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Combined Nodal Method and Finite Volume Method for Flow in Porous Media

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Abstract

This paper describes a numerical solution for two dimensional partial differential equations modeling (or arising from) a fluid flow and transport phenomena. The diffusion equation is discretized by the Nodal methods. The saturation equation is solved by a finite volume method. We start with incompressible single-phase flow and move step-by-step to the black-oil model and compressible two phase flow. Numerical results are presented to see the performance of the method, and seem to be interesting by comparing them with other recent results.

Keywords: Saturation Equation, Nodal Methods, Finite Volume Method, Two-Phase Simulation

1. Introduction

Nodal methods have long been one of the most popular discretization techniques employed within the reactor physics community to solve multigroup diffusion problem [1,2]. A survey of these methods can be founds in [3].

The Finite volume method (FV) has been proposed initially by Durlofsky *et al.* in 1990 for the advection equations and Burgers. Other works have been introduced by J. P. Cioni, in 1995, R. Eymard *et al.* in 1997 [4] and A. Shamsai and H. R. Vosoughifar [5].

In this paper we consider the model for incompressible two-phase flow in a porous medium. We consider Nodal methods for solving the diffusion equation and a general class of explicit finite volume upwind schemes for solving purely advective transport in the absence of gravity and capillary forces. We shall employ a Newton-Raphson method to solve the implicit system.

Section 2 presents the model problem used in this paper. The discretization by Nodal methods described is in Section 3. The discretization by finite volume method for the diffusion equation described is in Section 4. Section 5 shows the discretization by finite volume method for the saturation equation. Numerical experiments carried out within the framework of this publication and their comparisons with other results are shown in Section 6.

2. Governing Equations

Here we consider incompressible two-phase flow in domain $\Omega \subset IR^2$.

The pressure equation is given by

$$\begin{cases} \operatorname{div}(U) = f & on \quad \Omega, \\ U = -K\lambda(S)\nabla P & on \quad \Omega, \\ P = \overline{P} & on \quad \Gamma^D, \\ U \cdot n = 0 & on \quad \Gamma^N. \end{cases} \tag{1}$$

We consider the saturation equation in its simplest form (neglecting gravity and capillary forces)

$$\varphi \frac{\partial S}{\partial t} + \nabla \cdot (g(S)U) = \frac{f_w}{\rho_w} \quad on \quad \Omega \times]0,T[\tag{2}$$

We shall assume that the phases are oil (o) and water (w) and that the two phases together fill the void space completely so that

$$S_w + S_O = 1 \tag{3}$$

The source term for the saturation equation becomes:

$$\frac{f_w}{\rho_w} = \max(f,0) + g(S)\min(f,0).$$

$$\frac{f_w}{\rho_w} = \max(f,0) + g(S)\min(f,0). \tag{4}$$

where $g(S)=\dfrac{S^2}{S^2+(1-S^2)}$

To close the model, we must also supply expressions for the saturation-dependent quantities. Here we use simple analytical expressions:

$$\lambda_w(S)=\frac{(S^*)^2}{\mu_w},\quad \lambda_0(S)=\frac{(1-S^*)^2}{\mu_0},\quad S^*=\frac{S-S_{wc}}{1-S_{or}-S_{wc}}$$

K is the permeability, ϕ is the Rock porosity, μ_0 and μ_w Viscosities, S_{or} is the irreducible oil saturation, S_{wc} is the connate water saturation, and the total mobility is given by $\lambda=\lambda_w+\lambda_0$.

3. Discretization with Nodal Methods

The discretizations which follow assume that Ω is a rectangular domain and employ an $L\times M$ tensor product mesh having cells $\Omega_{i,j}=(x_{i-\frac{1}{2}},x_{i+\frac{1}{2}})\times(y_{j-\frac{1}{2}},y_{j+\frac{1}{2}})$. It is convenient to denote the mesh spacing by $\Delta x_i=x_{i+\frac{1}{2}}-x_{i-\frac{1}{2}}$ and $\Delta y_j=y_{j+\frac{1}{2}}-y_{j-\frac{1}{2}}$.

Common to all nodal discretizations is the choice of cell-and edge-based unknowns. Generally, these are taken to be moments up to some specified order; hence in the lowest-order case simple averages are employed. Specifically, the cell-based unknowns are averages defined by

$$P_{i,j}=\frac{1}{\Delta x_i\Delta y_j}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}P(x,y)dxdy \qquad (5)$$

while the edge-based scalar unknowns, namely, edge averages of the scalar flux, are given by

$$P_{i+\frac{1}{2},j}=P_j(x_{i+\frac{1}{2}}),\ P_j(x)=\frac{1}{\Delta y_j}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}P(x,y)dy \qquad (6)$$

with the analogous definitions of $P_{i,j+\frac{1}{2}}$ and $P_i(y)$. Similarly, the edge-averaged currents are written as

$$U^{\pm}_{i+\frac{1}{2},j}=\lim_{x\to x^{\pm}_{i+\frac{1}{2}}}U_j(x),\ U_j(x)=\frac{1}{\Delta y_j}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\left[-(K\lambda)\frac{\partial}{\partial x}\{P(x,y)\}\right]dy$$

while $U^{\pm}_{i,j+\frac{1}{2}}$ is defined analogously.

All lowest-order members of the nodal method family, such as the nodal expansion methods (NEM) [6], the nodal integration method (NIM) [7], the nodal Green's function method (NGFM) [8] and the coarse mesh expansion methods [9] yield equivalent discretization of (1). We have chosen to present a brief discussion of the NIM. The first step in the NIM discretization consists of posing the cell-based trans-verse integrated ODEs which govern $P_j(x)$ and $P_i(y)$. To this end we assume that a homogenized diffusion coefficient $(K\lambda)_{ij}$ is defined on each cell, and transverse integrate (1) to obtain

$$-\frac{\partial}{\partial x}\left\{(K\lambda)_{ij}\frac{\partial}{\partial x}P_j(x)\right\}=-\frac{1}{\Delta y_j}\left\{U^-_{j+\frac{1}{2}}(x)-U^+_{j-\frac{1}{2}}(x)\right\}+f_j(x),\quad x_{i-\frac{1}{2}}\le x\le x_{i+\frac{1}{2}},$$

$$-\frac{\partial}{\partial y}\left\{(K\lambda)_{ij}\frac{\partial}{\partial y}P_i(y)\right\}=-\frac{1}{\Delta x_i}\left\{U^-_{i+\frac{1}{2}}(y)-U^+_{i-\frac{1}{2}}(y)\right\}+f_i(y),\quad y_{j-\frac{1}{2}}\le y\le x_{j+\frac{1}{2}},$$

where $U^{\mp}_{j\pm\frac{1}{2}}(x)$ and $U^{\mp}_{i\pm\frac{1}{2}}$ are one-sided limits of the normal currents along the edges of the cell, and f_j is defined in analogy with the transverse averaged unknowns.

Thus, we have reduced the discretization of the PDE given in (1) to that of two ODEs that are coupled through pseudo source terms. The definition of the edge averages (6) naturally yields Dirichlet boundary conditions for each cell. Moreover, for lowest-order or constant-constant NIM the pseudo source term (*i.e.* $U^{\pm}_{j\mp\frac{1}{2}}(x)$ and $U^{\pm}_{i\mp\frac{1}{2}}(y)$) are assumed to be constant along their respective cell edges. We assume that the source $f(x,y)$ is constant over each cell.

With expressions for $P_j(x)$ and $P_i(y)$ in hand we construct the discretization. First, note that two independent definitions of the cell average are possible:

$$P_{i,j}=\frac{1}{\Delta x_i}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}P_j(x)dx=\frac{1}{\Delta y_j}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}P_i(y)dy$$

Yielding 2LM equations, e.g.

$$P_{i,j}=\frac{1}{2}(P_{i+\frac{1}{2},j}+P_{i-\frac{1}{2},j})-\frac{\Delta x_i^2}{12}(K\lambda)^{-1}_{i,j}\left\{\frac{1}{\Delta y_j}\left(U^-_{i,j+\frac{1}{2}}-U^+_{i,j-\frac{1}{2}}\right)-f_{i,j}\right\}.$$

Furthermore, under the assumption of an homogenized diffusion coefficient and utilizing $U=-K\lambda(S)\nabla P$ we obtain expressions for $U^{\pm}_{i\mp\frac{1}{2},j}$, $U^{\pm}_{i,j\mp\frac{1}{2}}$, on each cell, e.g.

$$U^-_{i+\frac{1}{2},j}=-\frac{(K\lambda)_{i,j}}{\Delta x_i}(P_{i+\frac{1}{2},j}+P_{i-\frac{1}{2},j})+\frac{1}{2\Delta x_i}\left\{\frac{1}{\Delta y_j}(U^-_{i,j+\frac{1}{2}}-U^+_{i,j-\frac{1}{2}})-f_{i,j}\right\}$$

Although these comprise for equations per cell, only three of them are linearly independent, as the same balance equation arises from both $U^-_{i+\frac{1}{2},j}-U^+_{i-\frac{1}{2},j}$ and $U^-_{i,j+\frac{1}{2}}-U^+_{i,j-\frac{1}{2}}$. Imposing continuity of $J\cdot n$ yields an equation for each interior edge (*i.e.* $(L\text{-}1)M+L(M\text{-}1)$ equations) while the boundary conditions give rise to $2L+2M$ discrete boundary equations. Thus we have $7LM+L+M$ equations as many unknowns.

Although the constant-constant NIM discretization is complete at this point, it is seldom used in this form. Typically in the literature one proceeds by eliminating

the edge currents $U^{\pm}_{i\mp\frac{1}{2},j}$, $U^{\pm}_{i,j\mp\frac{1}{2}}$, followed by the trivial elimination of the averages $P_{i,j}$ to obtain the 7-point nearest neighbor hexagonally coupled stencils that govern the edge-based fluxes $P_{i+\frac{1}{2}j}$:

$$(K\lambda)_{i,j}\left(\frac{\Delta y_j}{\Delta x_i}\right)\left(P_{i+\frac{1}{2},j}-P_{i-\frac{1}{2},j}\right)-(K\lambda)_{i+1,j}\left(\frac{\Delta y_j}{\Delta x_{i+1}}\right)\left(P_{i+\frac{3}{2},j}-P_{i+\frac{1}{2},j}\right)$$
$$+\frac{3(K\lambda)_{i,j}\Delta x_i\Delta y_j}{\Delta x_i^2+\Delta y_j^2}\left\{P_{i+\frac{1}{2},j}+P_{i-\frac{1}{2},j}-P_{i,j+\frac{1}{2}}-P_{i,j-\frac{1}{2}}\right\}$$
$$+\frac{3(K\lambda)_{i+1,j}\Delta x_{i+1}\Delta y_j}{\Delta x_{i+1}^2+\Delta y_j^2}\left\{P_{i+\frac{3}{2},j}+P_{i+\frac{1}{2},j}-P_{i+1,j+\frac{1}{2}}-P_{i+1,j-\frac{1}{2}}\right\}$$
$$=\frac{1}{2}\frac{\Delta x_i\Delta y_j^3}{\Delta x_i^2+\Delta y_j^2}f_{i,j}+\frac{1}{2}\frac{\Delta x_{i+1}\Delta y_j^3}{\Delta x_{i+1}^2+\Delta y_j^2}f_{i+1,j}$$

With the y-oriented rotated analogue at $P_{i,j+\frac{1}{2}}$.

4. Finite Volume Method for the Diffusion Equation

In a finite volume method for (1) one introduces a family of control volumes (typically the given mesh) and imposes mass conservation locally for each control volume E: $\int_{\partial E}-K\lambda\nabla P.nds=\int_E fdx$, where n is the outward unit normal on ∂E. The cell-centered finite volume method, which may be expressed as a pressure stencil for each cell E_i,

$$\sum_j T_{ij}\left(P_i-P_j\right)=\int_{E_i}fdx, \tag{7}$$

where j loops over all cells neighboring cell E_i. The transmissibilities T_{ij} are associated with cell interfaces, and the expression $T_{ij}\left(P_i-P_j\right)$ is a discrete form of $-\int_{\partial E_i\cap\partial E_j}K\lambda\nabla P.n_{ij}ds$, where n_{ij} is the unit normal on $\Gamma_{ij}=\partial E_i\cap\partial E_j$ pointing into E_j. Thus, $T_{ij}\left(P_i-P_j\right)$ estimates the total flux across $\partial E_i\cap\partial E_j$. The system (7) is clearly symmetric, and a solution is, as for the continuous problem, defined up to an arbitrary constant. The system is made positive definite and symmetry is preserved, by forcing $P_1=0$, for instance. That is, by adding a positive constant to the first diagonal of the matrix $A=[a_{ik}]$,
where

$$a_{ik}=\begin{cases}\sum_j T_{ij} & if\ k=i,\\ -T_{ik} & if\ k\neq i.\end{cases} \tag{8}$$

The matrix A is sparse, consisting of a tridiagonal part corresponding to the *x*-derivative, and two off-diagonal bands corresponding to the *y*-derivatives.

5. Finite Volume Method for the Saturation Equation

For the saturation Equation (2), we use a finite volume scheme on the form

$$S_i^{n+1}=S_i^n+\left(\delta_x^t\right)_i\left(\max\left(f_i,0\right)-\sum_j g(S^m)_{ij}U_{ij}+g\left(S^m{}_i\right)\min\left(f_i,0\right)\right) \tag{9}$$

where $\left(\delta_x^t\right)_i=\frac{\Delta t}{\left(\varphi_i\left|\Omega_i\right|\right)}$, ϕ_i is the porosity in Ω_i, f_i denotes the source term, Δt is the time step, and S_i^k is the cell-average of the water saturation at time $t=t_k$,

$$S_i^k=\frac{1}{\left|\Omega_i\right|}\int_{\Omega_i}S\left(x,t_k\right)dx.$$

By specifying for time level m in the evaluation in the fractional flow functions, we obtain either an explicit ($m = n$) or implicit ($m = n + 1$) scheme. Here U_{ij} is the total flux (for oil and water) over the edge Γ_{ij} between two adjacent cells Ω_i and Ω_j, and g_{ij} is the fractional flow function at Γ_{ij}

$$g_w(S)_{ij}=\begin{cases}g_w(S_i) & if\ U.n_{ij}\geq 0,\\ g_w(S_j) & if\ U.n_{ij}<0.\end{cases} \tag{10}$$

The explicit solver is obtained by using $m = n$ in (9). Explicit schemes are only stable provided that the time step Δt satisfies a CFL condition. For the homogeneous transport equation (with $f\equiv 0$), the CFL condition for the first-order upwind scheme reads

$$\max_{S\in(0,1)}\left|g'(S)\right|\max_i\left(\delta_x^t\right)_i\sum_j\left|U_{ij}\right|\leq 2\left(1-S_{wc}-S_{or}\right) \tag{11}$$

For the inhomogeneous equation, we can derive a stability condition on Δt using a more heuristic argument.

Physically, we require that $S_{wc}\leq S_i^{n+1}\leq 1-S_{or}$.

This implies that the discretization parameters $\left|\Omega_i\right|$ and Δt must satisfy the following dynamic conditions:

$$S_{wc}\leq S_i^n+\left(\delta_x^t\right)_i\left(\max\left(f_i,0\right)-\sum_j g(S^n)_{ij}U_{ij}+g\left(S_i^n\right)\min\left(f_i,0\right)\right)\leq 1-S_{or} \tag{12}$$

The interface fluxes U_{ij} satisfy the following mass balance property:

$$U_i^{in} = \max(f_i,0) - \sum_j \min(U_{ij},0) = -\min(f_i,0) + \sum_j \max(U_{ij},0)$$
$$= U_i^{out}.$$

Thus, since $0 \le g(S) \le 1$ it follows that

$$-g(S_i^n)U_i^{in} = \max(f_i,0) - \sum_j g(S^n)_{ij}\, U_{ij} + g(S_i^n)\min(f_i,0)$$
$$\le (1-g(S_i^n))U_i^{in}$$

Then the general saturation dependent stability condition holds in Ω_i if the following inequality is satisfied:

$$\max\left(\frac{g(S_i^n)-0}{S_i^n - S_{wc}}, \frac{1-g(S_i^n)}{1-S_{or}-S_i^n}\right)\left(\delta_x^t\right)_i U_i^{in} \le 1 \quad (13)$$

Finally to remove the saturation dependence from (13), we invoke the mean value theorem and deduce that (13) holds whenever

$$\Delta t \le \frac{\varphi|\Omega_i|}{U_i^{in} \max\{g'(S)\}_{0\le S\le 1}} \quad (14)$$

6. Numerical Simulations

In this section we illustrate the performance of the Nodal methods. We restrict to quarter-five spot problems in two dimensions with uniform rectangular grids [10,11]. We compare the solutions obtained using Nodal Methods with the reference fine scale solution obtained using a two point finite volume method.

We compute a mean pressure error in the following manner:

$$\varepsilon(P) = \frac{\left\|P - P^{ref}\right\|^2}{\left\|P^{ref}\right\|^2} \quad (15)$$

where P and P^{ref} are array vectors that contain the average pressure in each fine element (Nodal Methods solution and reference finite volume solution, respectively).

We measure velocity solution errors using the following measure:

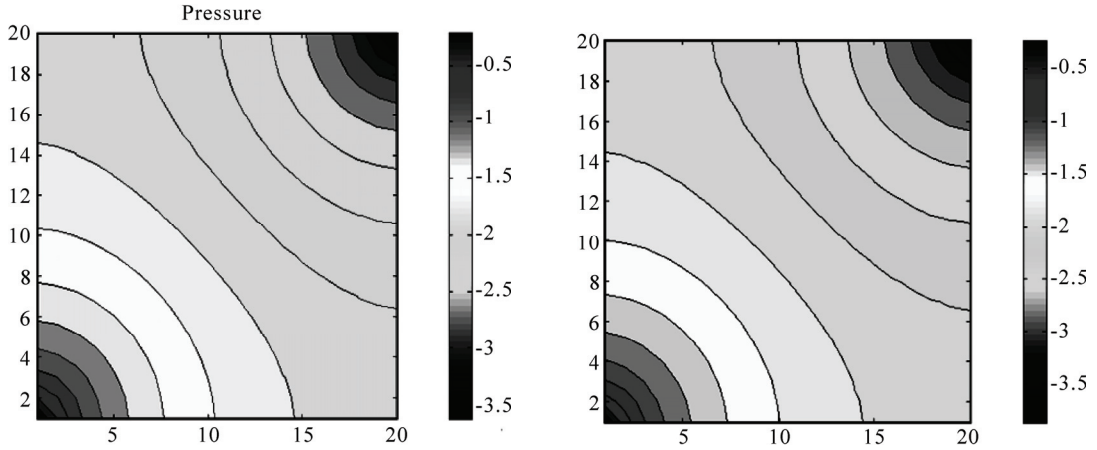
$$\varepsilon(U) = \frac{\left\|U_x - U_x^{ref}\right\|^2}{\left\|U_x^{ref}\right\|^2} + \frac{\left\|U_y - U_y^{ref}\right\|^2}{\left\|U_x^{ref}\right\|^2} \quad (16)$$

U_x and U_y are vectors containing the average velocities in the x-and y-directions, respectively, $\|.\|$ and is the Euclidean norm.

The pressure equation is discretized by the Nodal volume method (FV). We use an Explicit and Implicit upwind finite volume discretization for the saturation Equation (2).

Example: Here, we present a simulator for incompressible and immiscible Two-Phase flow system. We first want to look at the so called the quarter-five spot problems. The reservoir is $\Omega = [0,1]\times[0,1]$ with a unit injection well placed at (0; 0) and a production well with unit production rate placed at (1; 1). Let us revisit the quarter-five spot problems, but now assume that the reservoir is initially filled with pure oil. To produce the oil in the upper-right corner, we inject water in the lower left. We assume unit porosity, unit viscosities for both phases, and set $S_{or} = S_{wc} = 0$. We impose no-flow boundary conditions on $\partial\Omega$.

We obtain a similar result with J. E. Aarnes *et al.* [10]. The water saturation is increasing monotonically toward the injector, meaning that more oil is gradually displaced as more water is injected, and the pressure fields are symmetric.



Figuret 1. The left plot shows pressure contours for a homogeneous quarter-five spot obtained using Nodal Methods and the right plot shows pressure contours for a homogeneous quarter-five spot with the reference solution obtained by J. E. Aarnes *et al.* [10], with a 20 × 20 square grid.

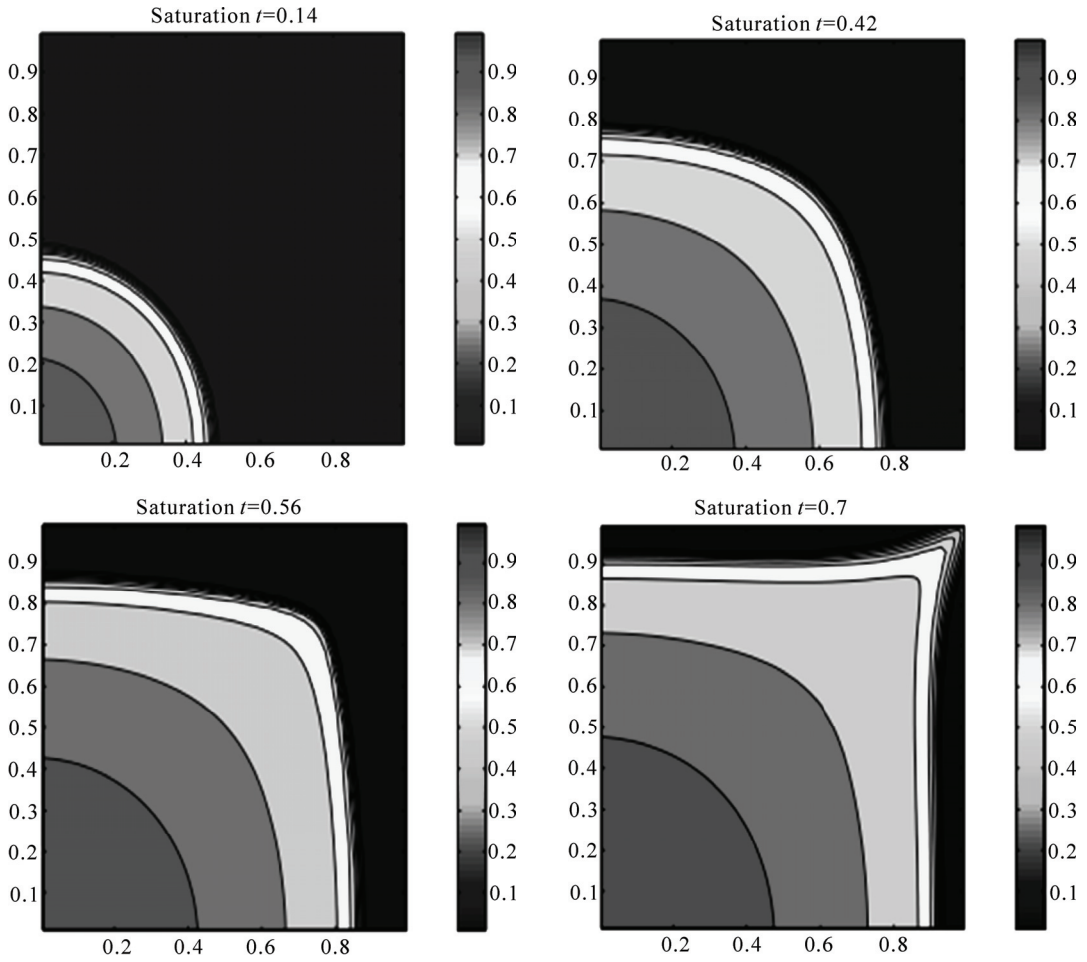


Figure 2. Saturation profiles for the homogeneous quarter-five spot at several different times: t = 0.14, t = 0.42, t = 0.56 and t = 0.7.

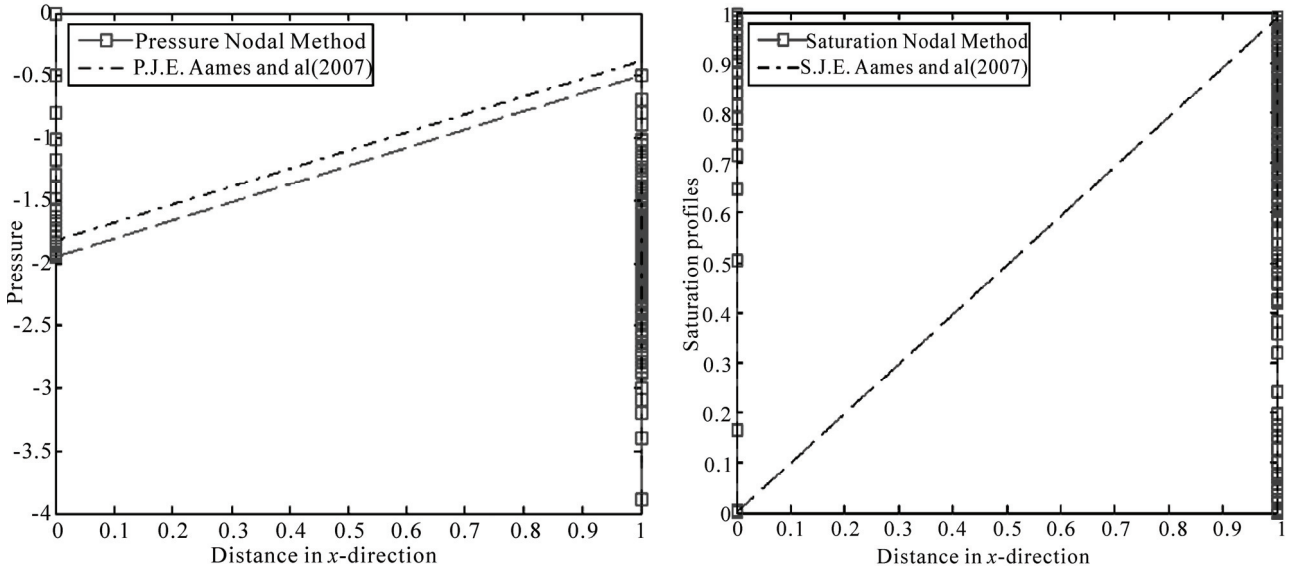


Figure 3. The left shows pressure obtained with Nodal Method and pressure obtained by J. E. Aarnes *et al.*, and the right shows saturation profiles obtained with Nodal Method and saturation profiles obtained by J. E. Aarnes *et al.*, with a 20 × 20 square grid.

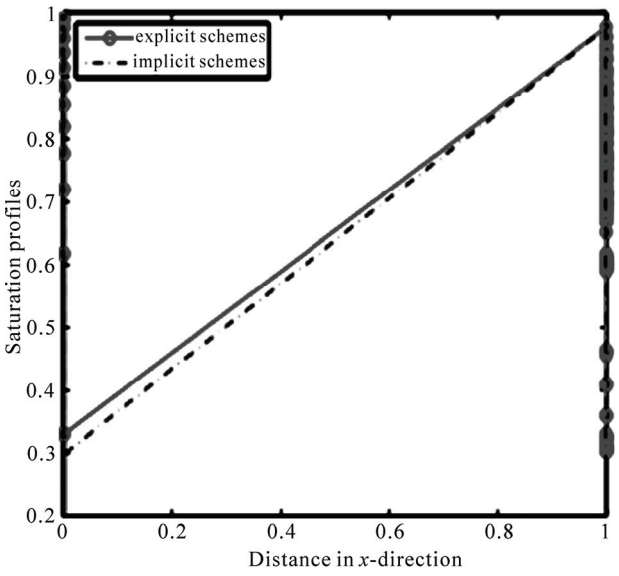


Figure 4. Saturation profiles for explicit and implicit schemes for the homogeneous quarter-five spot.

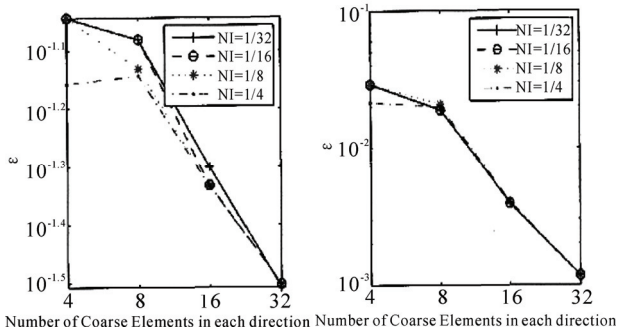


Figure 5. Pressure (left) and Velocity (Right) errors for different coarse discretizations and well length scales used to compute the fine well contributions. The error decreases with increasing numbers of coarse cells.

7. Conclusions

We were interested in this work in the numeric solution for two dimensional partial differential equations modeling a fluid flow and transport phenomena. In this article, numerical approximation of two-phase incompressible problems is studied. The diffusion equation is discretized by the Nodal Methods. The saturation equation is solved by a finite volume method. The pressure and velocity field are symmetric about both diagonals for the homogeneous field. The water saturation is increasing monotonically toward the injector for the homogeneous field, meaning that more oil is gradually displaced as more water is injected.

Numerical results are presented to see the performance of the method, and seem to be interesting by comparing them with other recent results.

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