)X + (a_0+\varepsilon_0) \in \mathcal{AH}_n$.

Boundary Crossing Theorems

Let $p(\lambda, X)$ be a family of degree n polynomials with real coefficients, which is continuous with respect to $\lambda \in [a,b]$. In other words, $p(\lambda, X)$ can be written as

$$p(\lambda, X) = a_n(\lambda)X^n + \cdots + a_1(\lambda)X + a_0(\lambda),$$

where $a_0(\lambda), a_1(\lambda), \cdots, a_n(\lambda)$ are continuous functions in λ and $a_n(\lambda) \neq 0$

for all λ.

Theorem 8. Suppose that $p(a,X)$ has all its roots in $S \subset \mathbb{C}$, where $p(b,X)$ has at leat one root in $U = \mathbb{C} \setminus S$. Then, there exist at least one $\rho \in (a,b]$ such that

1) $p(\rho,X)$ has all its roots in $S \cup \partial S$.
2) $p(\rho,X)$ has at least one root in ∂S.

The proof of the Theorem above can be see in [5]. A direct consequence of the last Theorem relevant for the case of Families of anti-Hurwitz polinomials is given in the following Corollary:

Corollary 2. Suppose $\{f(\lambda,X)\}_{\lambda\in[a,b]}$ is a family of real polinomials in the variable X which depends continuously on the $\lambda \in [a,b]$ and that the family is of degree constant. If $p(a,X)$ has all its roots on $\mathbb{C}^+$ and $p(b,X)$ has at least one root on $\mathbb{C}^-$, then there exist $\rho \in (a,b]$ such that

1) $p(\rho,X)$ has all its roots in $\mathbb{C}^- \cup i\mathbb{R}$.
2) $p(\rho,X)$ has at least one root in $i\mathbb{R}$.

Theorem 9. Let $\{f_n(X)\}_{n\in\mathbb{N}}$ be a sequence of anti-Hurwitz polynomials of degree least or equal to N such that $f_n(X) \to q(X)$. Then, the roots of $q(X)$ remain in $\mathbb{C}^+ \cup i\mathbb{R}$.

Proof. We consider the polynomials $q(X) = a_0 + a_1 X + \cdots + a_n X^N$, and $f_m(X) = a_0^m + a_1^m X + \cdots + a_N^m X^N$. We suppose that $q(X)$ has a root $\bar{X} \in \mathbb{C}^-$. We know that there is a circle C with center $\bar{X}$ such that $C \subset \mathbb{C}^-$. Then, by Theorem 7 there is $\varepsilon > 0$ such that if $|\varepsilon_i| < \varepsilon$, for all $i = 0,1,\cdots,N$, then

$$p(X) = (b_0 + \varepsilon_0) + (b_1 + \varepsilon_1)X + \cdots + (b_N + \varepsilon_N)X^N$$

has at least one root inside of C. How $f_n(X) \to q(X)$, then there is $f_k(X) = a_0^k + a_1^k X + \cdots + a_N^k X^N$ such that $|b_0 - a_0^k|, |b_1 - a_1^k|, \cdots, |b_N - a_N^k| < \varepsilon$. Then, the following polynomial

$$f_k(X) = b_0 + (a_0^k - b_0) + [b_1 + (a_1^k - b_1)]X + \cdots + [b_N + (a_N^k - b_N)]X^N,$$

has a root in $\mathbb{C}^-$, which is a contradiction with the fact that $\{f_n(X)\}_{n\in\mathbb{N}}$ is a sequence of anti-Hurwitz polynomials.

Theorem 10 (Zero exclusion principle). Let $\mathcal{AH}$ the set of the real anti-Hurwitz polynomials. Suppose $\{f(\lambda,X)\}_{\lambda\in\Omega}$ is a family of real polinomials in the variable X wich depends continuously on the $\lambda \in \Omega \subset \mathbb{R}^n$, with Ω pathwise connected. Suppose moreover that the family $f(\lambda,X)$ is of degree constant and there is at least one anti-Hurwitz polynomial. Then, the family $\{f(\lambda,X)\}_{\lambda\in\Omega} \subset \mathcal{AH}$, if and only if $p(\lambda,iw) \neq 0$, for all $w \in \mathbb{R}$ and $\lambda \in \Omega$.

Proof.

- This is a direct consequence of Theorem 6.
- Let $f(\lambda,X) \in \{f(\lambda,X)\}_{\lambda\in\Omega}$ an arbitrary polynomial and $f(\lambda_0,X)$ the anti-Hurwitz polynomial on $f(\lambda_0,X) \in \{f(\lambda,X)\}_{\lambda\in\Omega}$. We conside the path $\gamma:[a,b] \to \Omega$ such that $\gamma(a) = \lambda_0$ and $\gamma(b) = \lambda$ and the subfamily: $f(\gamma(X),X) = p(\lambda,X)$. We can see that $p(a,X) = f(\lambda_0,X)$ is anti-Hurwitz. Suppose that $p(b,X) = f(\lambda_0,X)$ does not an anti-Hurwitz

polynomial, and hence has a root in $\mathbb{C}^- \cup i\mathbb{R}$. If $p(b,iw) = f(\lambda_0, iw) = 0$ is a contradiction. If $p(b,X)$ has a root in $\mathbb{C}^-$, then by Theorem 8 there is $\rho \in (a,b]$ such that

1) $p(\rho, X)$ has all its roots in $\mathbb{C}^- \cup i\mathbb{R}$.

2) $p(\rho, X)$ has at least one root in $i\mathbb{R}$.

By 2) there is w_0 such that $p(\rho, iw_0) = f(\gamma(\rho), iw_0) = 0$, but this is a contradiction. Therefore $p(b,X) = f(\lambda, X)$ is anti-Hurwitz for all $\lambda \in \Omega$.

Given a real polinomial $p(X) = a_0 + a_1 X + \cdots + a_n X^n$, we note that

$$p(X) = (a_0 + a_2 X^2 + a_4 X^4 + \cdots) + X(a_1 + a_3 X^2 + a_5 X^4 + \cdots).$$

By evaluate iw, we obtain

$$p(iw) = (a_0 - a_2 w^2 + a_4 w^4 + \cdots) + iw(a_1 - a_3 w^2 - a_5 X^4 + \cdots).$$

Considering this, we consider the following notations

- $p^e(X) = a_0 - a_2 X^2 + a_4 X^4 + \cdots$.
- $p^o(X) = a_1 - a_3 X^2 - a_5 X^4 + \cdots$.
- $p^{even}(X) = a_0 + a_2 X^2 + a_4 X^4 + \cdots$.
- $p^{odd}(X) = a_1 + a_3 X^2 + a_5 X^4 + \cdots$.

Definition 4.2. A real polynomial $p(X) = a_0 + a_1 X + \cdots + a_n X^n$ satisfies the anti-alternancy property if and only if

1) The principal coeffients of $p^{even}(X)$ and $p^{odd}(X)$ has different sign.

2) All the roots of $p^e(X)$ and $p^o(X)$ are reals and its negatives roots are interspersed, *i.e.*

$$0 > X_{e,1} > X_{o,1} > X_{e,2} > X_{o,2} > \cdots.$$

Theorem 11 (Dual version of the Theorem of Hermite-Biehler). A real polinomial $p(X)$ is Anti-Hurwitz, if and only if satisfies the anti-alternancy property.

Proof. By Theorem 6 that the phase of $p(iw)$ strictly decreases for $w \in \mathbb{R}$ from $\frac{n\pi}{2}$ to $-\frac{n\pi}{2}$ and the change in the phase is $2m\pi$, which is equivalent to m turns in $w \in (-\infty, +\infty)$, or $m/2$ turns on $w \in (-\infty, 0)$. We note that for $w \in (-\infty, 0)$ the roots of $p^e(w)$ and $p^o(w)$ must be ordering in the following manner:

$$0 > X_{e,1} > X_{o,1} > X_{e,2} > X_{o,2} > \cdots > X_{o,m-1} > X_{e,m}. \tag{4.3}$$

In fact, in every turn it goes through by two roots of $p^e(w)$, and by two roots of $p^o(w)$. Then, in $m/2$ turns it goes through by m roots of of $p^e(w)$ and by m roots of $p^o(w)$. We note that, everyone is real and negative, and then, we obtain part (2) of property of anti-alternancy. For the converse, assume that $p(X)$ satisfies the anti-intelacing property, and suposes without loss of generality of p is of degree $n = 2m$ and that the coefficient a_{2m} is positive. Let us consider the roots of $p^{even}(X)$ and $P^{odd}(X)$ in the form

$$0 > X^p_{R,1} > X^p_{L,1} > X^p_{R,2} > X^p_{L,2} > \cdots > X^p_{L,m-1} > X^p_{R,m}. \tag{4.4}$$

Now, let us consider a polynomial $q(X) = q_0 + \cdots + q_{2m}X^{2m}$ that is known to be anti-Hurwitz, of the same degree $2m$, and with its leader coefficients positive. With this assumption on $q(X)$, we know from the first part of that $q(X)$ satisfies the anti-interlacing Theorem so that $q^{even}(X)$ has m negatives roots and q^{odd} has $m-1$ negative roots, both set of roots such that

$$0 > X^q_{R,1} > X^q_{L,1} > X^q_{R,2} > X^q_{L,2} > \cdots > X^q_{L,m-1} > X^q_{R,m}. \tag{4.5}$$

We note that for $q(iw)$, it has no imaginary roots, then for any $w \in \mathbb{R}$, $f(iw) \neq 0$. By taking $\lambda \in (0,1)$, we have

$$0 > \lambda X^p_{e,1} + (1-\lambda) X^q_{e,1} > \lambda X^p_{o,1} + (1-\lambda) X^q_{o,1} > \cdots > \lambda X^p_{e,m-1} + (1-\lambda) X^q_{e,m}.$$

Consider now the polynomial $p_\lambda(X)$ given by

$$p^e_\lambda(X) := \left((1-\lambda) q_{2m} + \lambda p_{2m}\right) \prod_{i=1}^{m} \left(X^2 - \left((1-\lambda) X^q_{e,i} + \lambda X^p_{e,i}\right)^2\right),$$

$$p^o_\lambda(X) := \left((1-\lambda) q_{2m-1} + \lambda p_{2m-1}\right) \prod_{i=1}^{m-1} \left(X^2 - \left((1-\lambda) X^q_{o,i} + \lambda X^p_{o,i}\right)^2\right).$$

We can see that the coefficients of p_λ are a family of polynomial functions in λ, wich are continuous on $[0,1]$. Moreover, the coefficient of the leader degree term of $p_\lambda(X)$ remains positive as $\lambda \in [0,1]$. Moreover, we note that for $\lambda = 0$, we have $p_0(X) = q(X)$. Then, how q is an anti-Hurwitz polynomial. This implies that the family $p_\lambda(X)$ has al least an element that is anti-Hurwitz. Hence, by the principle of exclusion of zero all the elements of the family are anti-Hurwitz polynomials, in particular $p_1(X) = p(X)$.

5. Applications

In this section we consider a family of reals polynomials called *viscous polynomials* introduced by Vergara-Hermosilla *et al.* in [1].

$$\begin{aligned} P_T(\lambda) = {} & \left(1 + \frac{l^3}{12}\right)\lambda^4 + l^2\sqrt{\mu}\lambda^3 + \left(l\mu - \frac{2}{\mu}\left(1 + \frac{l^3}{12}\right)\right)\lambda^2 \\ & - \frac{l^2}{\sqrt{\mu}}\lambda + \frac{1}{\mu^2}\left(1 + \frac{l^3}{12}\right), \end{aligned} \tag{5.1}$$

where l and μ are free parameters in $\mathbb{R}^+$. The viscous polynomials arises naturally when considering the transfer function of a system that models the vertical movement of a solid floating in a viscous fluid, studied by Vergara-Hermosilla *et al.* in [9] and [1], in fact, the name of the family of polynomials is originally due to the fact that the parameters l and μ represent a measure associated with the size of the floating structure and the viscosity coefficient, respectively. Our objective in this section is to use the criteria developed in Section 2 to obtain information on the location of the roots.

To this end, we can check easily that:

1) When dividing $P_T(\lambda)$ by the coefficient of the term with exponent 4, we obtain the equivalent polynomial.

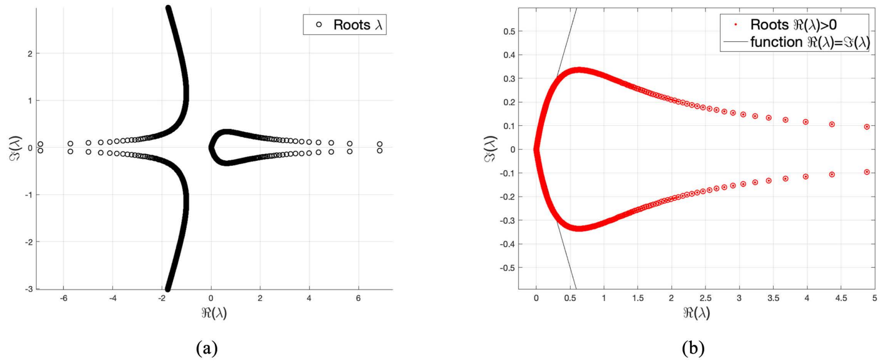

Figure 1. Evolution of the four roots λ_i in the complex plane, as a function of μ. (a): global picture with 4 trajectories; (b): zoom in the right-half plane $\mathrm{Re}(\sigma)>0$, here 2 trajectories are crossing the segment $\mathrm{Re}(\lambda)=|\Im(\lambda)|$ for a critical value of μ referent to the viscosity coefficient.

$$Q_T(\lambda)=\lambda^4+\frac{l^2\sqrt{\mu}}{\left(1+\frac{l^3}{12}\right)}\lambda^3+\frac{\left(l\mu-\frac{2}{\mu}\left(1+\frac{l^3}{12}\right)\right)}{\left(1+\frac{l^3}{12}\right)}\lambda^2-\frac{l^2}{\sqrt{\mu}\left(1+\frac{l^3}{12}\right)}\lambda+\frac{1}{\mu^2}. \quad (5.2)$$

In this polynomial we can see that the coefficients of the terms with exponents 3 and 2 have the same sign, by considering the criteria developed in Section 3, we can conclude that the viscous polynomial is not anti-Hurwitz.

2) In a similar form, we can see that the there are coefficients in $P_T(\lambda)$ with different sign, then using the Stodola condition give in Theorem 1, we conclude that the viscous polynomial is not Hurwitz.

In conclusion, due to the polynomial $P_T(\lambda)$ have degree 4, is not Hurwitz and is not anti-Hurwitz, we will always have two roots in the right complex half plane and two roots in the left complex half plane. In fact, in the **Figure 1** we can see numerical evidence about the behavior of the roots of the viscous polynomial with suitable parameters.

6. Conclusions

In this paper we present simple explicit criteria for determining the classification of the roots of real polynomials in one variable in the right half complex plane. These results appear as natural extensions of the classical theory of Hurwitz polynomials over the family of anti-Hurwitz polynomials introduced in [1]. More precisely, the results introduced in this work follow an implicit use of the linear transformation $z\to -z$ into the properties that define the theory of Hurwitz polynomials, and define our notion of duality. Considering this, we can summarize our contribution in two important results: A dual version of the Theorem of Routh-Hurwitz and a version dual of the Boundary Crossing Theorems. These ideas are applied on a family of polynomials associated to a system that describes the vertical movement of a solid floating in a viscous fluid, called viscous polynomials.

In a subsequent work we will extend the ideas developed in this paper in order to explore the classification of roots of real polynomials on subregions of the complex plane limited for the intersection of finite number of graphs of convex functions.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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