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A Closed-Form Formulation for the Build-Up Factor and Absorbed Energy for Photons and Electrons in the Compton Energy Range in Cartesian Geometry

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ABSTRACT

In this work, we report on a closed-form formulation for the build-up factor and absorbed energy, in one and two dimensional Cartesian geometry for photons and electrons, in the Compton energy range. For the one-dimensional case we use the LTS_N method, assuming the Klein-Nishina scattering kernel for the determination of the angular radiation intensity for photons. We apply the two-dimensional LTS_N nodal solution for the averaged angular radiation evaluation for the two-dimensional case, using the Klein-Nishina kernel for photons and the Compton kernel for electrons. From the angular radiation intensity we construct a closed-form solution for the build-up factor and evaluate the absorbed energy. We present numerical simulations and comparisons against results from the literature.

Keywords: Build-Up Factor; Compton Energy; Cartesian Geometry; Fokker-Plank Equation

1. Introduction

In radiological protection the effectiveness of a material as a biological shield is related to its cross-section for scattering and absorption which is cast into physical parameters. The determination of those parameters require the solution of a systems of linear transport equations, *i.e.* the fluence for photons and electrons. Established methods that solve the transport equations are the PN approximation [1], the discrete ordinate method and their variants [2]. The S_N method has been used successfully in photon transport calculation, whereas recently the P_N approximation was applied to electron transport. In this work we present the solution of a couple system of linear transport equations, using the LTS_N method for a rectangular domain considering the Klein-Nishina scattering kernel and a multi-group model for photons. The electron contribution to energy deposition induced by incident photons is quantified solving the two-dimensional Fokker-Planck equation for electron transport [3,4] by the P_N approximation in the angular variable followed by applying the Laplace Transform to one of the spatial variables (here x). This procedure leads to a closed-form formulation for the build-up factor and absorbed energy, in one and two dimensional Cartesian geometry for photons and electrons, in the energy range where Compton scattering is dominant [5].

2. The LTS_N Nodal Solution in Two Dimensional for Photons

The two-dimensional S_N nodal problem for photons, assuming Klein-Nishina scattering kernel and a multi-group model is

$$\begin{aligned}&\mu_n \frac{\partial}{\partial x} I_{jn}(x,y) + \eta_n \frac{\partial}{\partial y} I_{jn}(x,y) + \mu_{1j} I_{jn}(x,y) \\ &= \frac{\Delta}{3} \sum_{l=0}^{L} \frac{2l+1}{2} \sum_{r=1}^{G} c_r \alpha k_{rj} P_l(1+\lambda_r-\lambda_j) P_l(\mu_n) \\ &\times \sum_{i=1}^{N} P_l(\mu_i) I_{ri}(x,y) w_i\end{aligned} \quad (1)$$

subject to vacuum boundary conditions in a rectangle $0 \le x \le a$ and $0 \le y \le b$. Where $j = 1, \cdots, G; n = 1, \cdots, N$; $N = M(M+2)/2$ is the cardinality of the discrete ordi- nate set, M represents the order of the angular quadrature, G is the number of energy groups in units of wavelengths, μ_{lj} is the linear attenuation coefficient, $I_{jn}(x,y) = I(x, y, \lambda_j, \Omega_n)$ is the angular flux into the discrete direction $\Omega_n = (\mu_n, \eta_n)$ for the j-th energy group, w_i are the Lewis-Miller quadrature weights and $k_{rj} = k(\lambda_r, \lambda_j)$ is the Klein-Nishina scattering kernel, defined as

$$k_{rj} = \frac{3}{8} \frac{\lambda_r}{\lambda_j} \left(\frac{\lambda_r}{\lambda_j} + \frac{\lambda_j}{\lambda_r} - \sin^2 \sin^2 \theta \right) \quad (2)$$

The LTS_N nodal approach for photons yields the following S_N equation system

$$\eta_n \frac{\partial}{\partial y} I_{jny}(y) + \frac{\mu_n}{a}\left[I_{jn}(a,y) - I_{jn}(0,y)\right] + \mu_{lj} I_{jny}$$
$$= \frac{\Delta}{3}\sum_{l=0}^{L} \frac{2l+1}{2} \sum_{r=1}^{G} c_r \alpha k_{rj} P_l\left(1+\lambda_r-\lambda_j\right) P_l(\mu_n) \quad (3)$$
$$\times \sum_{i=1}^{N} P_l(\mu_i) I_{riy}(x,y) w_i$$

For $j = 1,\cdots,G$ and $=1,\cdots,N$. Here $I_{jn}(a,y)$ and $I_{jn}(0,y)$ are the angular fluxes at the boundary edges with $x = 0$, $x = a$ and the average angular flux is

$$I_{jny}(y) = \frac{1}{a}\int_0^a I_{jn}(x,y)\,\mathrm{d}x \quad (4)$$

Further,

$$\mu_n \frac{\partial}{\partial x} I_{jnx}(x) + \frac{\eta_n}{b}\left[I_{jn}(x,0) - I_{jn}(x,b)\right] + \mu_{lj} I_{jnx}$$
$$= \frac{\Delta}{3}\sum_{l=0}^{L} \frac{2l+1}{2} \sum_{r=1}^{G} c_r \alpha k_{rj} P_l\left(1+\lambda_r-\lambda_j\right) P_l(\mu_n) \quad (5)$$
$$\times \sum_{i=1}^{N} P_l(\mu_i) I_{rix}(x) w_i$$

For $j = 1,\cdots,G$ and $=1,\cdots, N$. In Equation (5), $I_{jn}(x,b)$ and $I_{jn}(x,0)$ are the angular fluxes at the boundary edges with $y = 0$, $y = b$ and the average angular flux is

$$I_{jnx}(x) = \frac{1}{b}\int_0^b I_{jn}(x,y)\,\mathrm{d}y \quad (6)$$

Application of the LTS_N method consists in Laplace transform of the set of S_N equations and solving the resulting algebraic equations by matrix diagonalization and subsequent inversion of the transformed angular flux by standard procedures of integral transform theory. The Laplace transform of Equation (3) is

$$s\overline{I}_{jny}(s) + \frac{\mu_{lj}}{\eta_n}\overline{I}_{jny}(s) - \frac{\Delta}{3\eta_n}\sum_{l=1}^{L}$$
$$\times \frac{2l+1}{2}\sum_{r=1}^{G} c_r \alpha k_{rj} P_l(\mu_n) \sum_{i=1}^{N} P_l(\mu_i)\overline{I}_{riy}(s) w_i \quad (7)$$
$$= I_{jny}(0) - \frac{\mu_n}{a\eta_n}[\overline{I}_{jn}(a,s) - \overline{I}_{jn}(0,s)]$$

For $j = 1,\cdots,G$ and $n = 1,\cdots,N$ which can be cast in matrix form,

$$\left(sI - B_{jny}\right)\overline{I}_{jny}(s) = I_{jny}(0) + \overline{Z}_{(j-1)y}(s) + \overline{S}_{jny}(s) \quad (8)$$

Here, $\overline{I}_{jny}(s)$ are the N componentes of the Laplace transformed angular flux in the y variable and $\overline{I}_{jny}(0)$ are the respective components of the angular flux at the edge $y = 0$. The components have the following forms,

$$\overline{I}_{jny}(s) = \left[\overline{I}_{j1y}(s), \overline{I}_{j2y}(s), \cdots, \overline{I}_{jNy}(s)\right]^T \quad (9)$$

and

$$\overline{I}_{jny}(0) = \left[\overline{I}_{j1y}(0), \overline{I}_{j2y}(0), \cdots, \overline{I}_{jNy}(0)\right]^T \quad (10)$$

The components of the matrix B_{jny} are given by,

$$b_y(p,q) = \begin{cases} -\dfrac{\mu_{lj}}{\eta_p} + \dfrac{\Delta}{3\eta_p}\sum_{l=1}^{L} \dfrac{2l+1}{2} c_j \alpha k_{jj} P_l(\mu_p) P_l(\mu_p) w_p, & p = q \\ \dfrac{\Delta}{3\eta_p}\sum_{l=0}^{L} \dfrac{2l+1}{2} c_j \alpha k_{jj} P_l(\mu_p) P_l(\mu_q) w_p, & p \neq q \end{cases}$$

The scattering term is,

$$\overline{Z}_{(j-1)y}(s) = \sum_{i=1}^{j-1} H_{ij}\overline{I}_{iny}(s) \quad (11)$$

with the components of matrix H_{ij} (see the equation below).

The vector $\overline{S}_{jny}(s)$ has the components,

$$\overline{S}_{jiy}(s) = -\frac{\mu_{lj}}{a\eta_i}\left[\overline{I}_{ji}(a,s) - \overline{I}_{ji}(0,s)\right] \quad (12)$$

A similar methodology in the x variable leads to the linear algebric system which can be matrix form,

$$\left(sI - A_{jnx}\right)\overline{I}_{jnx}(s) = I_{jnx}(0) + \overline{Z}_{(j-1)x}(s) + \overline{S}_{jnx}(s) \quad (13)$$

where $\overline{I}_{jnx}(s)$ has the following form,

$$\overline{I}_{jnx}(s) = \left[\overline{I}_{j1x}(s), \overline{I}_{j2x}(s), \cdots, \overline{I}_{jNx}(s)\right]^T \quad (14)$$

The matrix elements of A_{jnx} are,

$$a_x(p,q) = \begin{cases} -\dfrac{\mu_{lj}}{\eta_p} + \dfrac{\Delta}{3\eta_p}\sum_{l=0}^{L} \dfrac{2l+1}{2} c_j \alpha k_{jj} P_l(\mu_p) P_l(\mu_p) w_p, & p = q \\ \dfrac{\Delta}{3\eta_p}\sum_{l=0}^{L} \dfrac{2l+1}{2} c_j \alpha k_{jj} P_l(\mu_p) P_l(\mu_q) w_p, & p \neq q \end{cases}$$

and the scattering term is

$$\overline{Z}_{(j-1)x}(s) = \sum_{i=1}^{j-1} H_{ix}\overline{I}_{inx}(s) \quad (15)$$

where the matrix elements of H_{ix} are given

$$h_x(p,q) = \begin{cases} \dfrac{\Delta}{3\eta_p}\sum_{l=0}^{L} \dfrac{2l+1}{2} c_j \alpha k_{ij} P(1+\lambda_i-\lambda_j) P_l(\mu_p) P_l(\mu_p) w_p \\ -\dfrac{\Delta}{3\eta_p}\sum_{l=0}^{L} \dfrac{2l+1}{2} c_j \alpha k_{jj} P(1+\lambda_i-\lambda_j) P_l(\mu_p) P_l(\mu_q) w_p \end{cases}$$

$$h_y(p,q) = \begin{cases} \dfrac{\Delta}{3\eta_p}\sum_{l=0}^{L} \dfrac{2l+1}{2} c_j \alpha k_{ij} P\left(1+\lambda_i-\lambda_j\right) P_l(\mu_p) P_l(\mu_p) w_p, & p = q \\ -\dfrac{\Delta}{3\eta_p}\sum_{l=0}^{L} \dfrac{2l+1}{2} c_j \alpha k_{jj} P\left(1+\lambda_i-\lambda_j\right) P_l(\mu_p) P_l(\mu_q) w_p, & p \neq q \end{cases}$$

and the vector $\bar{S}_{jnx}(x)$ is

$$\bar{S}_{jix}(s) = -\frac{\eta_i}{b\mu_i}\left[\bar{I}_{ji}(s,b) - \bar{I}_{ji}(s,0)\right] \tag{16}$$

The LTS_N solution for Equations (8) and (16) are given by

$$\bar{I}_{jny}(s) = \left(sI - B_{jny}\right)^{(-1)}\left[I_{jny}(0) + \bar{Z}_{(j-1)y}(s) + \bar{S}_{jny}(s)\right] \tag{17}$$

and

$$\bar{I}_{jnx}(0) = \left(sI - A_{jnx}\right)^{-1}\left[I_{jnx}(0) + \bar{Z}_{(j-1)x}(s) + \bar{S}_{jnx}(s)\right]^T \tag{18}$$

Taking the Laplace inversion by applying the Heaviside expansion yields

$$\begin{aligned} I_{jny}(y) = &\sum_{k=1}^{jn}\beta_k e^{s_k y} I_{jny}(0) + Z_{(j-1)y}(y) * \mathcal{L}^{-1}\left\{\left(sI - B_{jny}\right)^{-1}\right\} \\ &+ S_{jny}(y) * \mathcal{L}^{-1}\left\{\left(sI - B_{jny}\right)^{-1}\right\} \end{aligned} \tag{19}$$

and

$$\begin{aligned} I_{jnx}(x) = &\sum_{k=1}^{jn}\beta_k e^{s_k x} I_{jny}(0) + Z_{(j-1)x}(x) * \mathcal{L}^{-1}\left\{\left(sI - A_{jnx}\right)^{-1}\right\} \\ &+ S_{jnx}(x) * \mathcal{L}^{-1}\left\{\left(sI - A_{jny}\right)^{-1}\right\} \end{aligned} \tag{20}$$

For the fluxes at boundary one may use reasonable approximation [6]

$$I_{jn}(x,0) = F_{jn}e^{-sign(\mu_n)\Lambda x} \tag{21}$$

$$I_{jn}(0,y) = G_{jn}e^{-sign(\eta_n)\Lambda y} \tag{22}$$

$$I_{jn}(x,b) = O_{jn}e^{-sign(\mu_n)\Lambda x} \tag{23}$$

$$I_{jn}(a,y) = P_{jn}e^{-sign(\eta_n)\Lambda y} \tag{24}$$

Here the signal function follows the usual definition $sign(\mu) = 1$ for $\mu > 0$ and $sign(\mu) = -1$ for $\mu < 0$ and Λ represents an attenuation parameter, here the macroscopic absorption crosssection.

The generalization of the LTS_N nodal solution for a heterogeneous rectangular geometry assuming the Klein-Nishina scattering kernel and a multi-group model may be implemented in a completely analogue procedure. In such a problem the LTS_N solution is determined for each subdomain and the integration constants are evaluated upon applying the boundary and interface.

3. The Two-Dimensional Fokker-Planck Equation Solution for Electrons

The purpose of radiation transport is to determine how particles move through materials and what effects their propagation have on the material through the mechanisms of deposited energy and charge deposition. The angular flux of electrons in a rectangular domain can be determined by solving the following two-dimensional, time independent electron transport equation, in a rectangle $0 \le x \le a$ and $0 \le y \le b$, subject to vacuum boundary conditions.

$$\begin{aligned} &\mu\frac{\partial\psi\left(x,y,\bar{\Omega},E\right)}{\partial x} \\ &+\eta\frac{\partial\psi\left(x,y,\bar{\Omega},E\right)}{\partial y} + \sigma_{\mathrm{t}}(E)\psi\left(x,y,\bar{\Omega},E\right) \\ &= \int_{4\pi}\mathrm{d}\bar{\Omega}\sigma_s\left(E'\to E,\bar{\Omega}'\cdot\bar{\Omega}\right)\psi\left(x,y,\bar{\Omega}',E'\right) \end{aligned} \tag{25}$$

Here, the angular flux (x,y,E,Ω), represents the flux of particles at position (*x*, *y*), with energy *E* travelling in direction $\Omega = (\mu,\eta)$. The quantity σ_s in Equation (25) is the differential scattering cross-section, in this work, we focus on screened Rutherford scattering which is

$$\sigma_s(E,\mu_0) = \frac{\sigma_t(E)\eta^*(\eta^*+1)}{\pi\left(1+2\eta^*-\mu_0^2\right)} \tag{26}$$

where $\eta^* > 0$ is typically a small constant called the screening parameter parameter is given by

$$\eta^* = \frac{h^2 Z^{2/3}}{4\left(a_H\right)^2\left(m_e v\right)^2} \tag{27}$$

with *Z* the atomic number of the nucleus, *mv* the relativistic momentum of the scattered electron and

$$C = \frac{\bar{h}^2}{4a_H^2} \tag{28}$$

which $\bar{h}$ is Planck's constant and a_H is the Bohr radius.

We assume that the Fokker-Planck (FP) scattering description is appropriate, so that the transport problem (25) is given by,

$$\begin{aligned} &\mu\frac{\partial\psi^{FP}\left(x,y,\bar{\Omega}',E\right)}{\partial x} + \eta\frac{\partial\psi^{FP}\left(x,y,\bar{\Omega}',E\right)}{\partial y} \\ &= \frac{\sigma_{tr}}{2}\frac{\partial}{\partial\mu}\left[\left(1-\mu^2\right)\frac{\partial}{\partial\mu}\right]\psi^{FP}\left(x,y,\bar{\Omega}',E\right) \end{aligned} \tag{29}$$

Here the coefficient σ_{tr} is he transport cross-sectionand defined as,

$$\sigma_{tr} = 2\pi\int_{-1}^{1}\int_{0}^{1}\sigma_s(E,\mu_0)(1-\mu_0)\mathrm{d}\mu_0\mathrm{d}\eta \tag{30}$$

Upon multiplying Equation (29) by $P_N(\mu)$, integrating over μ and using the recursion formula, as well as general properties of Legendre polynomials leads to the following P_N equations

$$\begin{aligned} &\frac{n+1}{2n+1}\frac{\partial}{\partial x}\psi_{n+1}^{FP}(x,y,E) + \frac{n}{2n+1}\frac{\partial}{\partial x}\psi_{n-1}^{FP}(x,y,E) \\ &+\frac{2n+1}{n}\frac{\partial}{\partial y}\psi_n^{FP}(x,y,E)T_n = \frac{\sigma_{tr}}{2}\left[-n(n+1)\right]\psi_n^{FP}(x,y,E) \end{aligned} \tag{31}$$

The angular flux moments in discrete ordinates may be expressed using a quadrature approximation,

$$\psi^{FP}\left(x,y,\bar{\Omega}'E\right)=\sum_{l=0}^{L}\frac{2l+1}{2}\psi_n^{FP}\left(x,y,E\right)P_n\left(\mu\right) \quad (32)$$

For $n=0,\cdots,N$ with $\psi_{N+1}^{FP}\left(x,y,E\right)=0$ in the P_N approximation and T_n represented by an integral term which can be solved analytically,

$$T_n=\int_{(-1)}^{1}\sqrt{1-\mu^2}P_n\left(\mu\right)P_{(n+1)}\left(\mu\right)\mathrm{d}\mu \quad (33)$$

After applying the Laplace transform in Equation (31) in the spatial variable *x*, we came out with the linear algebric system,

$$\begin{aligned}&\frac{n+1}{2n+1}\left[s\overline{\psi_{n+1}^{FP}}\left(s,y,E\right)-\overline{\psi_{n+1}^{FP}}\left(0,y,E\right)\right]\\&+\frac{n}{2n+1}\left[\overline{\psi_{n-1}^{FP}}\left(s,y,E\right)-\overline{\psi_{n-1}^{FP}}\left(0,y,E\right)\right]\\&+\frac{2n+1}{2}\frac{\partial}{\partial y}\overline{\psi_n^{FP}}\left(s,y,E\right)T_n=\frac{\sigma_{tr}}{2}\left[-n\left(n+1\right)\right]\overline{\psi_n^{FP}}\end{aligned} \quad (34)$$

For $n=0,\cdots,N$ and where $\overline{\psi_{n-1}^{FP}}\left(x,y,E\right),\overline{\psi_n^{FP}}\left(x,y,E\right)$ and $\psi_{n+1}^{FP}\left(x,y,E\right)$ are the transformed angular fluxes in the spatial *x* variable. Equation (36) can be cast in matrix form,

$$\begin{aligned}&A_n\overline{\psi_n^{FP'}}\left(s,y,E\right)\\&+B_n\left(s\right)\overline{\psi_n^{FP}}\left\{\left(s,y,E\right)-C_n\overline{\psi_n^{FP}}\left(0,y,E\right)=0\right.\end{aligned} \quad (35)$$

where $\overline{\psi_n^{FP'}}$ are the *N* components of the derivative of the Laplace transformed angular flux in the *x* variable.

$$\overline{\psi_n^{FP'}}\left(s,y,E\right)=\left[\overline{\psi_n^{FP'}}\left(s,y,E\right),\cdots,\overline{\psi_n^{FP'}}\left(s,y,E\right)\right]^T \quad (36)$$

The components of matrices $A_N,B_N\left(s\right)$ and C_N are for $i\in\{1,2,\cdots,N\}$, respectively

$$a_{ij}=\begin{cases}\left(2i+1\right)^2T_i,\ i=j\\0,\ i\neq j\end{cases}$$

$$b_{ij}=\begin{cases}i\left(i+1\right)\left(2i+1\right)\sigma_{tr},i=j\\2is,\ i=\left|j-1\right|\\0,\ \text{otherwise}\end{cases}$$

$$c_{ij}=\begin{cases}2i,\ i=\left|j-1\right|\\0,\ \text{otherwise}\end{cases}$$

The solution of Equation (35) is

$$\begin{aligned}\overline{\psi_n^{FP}}\left(s,y,E\right)&=c_1\left(s\right)e^{-\left(B_n\left(s\right)A_n^{-1}\right)}\\&+C_n\left(B_n\left(s\right)\right)^{-1}\psi_n^{FP}\left(0,y,E\right)\end{aligned} \quad (37)$$

where $c_1\left(s\right)$ is an arbitrary constant determined by applying the boundary and interface conditions. Due to the linear character of the inverse Laplace transform operator, the solution is composed by

$$\begin{aligned}\psi_n^{FP}\left(x,y,E\right)&=\mathcal{L}^{-1}\left\{c_1\left(s\right)e^{-\left(B_n\left(s\right)A_n^{-1}\right)}\right\}\\&+C_n\cdot\mathcal{L}^{-1}\left\{\left(B_n\left(s\right)\right)^{-1}\right\}\cdot\psi_n^{FP}\left(0,y,E\right)\end{aligned} \quad (38)$$

4. Numerical Results

In the following we apply the closed-form formulation to the build-up factor and absorbed energy for photons and electrons in the Compton energy range and in Cartesian geometry. Moreover, we the one-dimensional, two-dimensional problem for photons and two-dimensional problem for photons and electrons. To this end we evaluate the exposure build-up factor, considering a scalar flux of the photons with energy 1 MeV, incident in a multi-layered slab with two regions, composed of water and lead, water and iron, lead and iron. Assuming that the kernel is described by the Klein-Nishina differential scattering cross-section and the energy variables may be simplified in form of a multi-group model in the wavelength (energy) variable.

Here the exposure build-up factor is defined as the sum of the product of the attenuation coefficient of the air with the scalar flux for all photons, including the incident flux, divided by the attenuation coefficient of the air for the incident flux multiplied by the incident scalar flux. The numerical results for three problems are shown in **Table 1** for water and lead, in **Table 2** for water and iron and in **Table 3** for lead and iron. In **Table 1** we present the LTS_N numerical simulations for the exposure build-up factor and comparisons with results from the EGS_4 code [7] generated for the one-group model. We consider a multi-layered slab with two regions, composed of water (μ_{ij} = 0.0707 cm^2/g and 1.0 mpf) and lead (μ_{ij} = 0.06848 cm^2/g and depth in multiples of the mean free path, 4.0, 5.0, 10.0, 20.0, 30.0 and 40.0 mfp) together with the afore mentioned vacuum boundary conditions.

Table 1. Numerical exposure buildup factor simulations for a multilayered slab composed of water (1.0 mfp) and lead.

Iron (mfp)	LTS_{16}	EGS_4
4.0	2.30	2.31
5.0	2.07	2.08
10.0	3.57	3.59
20.0	5.29	5.31
30.0	6.77	6.79
40.0	8.26	8.27

Table 2. Numerical exposure buildup factor simulations for a multilayered slab composed of water (1.0 mfp) and iron.

Iron (mfp)	LTS_{16}	EGS_4
4.0	4.99	5.01
5.0	6.21	6.23
10.0	13.9	13.9
20.0	36.3	36.3
30.0	67.6	67.5
40.0	101.0	101.0

Table 3. Numerical exposure buildup factor simulations for a multilayered slab composed of lead (1.0 mfp) and iron.

Iron (mfp)	LTS_{16}	EGS_4
4.0	4.87	4.86
5.0	6.34	6.28
10.0	15.4	15.3
20.0	41.5	41.4
30.0	78.4	78.3
40.0	118.0	117.0

In **Table 2** we present the LTS_N numerical simulations for the exposure build-up factor and comparisons with the results from EGS_4 generated for the one-group model. We consider a multi-layered slab with two regions, composed of water (μ_{ij} = 0.0707 cm^2/g with depth of 1.0 mpf) and iron (μ_{ij} = 0.0596 cm^2/g with depth 4.0, 5.0, 10.0, 20.0, 30.0 and 40.0 mfp) and vacuum boundary conditions.

From the analysis of the results encountered in **Tables 1-3**, one realizes a fairly good agreement between the LTS_{16} and EGS_4 results. The numerical convergence of the LTS_N results showed for increasing N a coincidence of six significant digits for N = 14 and N = 16. For two-dimensional problems for photons and electrons, we applied the LTS_N method, for N = 8, in the transport equation for photons and used N = 9 in the P_N approximation for the angular variable of the Fokker-Planck equation for electrons. We considered homogeneous rectangular geometries composed of water liquid (Z/A = 0.55508, ρ = 1 g/cm^2), soft tissue (ICRU44, Z/A = 0.54996, ρ = 1.06 g/cm^2) and cortical bone (ICRU44, Z/A = 0.51478, ρ = 1.92 g/cm^2). Further, we assumed a mono-energetic (E = 1.25 MeV) and monodirectional photon beam incident on the edge of a rectangle with dimension 20 cm × 20 cm and vacuum boundary conditions. The incoming photons were tracked until their whole energy was deposited and/or they left the domain of interest. In this problem also the energy deposited by the secondary electrons, generated by the Compton Effect, were considered. Other possible effects, however with small or spurious contributions were not taken into account. The numerical results encountered for absorbed energy in the domain were compared with simulations obtained with the program Geant4 v8 [8], using the low energy libraries and are presented in **Tables 4** and **5**.

The numerical convergence of the LTS_N results showed for increasing N a coincidence of three significant digits for N = 4, 6 and 8. Notice, the coincidence of four significant digits for the P_N approximation with N = 7 and 9. These results were obtained in the homogeneous domain with dimension 20 cm × 20 cm that was composed of water. In **Table 4** we present the LTS_8 nodal numerical simulations for the absorbed energy induced by photons incident on a homogeneous rectangular domain, that is composed of a variety of different materials. In **Table 5** we present numerical simulations for absorbed energy by the P_9 approximation, due to free electrons, arising from Compton scattering in a homogeneous rectangular domain composed of different materials. These results were compared with simulations obtained by Geant4.

In spite of the fact, that two different methods were used to simulate energy deposition, the Monte Carlo method with Geant4 and our closed form solution the results in **Tables 4** and **5** show a fairly good agreement. From the results, we notice that the maximum discrepancy found is 3.4% in the simulations for photons and 8.3% for electrons. The difference of our numerical results in comparison to the Geant simulations, that contain a catalogue of processes, demonstrate that other effects shall be taken into account. As the material density increases, the

Table 4. Absorved energy (KeV/photon emitted from the source) by the photons incident in a homogeneous domain dimension 20 cm × 20 cm, composed of materials different.

Domain composition	LTS_N	Geant4	Discrepancy (%)
Water liquid	0.00457	0.00468	2.3
Soft tissue	0.00531	0.00542	2.0
Cortical bone	0.0987	0.09487	3.4

Table 5. Absorved energy (KeV/photon emitted from the source) by the free electrons in a homogeneous domain dimension 20 cm × 20 cm, composed of materials different.

Domain composition	LTS_N	Geant4	Discrepancy (%)
Water liquid	0.03379	0.03609	6.4
Soft tissue	0.03317	0.03542	6.4
Cortical bone	0.79284	0.86380	8.3

number of interactions increases as well as the possibility of other production processes involving secondary electrons, responsible for more than 86% of the total energy absorbed in domain.

5. Conclusion

Concluding, we were successful in determinig the LTS_N solution in closed form forenergy deposition induced by photons in Cartesian geometry. From the solution we obtained the buildup factor and absorbed energy, for photons and electrons, in the Compton energy range. It is worth mentioning, that the LTS_N procedure maintains an analytical character of the solution and the unique approximation made was in the leakage angular flux at the boundary. The P_N solution of the Fokker-Planck equation remains analytical in the sense that no approximation is made along its derivation from P_N equations, except for the truncation. Finally, a variety of additional numerical experiments have shown us that the presented method is robust for problems of the considered transport equation type.

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