

Nonlinear Oscillations of a Pair of Charged Rings

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Abstract

Two electrically charged rings of different sizes are assembled along their common vertical symmetry axis through their centers. The bottom ring is secured on a horizontal support while the top one is loose. For a set of practical values characterizing the charged rings we envision a scenario where the mutual electric repulsion between the rings and the weight of the top ring results in stable nonlinear oscillations. To quantify the characteristics of the oscillations, we utilize a Computer Algebra System specifically *Mathematica* [1]. We accompany the analysis with a simulation for a comprehensive visual understanding.

Keywords

Nonlinear Oscillations, Charged Ring-Ring Electrostatic Interaction, Computer Algebra System, *Mathematica*

1. Introduction, Motivations and Goals

Reviewing and instructing off the popular undergraduate [2] [3] and graduate college physics texts [4] reveal the incompleteness of the topics covered in many areas of physics such as issues in electrostatic. The incompleteness of the topics provides opportunities to augment the domain of interest contributing to enriching physics. The same comment is true for popular mathematical physics texts e.g. [5]. Here in this investigating research article as a sample example we show how we have extended a topic that is hardly discussed in both references [2] [3] and how we have intertwined the analysis in the spirit of a mathematical physics issue [5] with a Computer Algebra System (CAS), *Mathematica* [1].

The project as briefly outlined in the abstract concerns with the oscillations of a charged ring that is placed horizontally floating above another charged ring clamped on a horizontal table along their common vertical symmetry axis.

Charging both rings positively (negatively) creates repulsive electrostatic force. The floating ring being massive gets pulled down by its weight. Adjusting the relevant physical parameters makes it feasible to balance these two forces making the latter motionless yielding to the equilibrium along the vertical axis, and therefore, intuitively speaking dropping or pushing the floating charged ring above or below the equilibrium in the absence of non-conservative agents such as air friction results in stable oscillations. One would also speculate because of the nature of the charged ring-ring interaction, the oscillation would be nonlinear. To quantify the oscillations conducive to the relevant quantities such as: frequency, period, and amplitude, one needs to establish the equation of motion. The latter requires a methodology, meaning either the mutual interactive force between the rings are to be considered or from energy point of view of their electrostatic potential energy. The former requires calculation of the electric field of the bottom ring at off axis region where the floating ring is at, where the latter calls for evaluation of associated electrostatic potential. Contemplating these two we favored the latter. Information cited in [2] [3] [4] is confined to issues such as electrostatic potential of a charged ring along its symmetry axis, so the challenge is to formulate the needed quantities of the symmetry axis. For clarification of the issues on hand **Figure 1** assists depicting the scenario.

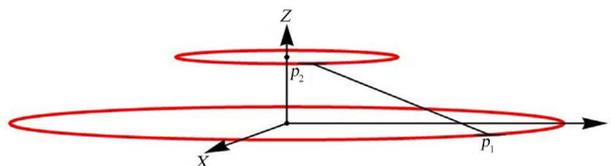


Figure 1. Depicts two parallel charged rings of different sizes. The bottom ring is clamped on a horizontal support while the upper one is placed above it along the mutual symmetry z -axis.

2. Mathematical Analysis—*Mathematica* Codes—Simulation

As shown in **Figure 1** the 3D slanted line connects virtually a differential charge length located at point p_1 on the lower ring to a similarly characterized differential charged length on the upper ring, p_2 . Intuitively speaking any physical quantity such as electrostatic potential energy comes about from summing a distance dependent expression between these two points, $\overline{p_1 p_2}$. And because the charges are distributed on each ring, the latter procedure for a chosen separation distance between the rings requires double integration over azimuthal angles of each ring. Assuming analytically this goal is achievable the output of the integration being a function of separation distance is to be differentiated with respect to the separation distance result the interactive force between the rings.

The distance $\overline{p_1 p_2}$ is

$$\overline{p_1 p_2} = \sqrt{R^2 + r^2 + z^2 - 2Rr \cos[\alpha - \beta]}, \quad (1)$$

here, R , and r are the radii of the lower and upper circles: α , and β are their azimuthal angles, respectively. The physical quantity of interest is the electrostatic

potential energy,

$$U(z) = \frac{1}{(2\pi)^2} k q_1 q_2 \int_0^{2\pi} \int_0^{2\pi} \frac{1}{p_1 p_2} d\alpha d\beta. \tag{2}$$

here, q_1 and q_2 are the charges of the lower and the upper rings, respectively. The k is the electrostatic coupling constant, $k = \frac{1}{4\pi\epsilon_0}$ in metric units it is 8.99×10^9 Nm²/C².

For a chosen separation height z , integrating (2) applying *Mathematica* [1] result a complicated almost useless CPU expensive expression, the same is true applying Maple [6]. We avoid including them as they clutter the text. Realizing this unexpected hurdle we devise our own practical approach that also may be applied to similar scenarios. In the next paragraph the procedure is outlined in seven sequential steps.

- 1) Select values for radii of the hoops and other relevant parameters.
- 2) Select a range for height z , $0 \leq z \leq \infty$.
- 3) For the list of chosen z 's subject to step 2 integrate (2). The output is a list of paired $\{z, \text{integrated (2)}\}$.
- 4) Plot the paired list of step 3 and fit the data with a physically meaningful z -dependent analytic function.
- 5) Utilize the fitted function in step 5, apply the fundamental relationship, $\vec{F} = -\hat{k} \left[\frac{d}{dz} U(z) \right]$ conducive the interactive force.
- 6) Apply result of step 5, set up the equation of motion.
- 7) Solve the equation of motion. Utilize the solution plot the $z(t)$.

Foreseeing the ultimate goal is to form the equation of motion of the upper ring, among the relevant physical parameters the **values** includes the mass of the floating ring m . The radii of the rings, their respective charges, and the constants such as k and g are included as well; units are metric.

$$\text{values} = \left\{ k \rightarrow 9 \times 10^9, q_1 \rightarrow 2 \times 10^{-6}, q_2 \rightarrow 2 \times 10^{-6}, R \rightarrow 20 \times 10^{-2}, \right. \\ \left. r \rightarrow 10 \times 10^{-2}, m \rightarrow 1 \times 10^{-3}, g \rightarrow 9.8 \right\};$$

Compatible with the radii of the rings we consider $z_\infty \rightarrow 10$. According to step 3, the numeric double integration of (2) yields,

$$\text{table Integrations} \\ = \text{Table} \left[\left\{ z, \text{factor } N \text{ Integrate} \left[\left(\sqrt{R^2 + r^2 + z^2 - 2Rr \cos[\alpha - \beta]} \right)^{-1} \right. \right. \right. \\ \left. \left. \left. / .\text{values}, \{ \alpha, 0, 2\pi \}, \{ \beta, 0, 2\pi \} \right] \right\}, \{ z, 0, 10, 0.5 \} \right];$$

It is assuring to note in the limit of $r \rightarrow 0$ i.e. by shrinking the small ring into a point-like charge the integrand and consequently the integration converges to the well-known result of potential energy of a charged ring and a point-like charge.

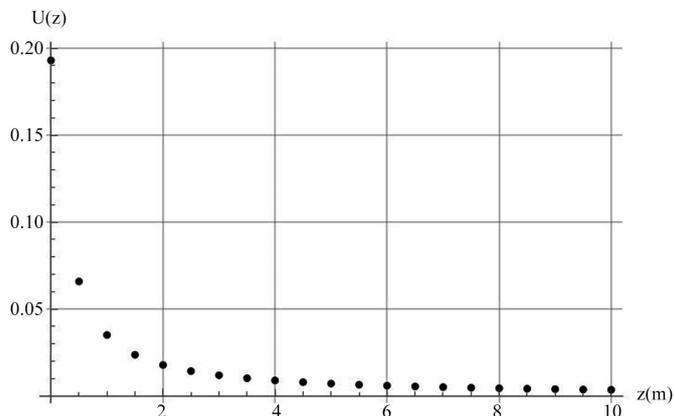


Figure 2. Display of the paired data, {z, table Integrations}.

The look of **Figure 2** suggests proposing a modified exponential z -dependent function fitting the data—we model the function,

$$\text{model } z[z_]= a + bz + ce^{-dz};$$

$$\text{fit } z = \text{Find Fit}[\text{table Integrations}, \text{model } z[z], \{a, b, c, d\}, z];$$

$$\text{plotfit } z = \text{Plot}[\text{model } z[z] / .\text{fit } z, \{z, 0, 10\}, \text{PlotStyle} \rightarrow \text{Red}, \text{PlotRange} \rightarrow \text{All}];$$

Its plot along with the fitted paired data is displayed in **Figure 3**.

$$\text{Show}[\text{list Plot table Integrations}, \text{plotfit } z]$$

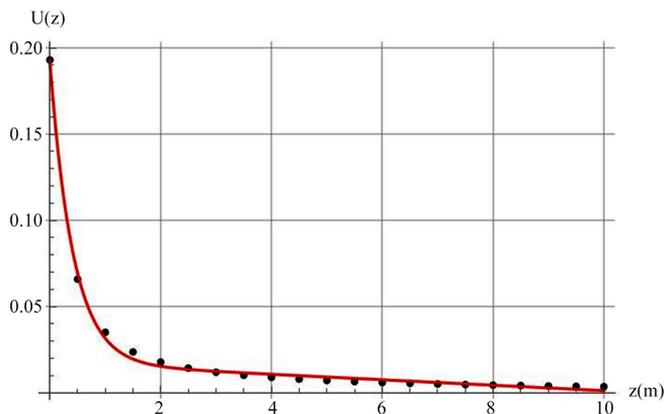


Figure 3. The black dots are the data, the red curve is the fitted analytic function.

As shown in **Figure 3**, the fitted function is satisfactory, it passes through the data points. Its analytic functional form assists to fulfill the aimed goal, namely the electrostatic force. According to procedural step 5 this is,

$$\text{Force } z = -D[\text{model } z[z] / .\text{fit } z, \{z, 1\}];$$

$$\text{Plot}[\text{Force } z, \{z, 0, 5\}, \text{AxesLabel} \rightarrow \{“z(m)”, “F(N)”\},$$

$$\text{PlotStyle} \rightarrow \text{Black}, \text{GridLines} \rightarrow \text{Automatic}]$$

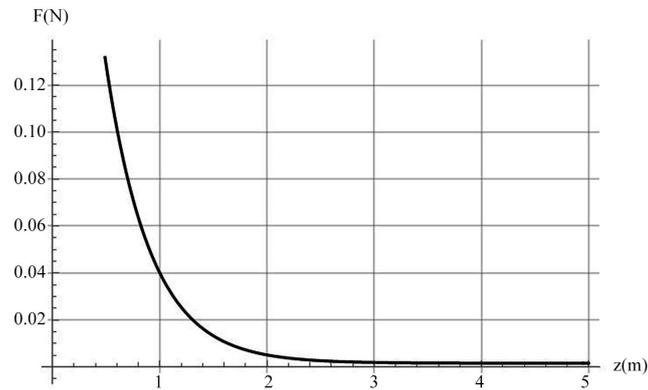


Figure 4. Display of the electrostatic mutual force between the rings.

As displayed, for small separation distances the force is strong and as the separation distance increases the force is weaker. It also shows the force is not a linear function of separation distance. Hence, nonlinear oscillations should be expected. More on this in forthcoming paragraphs.

Here we search for the separation equilibrium distance. Knowing this distance assists to determine how far up (down) the upper ring needs to be dropped (dropped) results oscillations about the equilibrium. To this end we solve the static equation setting the net force $F_{\text{electrostatic}} - mg = 0$.

$$\text{Solve}[\text{Force } z == mg /. \text{values}, z] \\ \{z \rightarrow 1.64658\}$$

And then according to step 6 we form the equation of motion. Noting the force according to **Figure 4** is non-linear we expect a non-linear oscillation. The solution of the equation of motion is achieved applying numeric solution of the differential equation with appropriately chosen initial conditions with the insight gained via equilibrium z -height. Irrespective to the initial height, we consider scenarios where the upper ring is dropped freely, *i.e.* with initial zero velocity. Of course, if interested this condition easily may be altered resulting different physical outcomes.

$$\text{Equation Of Motion} = z''[t] - (\text{Force } z /. z \rightarrow z[t]) \\ \left(\frac{1}{m} /. \text{values} \right) + g /. \text{values};$$

sol Equation Of Motion

$$= \text{NDSolve}[\{\text{Equation Of Motion} == 0., z[0] == 4., z'[0] == 0\}, z[t], \{t, 0, 10\}];$$

We confirm our physical intuition, *i.e.* the upper ring dropped from a reasonable height above the equilibrium results oscillations.

plot ring Oscillations

$$= \text{Plot}[\text{Evaluate}[z[t] /. \text{sol Equation Of Motion}], \{t, 0, 10\}, \\ \text{AxesLabel} \rightarrow \{“t(s)”, “z(m)”\}, \text{PlotStyle} \rightarrow \text{Black}, \\ \text{GridLines} \rightarrow \text{Automatic}, \text{PlotRange} \rightarrow \{0, 5\}]$$

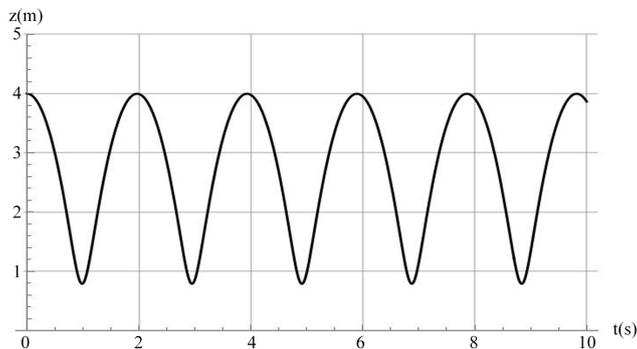


Figure 5. Position of the upper ring, $z(t)$ vs. time. The ring oscillates almost about the equilibrium position.

Figure 5 shows the ring oscillates almost about the equilibrium; the nonlinearity of the force prevents its perfect oscillation about the equilibrium. Utilizing **Figure 5** oscillation properties such as frequency, period and amplitude may be read off.

We also craft a code simulating the oscillating ring. Simulation of the oscillations along with the display of the $z(t)$ helps to form a visual understanding of the problem at hand, this is shown in screen shot of the situation in **Figure 6**.



Figure 6. Screen shot of the simulation of the oscillations of the floating ring. The left panel is the actual oscillator, the right panel is the oscillating position of the ring vs. time.

As shown in **Figure 6** the freely dropped ring from the initial height at $z = 4$ m oscillates along the z -axis. The impact of the nonlinearity of the interactive force makes the oscillations somewhat different from the counterpart of linear force as it doesn't symmetrically oscillate about the equilibrium position. It is interesting to realize the oscillations are confined to the upper side of the lower ring. By trial and error, we realized by increasing the initial height the ring may come very close to the bottom one. However, if the oscillations are the goal this puts a limit to the maximum initial height. Exceeding the maximum height, the ring passes through the bottom one. For such extreme scenarios since underneath the lower ring the repulsive electrostatic force and the gravity force are additive the ring continues falling without a possibility of return, consequently no oscillation occurs.

3. Discussion and Conclusion

Critically looking over and instructing off the material published in most of the

undergraduate and graduate popular physics as well as mathematical physics texts reveal the incompleteness of most of the presented topics. Although online resources are complementary nonetheless the topic of the interest discussed in this exploratory investigation has not been reported in scientific literature. As described in the abstract and the body of the article technically speaking evaluation of the electric field created by a charged loop in the region of space off the symmetry axis brings about mathematical challenges. Interacting charge distributions on two different spatially separated entities calls for a double integration. Despite the powerful computational algorithms built in *Mathematica* and Maple their symbolic integrated double integration is complicated and almost useless. In this article we have devised a technical approach overcoming this issue. Accordingly, we have been successful evaluating the double integration numerically converting the output to an analytic continuous useful function. Interested readers may find our newly proposed approach applicable and useful for similar scenarios. *Mathematica* codes are imbedded in the article so that users familiar with this software [7] readily may produce our result. Utilizing the result of our calculation we quantify our physical intuitive foreseen oscillations. For a comprehensive understanding we supplement the numerical solution of the nonlinear differential equation characterizing the equation of motion simulating the oscillations. Interested readers will find [8] resourceful crafting the simulation as well as figures in this article. Characteristics of the oscillations such as: frequency, period and amplitude may be read of the depicted simulation.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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