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A Neighborhood Condition for Graphs to Have Special [a,b]-Factor

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ABSTRACT

Let G be a graph of order n, and let a and b be integers, such that $1 \le a < b$. Let H be a subgraph of G with $m(\le b)$ edges, and $\delta(G)$ be the minimum degree. We prove that G has a $[a,b]$-factor containing all edges of H if $\delta(G) \ge a+m$, $NC(G) \ge \frac{an+2m}{a+b}$, and when $a \le 2$, $n \ge \frac{2(a+b)(a+b-1)}{b} - \frac{a+b}{b(a-1)} + \frac{2m}{b}$.

KEYWORDS

Graph; Factor; [*a*,*b*]-Factor; The Minimum Degree; Neighborhood Condition

1. Introduction

We consider the finite undirected graph without loops and multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Given $x \in V(G)$, the set of vertices adjacent to x is said to be the neighborhood of x, denoted by $N_G(x)$. $d_G(x) = |N_G(x)|$ is called the degree of x and we write $N_G[x]$ for $N_G(x) \cup \{x\}$. Furthermore we define $\delta(G) = \min\{d_G(x) | x \in V(G)\}$, $NC(G) = \min_{xy \notin E(G)} \{|N_G(x) \cup N_G(y)|\}$. For a subset $S \subset V(G)$, let $G - S$ denote the subgraph obtained from G by deleting all the vertices of S together with the edges incident with the vertices of S.

Let a and b be integers such that $1 \le a < b$. A $[a,b]$-factor of G is defined as a spanning subgraph F of G such that $a \le d_F(x) \le b$ for all $x \in V(G)$. Other notations and terminology are the same as those in [1]

The existence of a factor for a graph G is closely related to the degree of vertices. Concerning the minimum degree and the existence of *k*-factor Egawa, Enomoto [2] and Katerinis [3] proved that there exists *k* factor when $n \ge 4k-5$ and $\delta(G) \ge \frac{n}{2}$ for a graph G. Iida and Nishimura [4] proved that if $n \ge 4k-5$ and $\sigma_2(G) \ge n$ there exists *k*-factor for a graph G.

H. Y. Pan [5] generalized the result of Iida and Nishimura to [*a*, *b*] -factor: if $\delta(G) \ge a, n \ge \frac{(a+b)^2 - (a+b)}{b}$ and $\sigma_2(G) \ge \frac{2an}{a+b}$, G has an $[a,b]$-factor.

Concerning adjacent set union and $[a,b]$-factor, in 2000 H. Matsuda gave the following result:

Theorem 1 [5]: Let a, b be integer such that $1 \le a < b$, and G be a graph of order n with $n \ge \frac{2(a+b)(a+b-1)}{b}$ and $\delta(G) \ge a$.

If $|N_G(x) \cup N_G(y)| \ge \frac{an}{a+b}$ for any two non-adjacent vertices x and y of G, then G has a $[a,b]$-factor We prove the following theorem for a graph to have a $[a,b]$-factor with given properties, which is an extension of theorem 1.

Theorem 2: Let a and b be integers such that $1 \le a < b$, G be a graph of order n, and H be a subgraph of G with $m(\le b)$ edges. If $\delta(G) \ge a+m$, and $|N_G(x) \cup N_G(y)| \ge \frac{an+2m}{a+b}$ for any two non-adjacent vertices x and y of G, when $a \ge 2$, we suppose

$$n \ge \frac{2(a+b)(a+b-1)}{b} - \frac{a+b}{b(a-1)} + \frac{2m}{b}.$$

Then G has a $[a,b]$-factor containing all edges of H.

2. Proof of Theorem 2

Let S and T be two disjoint subset of $V(G)$, E_1 and E_2 be two disjoint subset of $E(G)$. Let $W = V(G) \setminus (S \cup T)$, $E(S) = \{xy \in E(G) : x, y \in S\}$, $E(S,T) = \{xy | xy \in E(G), x \in S, y \in T\}$, $E(T) = \{xy \in E(G) : x, y \in T\}$.

$E_1' = \{xy \in E_1; x, y \in S\}$, $E_1'' = \{xy \in E_1; x \in S, y \in V(G) \setminus (S \cup T)\}$

$$E_2' = \{xy \in E_2 : x, y \in T\}, \quad E_2'' = \{xy \in E_2 : x \in T, y \in V(G) \setminus (S \cup T)\}$$

$\alpha_G(S,T;E_1,E_2) = 2|E_1'| + |E_1'|$, $\beta_G(S,T;E_1,E_2) = 2|E_2'| + |E_2''|$.

Lemma 1 [6]: Let G be a graph, and let g and f be two integer-valued functions defined on $V(G)$ such that $0 \le g(x) < f(x) \le d_G(x)$ for all $x \in V(G)$. Let E_1 and E_2 be two disjoint subsets of $E(G)$. Then G has a (g,f)-factor F such that $E_1 \subseteq E(F)$ and $E_2 \cap E(F) = \Phi$ if and only if for any two disjoint subsets S and T of $V(G)$.

$$\delta_G(S,T;g,f) = d_{G-S}(T) - g(T) + f(S) \ge \alpha_G(S,T;E_1,E_2) + \beta_G(S,T;E_1,E_2).$$

Lemma 2: Let a and b be integers such that $1 \le a < b$, and G be a graph, and H be a subgraph of G. Then G has a $[a,b]$-factor F such that $E(H) \subseteq E(F)$ if and only if

$$b|S| - a|T| + d_{G-S}(T) \ge \sum_{x \in s} d_H(x) - e_H(S,T).$$

Let $E_1 = E(H)$ and $E_2 = \Phi$, and we note that

$$\alpha(S,T;E_1,E_2) = 2|E_1'| + |E_1''| = \underset{x \in S}{d_H}(x) - |E_H(S,T)|$$

where $E_H(S,T) = \{xy \in E(H) : x \in H, y \in T\}$ and $\beta(S,T;E_1,E_2) = 2|E_2'| + |E_2''| = 0$.

It is easy to see Lemma 2 is an immediately result of Lemma 1.

Now we prove Theorem 2: Suppose that G satisfies the assumptions of Theorem 2, but it has no $[a,b]$-factor as described in Theorem 2. Then by Lemma 2 there exist two disjoint subsets S and T of $V(G)$ such that

$$b|S| - a|T| + d_{G-S}(T) - \sum_{x \in s} d_H(x) + e_H(S,T) + 1 \le 0. \tag{1}$$

We choose such S and T so that $|T|$ is minimum. If $T = \Phi$, then by (1) we get $b|S| - \sum_{x \in s} d_H(x) + 1 \le 0$, which is a contradiction. Since $d_H(x) \le m \le b$ for all $x \in H$, hence we have $T \ne \Phi$

Suppose that there exists a vertex $\omega \in T$ such that $d_{G-S}(\omega) + e_H(S,\omega) \ge a$, then S and $T - \{\omega\}$ satisfy (1), which contradicts the choice of T, therefore $d_{G-S}(x) + e_H(S,x) \le a-1$ for all $x \in T$.

Now we define

$$h_1 = \min\{d_{G-S}(x) + e_H(S,x) : x \in T\},$$

and let $x_1 \in T$ be a vertex such that

$$h_1 = d_{G-S}(x_1) + e_H(S,x_1).$$

Note that $h_1 \le a-1$ holds, we consider two cases.

Cases 1: $T = N_T[x_1]$

Note that $|S| + h_1 \ge d_G(x_1) \ge \delta(G) \ge a+m$, $\sum_{x\in S} d_H(x) \le 2m$, $a > h_1$, and $b > a \ge h_1 + 1 \ge |N_T[x_1]| = |T|$.

By (1), we obtain

$$\begin{aligned} 0 &\ge b|S| + \sum_{x\in T}\left(d_{G-S}(x) + e_H(S,x)\right) - a|T| - \sum_{x\in S} d_H(x) + 1 \\ &\ge b(a+m-h_1) + (h_1 - a)|T| - 2m + 1 \ge (a-h_1)(b-|T|) + mb - 2m + 1 \ge 1. \end{aligned}$$

This is a contradiction.

Cases 2: $T \ne N_T[x_1]$

It is clear that $T \setminus N_T[x_1] \ne \Phi$, then we defined $h_2 = \min\{d_{G-S}(x) + e_H(S,x) : x \in T \setminus N_T[x_1]\}$ and let $x_2 \in T$ be a vertex such that $h_2 = d_{G-S}(x_2) + e_H(S,x_2)$ by the condition of Theorem 2, the following inequality holds:

$$\frac{an+2m}{a+b} \le |N_G(x_1) \cup N_G(x_2)| \le |S| + h_1 + h_2$$

which implies

$$|S| \ge \frac{an+2m}{a+b} - (h_1 + h_2). \tag{2}$$

Note that the number of vertices in T which satisfies the equality $d_{G-S}(x) + e_H(S,x) = h_1$ is at most $h_1 + 1$, and the rest of vertices in T satisfy $d_{G-S}(x) + e_H(S,x) \ge h_2$.

So we obtain

$$\sum_{x\in T}\left(d_{G-S}(x) + e_H(S,x)\right) \ge h_1(h_1+1) + h_2(|T| - h_1 - 1).$$

And further by (1)

$$b|S| + h_1(h_1+1) + h_2(|T| - h_1 - 1) - a|T| - \sum_{x\in S} d_H(x) + 1 \le 0.$$

Note that $|S| = n - |T| - |W|$ and $\sum_{x\in S} d_H(x) \le 2m$, so we have

$$b(n - |T| - |W|) + h_1(h_1+1) + h_2(|T| - h_1 - 1) - a|T| - 2m + 1 \le 0$$

and hence

$$|T| \ge \frac{bn + (h_1 - h_2)(h_1+1) - 2m + 1}{a+b-h_2} - \frac{b|W|}{b+a-h_2} \ge \frac{bn + (h_1 - h_2)(h_1+1) - 2m + 1}{a+b-h_2} - |W|. \tag{3}$$

By (2) (3) we have

$$n=|S|+|T|+|W|\geq\frac{an+2m}{a+b}-h_1-h_2+\frac{bn+(h_1-h_2)(h_1+1)-2m+1}{a+b-h_2}-|W|+|W|$$
$$=\frac{an+2m}{a+b}-h_1-h_2+\frac{bn+(h_1-h_2)(h_1+1)-2m+1}{a+b-h_2}.\ \square \tag{4}$$

Let $f(h_1)=\frac{an+2m}{a+b}-h_1-h_2+\frac{bn+(h_1-h_2)(h_1+1)-2m+1}{a+b-h_2}$.

Let $\frac{\mathrm{d}f}{\mathrm{d}h_1}=0$, we have $h_1=\frac{a+b-1}{2}\geq a$, Note that $h_1\leq h_2\leq a-1$, it is easy to see that $f(h_1)$ is the minimum when $h_1=h_2$.

So we have

$$n\geq\frac{an+2m}{a+b}-2h_2+\frac{bn-2m+1}{a+b-h_2}. \tag{5}$$

If $h_2=0$, by (5) we have $n\geq\frac{an+2m}{a+b}+\frac{bn-2m+1}{a+b}=n+\frac{1}{a+b}$.

This is a contradiction.

So we suppose $h_2\geq 1$, and hence $a\geq 2$. By (5) we have

$$n\leq\frac{2(a+b)(a+b-h_2)}{b}-\frac{a+b}{bh_2}+\frac{2m}{b}\leq\frac{2(a+b)(a+b-1)}{b}-\frac{a+b}{b(a-1)}+\frac{2m}{b}.$$

This is a final contradiction. Therefore theorem 2 is proved.

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