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Vibration of Three-Layered FGM Cylindrical Shells with Middle Layer of Isotropic Material for Various Boundary Conditions

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Abstract

In the present study, vibration analysis of a three-layered cylindrical shell is performed whose inner and outer layers are composed of functionally graded materials whereas the middle one is assumed to be of isotropic material. This formation of a cylindrical shell influences stiffness moduli and the resultant material properties. The shell problem is formulated from the constitutive relations of stresses and strains with the displacement deformations and they are taken from Love's thin shell theory. This problem is transformed into the integral form by evaluating the expressions for the strain and kinetic energies of the shell. Rayleigh-Ritz method is employed to solve the shell dynamic equations. Vibration characteristics of these cylindrical shells are investigated for a number of physical parameters and configurations of the fabrication of shells. The axial modal dependence is approximated by the characteristic beam functions that satisfy the boundary conditions. Results evaluated, show good agreement with the open literature.

Keywords

Functionally Graded Material, Isotropic Material, Three-Layered Cylindrical Shell, Love's Thin Shell Theory Rayleigh-Ritz Method, Natural Frequency

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1. Introduction

Yamanouchi *et al.* [1] introduced the concept of functionally graded materials (FGMs), working as aerospace researchers in Japan. Rabin and Heaps [2] reported some methods for the manufacturing of FGMs. The idea of ceramic transactions for the fabrication of FGMs was introduced by Koizumi [3]. Miyamoto *et al.* [4] composed a book on FGMs, in which they gave a high quality discussion on the design and applications of FGMs. Rayleigh [5] analyzed the study of Sophie on the vibration of circular cylindrical shells. Love [6] at the end of 19^{th} century, gave the first linear shell theory based on Krichhoff's hypothesis for plates. Arnold and Warburton [7] [8] derived equations of motion for vibration of thin circular cylindrical shells. They used Lagrange equations with strain and kinetic energy expressions to derive these equations. Forsberg [9] studied shell equations to scrutinize the effect of boundary conditions on vibration characteristics of circular cylindrical shells. In his work, exponential axial modal dependence was measured. Sewall and Naumann [10] studied analytical and experimental frequencies and mode shapes for the vibrations using the Rayleigh-Ritz method. The characteristic beam functions were used to approximate the modal dependence in the axial direction. Warburton [11], Warburton and Higgs [12] and Goldman [13] studied the natural frequencies and mode shapes of thin cylindrical shells and selected exponential functions for the modal dependence in the axial direction. Sharma [14] explored the natural frequencies of fixed free circular cylindrical shells. He [15] also studied the problems of vibration characteristics of thin circular cylindrical shells with various end conditions with first order shell theory of Sanders. A simple variational technique was applied to give a cubic frequency equation. Loy and Lam [16] studied the vibration of thin cylindrical shells with ring supports, placed along the shell length and which imposed a zero lateral deflection. The study was carried out using Sander's shell theory. Naeem and Sharma [17] employed an analytical procedure to study the free vibration characteristics of thin cylindrical shells. Ritz polynomial functions were assumed to satisfy the axial modal dependence and the Rayleigh Ritz variational approach was employed to formulate the general eigenvalue problem. Influence of some commonly used boundary conditions and shell parameters on the vibration frequencies were examined. Loy *et al.* [18] studied the vibration of FGM cylindrical shell fabricated with the constituent materials stainless steel and nickel. They concluded that the frequency characteristics were similar to those of homogeneous isotropic cylindrical shells and the frequencies were affected by the constituent volume fractions and the configurations of the constituent materials but they found the response of frequencies of FGM cylindrical shells for only simply supported boundary condition. This work was extended by Pradhan *et al.* [19] by studying the vibration of FGM cylindrical shell for various boundary conditions. Arshad *et al.* [20] [21] calculated natural frequencies of the FGM cylindrical shell by various volume fraction laws and under various boundary conditions respectively. Najafizadeh and Isvandzibaei [22] studied the vibration of thin FGM cylindrical shells with ring supports. The study was carried out based on third order shear deformation shell theory. The objective was to observe the influence of the configurations of the constituent materials, positions of the ring support and different boundary conditions on the natural frequencies of the cylindrical shells. The analysis was carried out with strain displacement relations from Love's shell theory. The governing equations were obtained using energy functional with the Rayleigh Ritz method. Sofiyev *et al.* [23] studied the vibration and stability analysis of a three-layered conical shell with middle layer composed of functionally graded material. They applied Galerkin numerical technique to transform the governing equations of motion into a pair of time dependent partial differential equations. They concluded that the material parameters are directly affected by the diverse configurations of the FGM constituents and by the nature of materials used in the shell layers. He [24] extended this work to study the vibration and stability response of a composite cylindrical shell containing a functionally graded layer. Li and Batra [25] [26] investigated other dynamic aspects of shells like buckling of a three layered simply supported axially compressed laminated composite as well as isotropic thin cylindrical shells. They designed these shells in such a way that the inner and outer layers of the shells were composed of the composite and isotropic materials respectively and a layer of functionally graded material was inserted at the middle in the transverse direction. Arshad *et al.* [27] [28] studied the frequency spectra of bi-layered cylindrical shells by taking different materials in both layers such as isotropic as well as functionally graded materials and by taking two different functionally graded materials at the inner and outer layers of the cylindrical shells respectively.

2. Formulation of Shell Problem

2.1. Volume Fraction Law

Most of functionally graded materials are used in high temperature and possess temperature dependent properties. The material property P is expressed as a function temperature $T(K)$ by Touloukian [29] as:

$$P = P_0\left(P_{-1}T^{-1} + 1 + P_1T + P_2T^2 + P_3T^3\right) \quad (1)$$

where P_0, P_1, P_2 and P_3 are the coefficients of temperature $T(K)$ expressed in Kelvin and are unique to the constituent materials. The material properties P of FGMs are a function of the material properties and volume fractions of the constituents, and can be expressed as:

$$P = \sum_{j=1}^{k} P_j V_{fj} \quad (2)$$

where P_j and V_{fj} are the material property and volume fraction of the constituent material j respectively. For three-layered FGM cylindrical shells, the outer and inner layers are assumed to be functionally graded and the middle layer isotropic. The shell thickness is assumed to be distributed in three portions. With this assumption, the extensional, coupling and bending stiffness are modified in three layers. Let M_{in} and M_{out} represent the inner and outer constituent materials of the FGM layers used to fabricate the three layered FGM cylindrical shell with middle layer of isotropic material. For a functionally graded material layers, consisting of two materials M_{in} and M_{out}, volume fraction is written for an effective material property as:

$$P(z) = \left(P_{\text{in}} - P_{\text{out}}\right)V_{\text{in}} + P_{\text{out}} \quad (3)$$

where V_{in}, the volume fraction, is defined for a material M_{in} as:

$$V_{\text{in}} = \left(\frac{z - h_{\text{in}}}{h_{\text{out}} - h_{\text{in}}}\right)^p \quad (4)$$

where z is the radial variable, h_{in} and h_{out} are the inner and outer coordinates from the centre of the circular cylindrical shell. Also P_{in} and P_{out} are the material properties of M_{in} and M_{out} respectively.

2.2. Theoretical Considerations

Consider a cylindrical shell as shown in the **Figure 1(a)**. Let R is the radius, L is the length and h is the thickness of the cylindrical shell. The orthogonal coordinates system (x,θ,z) is taken to be at the middle surface of the shell. The x-coordinate is taken in the axial direction of the shell θ is in the circumferential and z-coordinate is in the radial direction of the shell. The deformations of the shell in axial, circumferential and radial directions are represented by $u(x,\theta,t)$, $v(x,\theta,t)$ and $w(x,\theta,t)$ respectively. For the study of thin cylindrical shell, three dimensional problems are converted in to two dimensional by applying plane stress condition. The constitutive relation of stress and strain of a thin cylindrical shell is given by Hook's law as:

$$\{\sigma\} = [Q]\{e\} \quad (5)$$

where $\{\sigma\}$ is the stress vector, $\{e\}$ is the strain vector and $[Q]$ is the reduced stiffness matrix. The stress vector and the strain vectors are defined as:

$$\{\sigma\}^{\mathrm{T}} = \{\sigma_x, \sigma_\theta, \sigma_{x\theta}\} \quad (6)$$

$$\{e\}^{\mathrm{T}} = \{e_x, e_\theta, e_{x\theta}\} \quad (7)$$

where σ_x and σ_θ are the normal stresses in x and θ directions, and $\sigma_{x\theta}$ is the shear stress on the $x\theta$-plane. Similarly e_x and e_θ are the normal strains in the x and θ directions, and $e_{x\theta}$ is the shear strain on the $x\theta$-plane. The reduced stiffness matrix is defined as:

$$[Q] = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \quad (8)$$

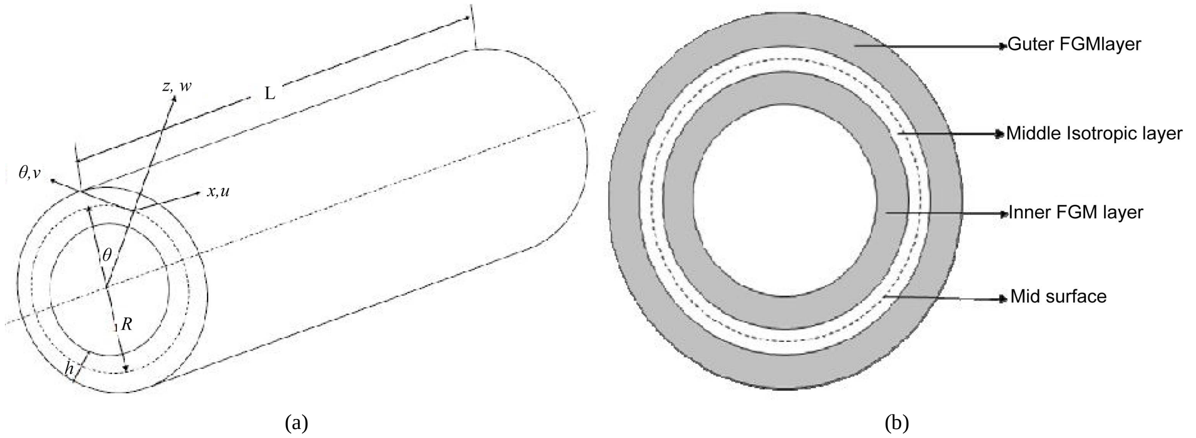


Figure 1. (a) Geometry of a Circular Cylindrical Shell; (b) Cross-sectional vision of three-layered cylindrical shell.

So the relation (5) can be expressed as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_\theta \\ \sigma_{x\theta} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} e_x \\ e_\theta \\ e_{x\theta} \end{Bmatrix} \quad (9)$$

For isotropic materials the reduced stiffness Q_{ij} are defined as:

$$Q_{11} = Q_{22} = \frac{E}{1-\nu^2}, \quad Q_{12} = \frac{\nu E}{1-\nu^2}, \quad Q_{66} = \frac{E}{2(1+\nu)} \quad (10)$$

where E is the Young's modulus and ν is the Poisson's ratio. According to Love's shell theory, the components in the strain vector $\{e\}$ are defined as:

$$\left.\begin{aligned} e_x &= e_1 + z\kappa_1 \\ e_\theta &= e_2 + z\kappa_2 \\ e_{x\theta} &= \gamma + 2z\tau \end{aligned}\right\} \quad (11)$$

where e_1, e_2 and γ are the reference surface strains. κ_1, κ_2 and τ are the surface curvatures. From Equations (9) and (11) the components in the stress vector $\{\sigma\}$ are defined as:

$$\left.\begin{aligned} \sigma_x &= (e_1 + z\kappa_1)Q_{11} + (e_2 + z\kappa_2)Q_{12} \\ \sigma_\theta &= (e_1 + z\kappa_1)Q_{12} + (e_2 + z\kappa_2)Q_{22} \\ \sigma_{x\theta} &= (\gamma + 2z\tau)Q_{66} \end{aligned}\right\} \quad (12)$$

For a thin cylindrical shell the force and moment resultants are defined as:

$$\{N_x, N_\theta, N_{x\theta}\} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \{\sigma_x, \sigma_\theta, \sigma_{x\theta}\} \mathrm{d}z \quad (13)$$

$$\{M_x, M_\theta, M_{x\theta}\} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \{\sigma_x, \sigma_\theta, \sigma_{x\theta}\} z\mathrm{d}z \quad (14)$$

where N_x, N_θ and $N_{x\theta}$ are force components in axial, circumferential and shear directions. M_x, M_θ and $M_{x\theta}$ are moment components in axial, circumferential and shear directions. Equations (12), (13) and (14) implies:

$$\{N\} = [S]\{\in\} \tag{15}$$

where $\{N\}$ and $\{\in\}$ are defined as:

$$\{N\}^{\mathrm{T}} = \{N_x, N_\theta, N_{x\theta}, M_x, M_\theta, M_{x\theta}\} \tag{16}$$

$$\{\in\}^{\mathrm{T}} = \{e_1, e_2, \gamma, \kappa_1, \kappa_2, 2\tau\} \tag{17}$$

and $[S]$ is defined as:

$$[S] = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \tag{18}$$

where $[A]$, $[B]$ and $[D]$ are the extensional, coupling and bending stiffness matrices given as:

$$[A] = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \quad [B] = \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & B_{66} \end{bmatrix} \quad [D] = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix}$$

where A_{ij}, B_{ij} and D_{ij} are the extensional, coupling and bending stiffness and defined as:

$$\left(A_{ij}, B_{ij}, D_{ij}\right) = \int_{\frac{-h}{2}}^{\frac{h}{2}} \left(1, Z, Z^2\right) Q_{ij} \mathrm{d}z \tag{19}$$

The coupling stiffness B_{ij} become zero for isotropic cylindrical shell and is non-zero for FGM cylindrical shells. The general equations for strain energy and kinetic energy of a cylindrical shell can be written as:

$$U = \frac{1}{2} \int_0^L \int_0^{2\pi} \{\in\}^{\mathrm{T}} [S] \{\in\} R \mathrm{d}\theta \mathrm{d}x \tag{20}$$

$$T = \frac{1}{2} \int_0^L \int_0^{2\pi} \rho_T \left\{ \left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2 \right\} R \mathrm{d}\theta \mathrm{d}x \tag{21}$$

where ρ_T is the mass density per unit length and is defined as follows:

$$\rho_T = \int_{\frac{-h}{2}}^{\frac{h}{2}} \rho \mathrm{d}z \tag{22}$$

where ρ is the mass density of the shell material.

By substituting $\{\in\}$, $\{\in\}^{\mathrm{T}}$ and $[S]$ from Equations (17) and (18) in Equation (20) implies:

$$\begin{aligned} U = \frac{1}{2} \int_0^L \int_0^{2\pi} \{ & A_{11} e_1^2 + A_{22} e_2^2 + 2A_{12} e_1 e_2 + A_{66} \gamma^2 + 2B_{11} e_1 \kappa_1 + 2B_{12} e_1 \kappa_2 + 2B_{12} e_2 \kappa_1 + 2B_{22} e_2 \kappa_2 + 4B_{66} \gamma \tau + D_{11} \kappa_1^2 \\ & + D_{22} \kappa_2^2 + 2D_{12} \kappa_1 \kappa_2 + 4D_{66} \tau^2 \} R \mathrm{d}\theta \mathrm{d}x. \end{aligned} \tag{23}$$

2.3. Strain-Displacement and Curvature-Displacement Relation

A number of shell theories have arisen and are used. Among these theories however the Love's shell theory is considered to be the first theory about shells and all other shell theories were derived from the Love's shell theory by amending some physical terms. The strain-displacement and the curvature-displacement relations which are adopted from Love's [6] shell theory are given as below:

$$e_1 = \frac{\partial u}{\partial x}, \quad e_2 = \frac{1}{R}\left(\frac{\partial v}{\partial \theta} + w\right), \quad \gamma = \frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial u}{\partial \theta} \tag{24}$$

$$\kappa_1 = -\frac{\partial^2 w}{\partial x^2}, \quad \kappa_2 = -\frac{1}{R^2}\left(\frac{\partial^2 w}{\partial \theta^2} - \frac{\partial v}{\partial \theta}\right), \quad \tau = -\frac{1}{R}\left(\frac{\partial^2 w}{\partial x \partial \theta} - \frac{\partial v}{\partial x}\right) \tag{25}$$

By substituting these values of strain displacement and curvature displacement from Equations (24) and (25) in Equation (23), we obtain the strain energy equation in the form of displacement functions u, v, w and their partial derivatives as:

$$\begin{aligned} U = \frac{1}{2}\int_0^L\int_0^{2\pi} \Bigg\{ & A_{11}\left(\frac{\partial u}{\partial x}\right)^2 + A_{22}\frac{1}{R^2}\left(\frac{\partial v}{\partial \theta} + w\right)^2 + 2A_{12}\frac{1}{R}\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial \theta} + w\right) + A_{66}\left(\frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial u}{\partial \theta}\right)^2 - 2B_{11}\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial^2 w}{\partial x^2}\right) \\ & - 2B_{12}\frac{1}{R^2}\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial^2 w}{\partial \theta^2} - \frac{\partial v}{\partial \theta}\right) - 2B_{12}\frac{1}{R}\left(\frac{\partial v}{\partial \theta} + w\right)\left(\frac{\partial^2 w}{\partial x^2}\right) - 2B_{22}\frac{1}{R^3}\left(\frac{\partial v}{\partial \theta} + w\right)\left(\frac{\partial^2 w}{\partial \theta^2} - \frac{\partial v}{\partial \theta}\right) \\ & - 8B_{66}\frac{1}{R}\left(\frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial u}{\partial \theta}\right)\left(\frac{\partial^2 w}{\partial x \partial \theta} - \frac{\partial v}{\partial x}\right) + D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + D_{22}\frac{1}{R^4}\left(\frac{\partial^2 w}{\partial \theta^2} - \frac{\partial v}{\partial \theta}\right)^2 \\ & + 2D_{12}\frac{1}{R^2}\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial \theta^2} - \frac{\partial v}{\partial \theta}\right) + 4D_{66}\frac{1}{R^2}\left(\frac{\partial^2 w}{\partial x \partial \theta} - \frac{\partial v}{\partial x}\right)^2 \Bigg\} R\,\mathrm{d}\theta\,\mathrm{d}x. \end{aligned} \tag{26}$$

2.4. Lagrangian Energy Functional

The Lagrangian energy functional Π, is the difference of the two types of shell energies and defined as:

$$\Pi = T - U \tag{27}$$

2.5. Rayleigh-Ritz Method

The energy variation methods *i.e.*, Rayleigh Ritz and Galerkin methods are the most frequently used ones to analyze the shell vibrational behavior. In the Rayleigh Ritz method, the energy variational functional is minimized with respect to the coefficients of an approximating series representing the displacement deformations. Many researchers such as Sewall and Naumann [10], Sharma and Johns [15], Loy *et al.* [18] and Naeem and Sharma [17] used this procedure to analyze the vibration characteristics of the cylindrical shells.

2.6. Axial Modal Dependence

The expressions for the modal displacement deformations are presumed in the form of product of functions of space and time variables. This leads to a system of ordinary differential equations of three unknown functions of the axial space variable. Different types of functions are chosen to approximate the axial modal dependence. These functions satisfy the boundary conditions. Well-known functions are beam functions, Ritz polynomial functions, orthogonal polynomials and Fourier series of the circular functions. The expression for modal displacement deformations are assumed as:

$$u(x,\theta,t) = A\frac{\mathrm{d}\phi}{\mathrm{d}x}\cos n\theta \sin \omega t \tag{28}$$

$$v(x,\theta,t) = B\phi \sin n\theta \sin \omega t \tag{29}$$

$$w(x,\theta,t) = C\phi \cos n\theta \sin \omega t \tag{30}$$

In the axial, circumferential and radial directions respectively, the coefficients A, B and C are the constants denoting the amplitudes of the vibrations in the x, θ and z directions respectively, n is the circumferential wave number and ω is the natural angular frequency for the cylindrical shell. The axial function $\phi(x)$ is chosen as the beam function as:

$$\phi(x) = \alpha_1 \cosh\left(\frac{\lambda_m x}{L}\right) + \alpha_2 \cos\left(\frac{\lambda_m x}{L}\right) - \varsigma_m\left(\alpha_3 \sinh\left(\frac{\lambda_m x}{L}\right) + \alpha_4 \sin\left(\frac{\lambda_m x}{L}\right)\right) \tag{31}$$

where $\alpha_i\left(i=1,4\right)$ are some constants with value 0 or 1 chosen according to the boundary condition. λ_m are the roots of some transcendental equations and ς_m are some parameters dependent on λ_m. Their values are given as in **Table 1**:

The geometric boundary conditions for clamped, free and simply supported boundary conditions can be expressed mathematically in terms of characteristic beam function $\phi(x)$ as:

Clamped boundary condition $\phi(x)=\phi'(x)=0$

Free boundary condition $\phi''(x)=\phi'''(x)=0$

Simply supported boundary condition $\phi(x)=\phi''(x)=0$

2.7. Derivation of Frequency Equation

On substituting the expressions for the deformation displacements u, v and w in the expression for the strain and kinetic energies of the cylindrical shells and employing the principle of minimization of the energy the expression for maximum strain and kinetic energies are obtained. The new form of the Lagrangian functional is formed as:

$$\Pi = T_{\max} - U_{\max} \tag{32}$$

where

$$\begin{aligned} U_{\max} = \frac{\pi R}{2}\int_0^L \Bigg[& A_{11}A^2\left(\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}\right)^2 + \frac{1}{R^2}A_{22}\left(nB+C\right)^2\varphi^2 + \frac{2}{R}A_{12}\left(nAB+AC\right)\varphi\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + A_{66}\left(B-\frac{nA}{R}\right)^2\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)^2 \\ & -2B_{11}AC\left(\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}\right)^2 + \frac{2}{R^3}B_{22}\left(nB+C\right)\left(n^2C+nB\right)\varphi^2 + \frac{2}{R^2}B_{12}\left(n^2AC+nAB\right)\varphi\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} \\ & -\frac{2}{R}B_{12}\left(nBC+C^2\right)\varphi\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + \frac{4}{R}B_{66}\left(B-\frac{nA}{R}\right)\left(nC+B\right)\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)^2 + D_{11}C^2\left(\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}\right)^2 \\ & +\frac{1}{R^4}D_{22}\left(n^2C+nB\right)^2\varphi^2 - \frac{2}{R^2}D_{12}\left(n^2C^2+nBC\right)\varphi\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + \frac{4}{R^2}D_{66}\left(nC+B\right)^2\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)^2\Bigg]\mathrm{d}x. \end{aligned} \tag{33}$$

$$T_{\max} = \frac{\pi R}{2}\rho_T\omega^2\int_0^L\left[\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2A^2 + \phi^2B^2 + \phi^2C^2\right]\mathrm{d}x \tag{34}$$

To derive the shell frequency equation, the energy functional is extremized with respect to the vibration amplitudes: *A*, *B* and *C*, resulting in three homogenous linear following equations:

$$\frac{\partial\Pi}{\partial A} = \frac{\partial\Pi}{\partial B} = \frac{\partial\Pi}{\partial C} = 0 \tag{35}$$

By re-arranging Equation (35), the shell frequency equation is written in the eigenvalue form as:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \rho h\omega^2\begin{bmatrix} I_2 & 0 & 0 \\ 0 & I_4 & 0 \\ 0 & 0 & I_4 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \end{bmatrix} \tag{36}$$

where the expressions for the terms C_{ij}'s, I_2 and I_4 are the given in **Appendix**.

3. Results and Discussion

A number of comparison of the results for isotropic and FGM cylindrical shells are presented to verify the validity, efficiency and accuracy of the present approach. The present analysis is carried out by using the energy variational procedure viz: Rayleigh-Ritz method. This method is based on the principle of minimization of energy. The numerical results for the following three frequently encountered sets of boundary conditions are evaluated to check the validity, efficiency and accuracy of the present technique.

Table 1. Six commonly used boundary conditions.

Boundary Conditions	$\alpha_i\,(i=1,4)$	Values for λ_m	ς_m
SS-SS	$\alpha_1=\alpha_2=\alpha_3=0,\ \alpha_4=-1$	$m\pi$	1
C-C	$\alpha_1=\alpha_3=1,\ \alpha_2=\alpha_4=-1$	$\cos\lambda_m\cosh\lambda_m=1$	$\frac{\cosh\lambda_m-\cos\lambda_m}{\sinh\lambda_m-\sin\lambda_m}$
F-F	$\alpha_1=\alpha_2=\alpha_3=\alpha_4=1$	$\cos\lambda_m\cosh\lambda_m=1$	$\frac{\cosh\lambda_m-\cos\lambda_m}{\sinh\lambda_m-\sin\lambda_m}$
C-SS	$\alpha_1=\alpha_3=1,\ \alpha_2=\alpha_4=-1$	$\tan\lambda_m=\tanh\lambda_m$	$\frac{\cosh\lambda_m-\cos\lambda_m}{\sinh\lambda_m-\sin\lambda_m}$
C-F	$\alpha_1=\alpha_3=1,\ \alpha_2=\alpha_4=-1$	$\cos\lambda_m\cosh\lambda_m=-1$	$\frac{\sinh\lambda_m-\sin\lambda_m}{\cosh\lambda_m+\cos\lambda_m}$
F-SS	$\alpha_1=\alpha_2=\alpha_3=\alpha_4=1$	$\tan\lambda_m=\tanh\lambda_m$	$\frac{\cosh\lambda_m-\cos\lambda_m}{\sinh\lambda_m-\sin\lambda_m}$

- Simply supported-simply supported (SS-SS)
- Clamped-clamped (C-C)
- Clamped-free (C-F)

3.1. Isotropic Cylindrical Shells

In **Table 2**, the frequency parameters $\Omega=\omega R\sqrt{(1-v^2)\rho/E}$ for an isotropic cylindrical shell is compared with those ones evaluated by Swaddiwudhipong [30] for simply supported boundary conditions. The shell parameters are listed in this table. The comparison is analyzed for the cases: $L/R=20$ and $L/R=0.25$ with circumferential mode $n=1$ to 5. The absolute differences between the two sets of frequencies are very minute.

In **Table 3**, frequency parameter (Ω) for a cylindrical shell with clamped-clamped edge conditions are compared with those evaluated by Joseph and Haim [30] Shell properties are described in the table. It is noticed that for $n\le 7$, the present frequencies are larger where as for $n\ge 8$, the two frequencies are approximately equal. The fundamental frequency is associated with the circumferential mode number $n=4$.

In **Table 4**, natural frequencies (Hz) for a clamped-free cylindrical shells are compared with those calculated experimentally by Sewall and Nauman [10] for the axial wave number $m=1,\ 2,\ 3$. Experimental values of the shell frequency are lower than the present theoretical ones. This difference may be due to the some imperfection in the experimental devices. The lowest frequency is associated with the circumferential wave numbers $n=5,\ 8,\ 10$ for $m=1,\ 2,\ 3$ respectively.

3.2. FGM Cylindrical Shells

Table 5 represents a comparison of natural frequencies (Hz) for type-I FGM cylindrical shell configured according to those ones evaluated by Loy *et al.* [18] for simply supported boundary condition and power law exponents $p=0.5$, 1, 5. The minimum frequency occurs at the circumferential wave number $n=3$ which is about 0.009%, 0.001% and 0.007% less than those given in [18] whereas in Type II shell, lowest frequency corresponds to circumferential wave number $n=3$ which is about 0.01%, 0.009% and 0.021%, less than those evaluated in [18]. It is concluded from the above comparisons of shell frequencies that the present method is valid and efficient and gives fast and accurate results.

3.3. Three-Layered FGM Cylindrical Shells with Middle Layer of Isotropic Material

A three-layered FGM cylindrical shell whose cross section is shown in **Figure 1(b)**, is fabricated in such a way that the inner and outer layers of the shells is fabricated with FGM layers while an isotropic material is inserted at the middle layer of the shell. The thickness of each layer is assumed to be equal. The shell material parameters include the Young's modulus, Poisson's ratio and the mass-density. The Young's modulus is the most in-

Table 2. Comparison of frequency parameter $\Omega = \omega R\sqrt{(1-v^2)\rho/E}$ for cylindrical shell with simply supported conditions: $h/R = 0.002$, $m = 1$, $E = 30\times10^6$ lbf in^{-2}, $v = 0.3$, $\rho = 7.35\times10^{-4}$ lbf s^2 in^{-4}.

L/R	n	Swaddiwudhipong [30]	Present
20	1	0.016101	0.016101
	2	0.005453	0.005450
	3	0.005042	0.005034
	4	0.008534	0.008525
	5	------	0.013623
0.25	1	0.951993	0.951976
	2	0.934461	0.934342
	3	0.906732	0.906435
	4	0.87076	0.870196
	5	------	0.827882

Table 3. Comparison of frequency parameter $\Omega = \omega R\sqrt{(1-v^2)\rho/E}$ for cylindrical shell with clamped-clamped conditions: $L = 304.8$ mm, $h = 0.254$ mm, $R = 76.2$ mm, $m = 1$, $E = 2.0668\times10^{11}$ N/m^2, $v = 0.3$, $\rho = 7.85\times10^3$ Kg/m^3.

n	Joseph and Haim[30]	Present
3	0.1030	0.1097
4	0.0681	0.0715
5	0.0515	0.0532
6	0.0475	0.0482
7	0.0528	0.0529
8	0.0639	0.0638
9	0.0788	0.0785
10	0.0964	0.0960

Table 4. Comparison of natural frequencies (Hz) for cylindrical shell with clamped-free conditions: $R = 0.2423$ m, $h = 0.000648$ m, $L = 0.6255$ m, $E = 68.95\times10^9$ N/m^2, $v = 0.315$, $\rho = 2.7145\times10^3$ Kg/m^3.

n	Sewall and Nauman [10]			Present		
	$m = 1$	$m = 2$	$m = 3$	$m = 1$	$m = 2$	$m = 3$
2	---	---	---	354	1500.6	2346
3	155.0 157.0	---	---	182	912	1715
4	107.0	---	---	114	588	1248
5	89.0 91.0	341.0	---	95	407	925
6	102.0	276.0	---	106	306	707
7	130.0	240.0	---	134	256	561
8	166.0	227.0 231.0	---	172	243	469
9	208.0	246.0	400.0	217	259	418
10	260.0	281.0	---	267	294	403
11	317.0	337.0	409.0 412.0	324	341	415
12	374.0	393.0 396.0	---	385	398	449

Table 5. Comparison of natural frequencies (Hz) for a simply supported-simply supported type-I, II FGM cylindrical shell: $L/R = 20$, $h/R = 0.002$, $m = 1$.

n	**Loy *et al.* [18]**			**Present**		
	p			p		
	0.5	1.0	5.0	0.5	1.0	5.0
			Type I Shell			
1	13.321	13.211	12.998	13.331	13.210	12.988
2	4.5168	4.480	4.4068	4.5175	4.4790	4.4045
3	4.1911	4.1569	4.0891	4.1909	4.1560	4.0883
4	7.0972	7.0384	6.9251	7.0965	7.0371	6.9244
5	11.336	11.241	11.061	11.3350	11.2404	11.0603
			Type II Shell			
1	13.154	13.3210	13.526	13.1545	13.3210	13.5052
2	4.4550	4.5114	4.5836	4.4550	4.5115	4.5759
3	4.1309	4.1827	4.2536	4.1308	4.1829	4.2450
4	7.0076	7.0903	7.2085	7.0034	7.0909	7.1945
5	11.189	11.3293	11.516	11.1896	11.3305	11.4944

fluencing the shell vibrations characteristics. In this study the Poisson's ratio is assumed to be constant for functionally graded materials whereas the Young's modulus is a function of the intrinsic thickness variable (z) as well as the Young's moduli of the constituent materials forming functionally graded layers. The thickness of each layer is supposed to be of $h/3$. This variation of material thickness distribution modifies the stiffness moduli such as:

$$\begin{aligned} A_{ij} &= A_{ij}^{\text{in(FGM)}} + A_{ij}^{m(\text{isotropic})} + A_{ij}^{\text{out(FGM)}}, \\ B_{ij} &= B_{ij}^{\text{in(FGM)}} + B_{ij}^{m(\text{isotropic})} + B_{ij}^{\text{out(FGM)}}, \\ D_{ij} &= D_{ij}^{\text{in(FGM)}} + D_{ij}^{m(\text{isotropic})} + D_{ij}^{\text{out(FGM)}}. \end{aligned} \tag{37}$$

where i, $j = 1, 2, 6$ and in(FGM), out(FGM) are associated with inner and outer functionally graded layers respectively and *m*(isotropic) is related with the middle isotropic layer of a cylindrical shell. Their values are given in Appendix I. If we take M_1, M_2 constituent materials at the inner FGM layer and M_3, M_4 at the outer FGM layer, the resultant material properties Young's modulii, Poisson ratios and mass density of inner and outer FGM layers are given as:

$$\left.\begin{aligned} E^{\text{in(FGM)}} &= E_2 - E_1\left(3\frac{z}{h}+\frac{3}{2}\right)^p + E_1 \\ \nu^{\text{in(FGM)}} &= \nu_2 - \nu_1\left(3\frac{z}{h}+\frac{3}{2}\right)^p + \nu_1 \\ \rho^{\text{in(FGM)}} &= \rho_2 - \rho_1\left(3\frac{z}{h}+\frac{3}{2}\right)^p + \rho_1 \end{aligned}\right] \tag{38}$$

$$\left.\begin{aligned} E^{\text{out(FGM)}} &= E_4 - E_3\left(3\frac{z}{h}-\frac{1}{2}\right)^p + E_3 \\ \nu^{\text{out(FGM)}} &= \nu_4 - \nu_3\left(3\frac{z}{h}-\frac{1}{2}\right)^p + \nu_3 \\ \rho^{\text{out(FGM)}} &= \rho_4 - \rho_3\left(3\frac{z}{h}-\frac{1}{2}\right)^p + \rho_3 \end{aligned}\right] \tag{39}$$

By keeping isotropic material at the middle layer and by the variation of the constituents in the FGM layer as shown in **Figure 1(b)**, four types of cylindrical shells can be formed as shown in **Table 6**. Material properties of the isotropic material as well as FGM constituents are given in the reference [18] [19].

3.4. Variation of Volume Fractions of FGM Constituents at the Inner and Outer FGM Layers

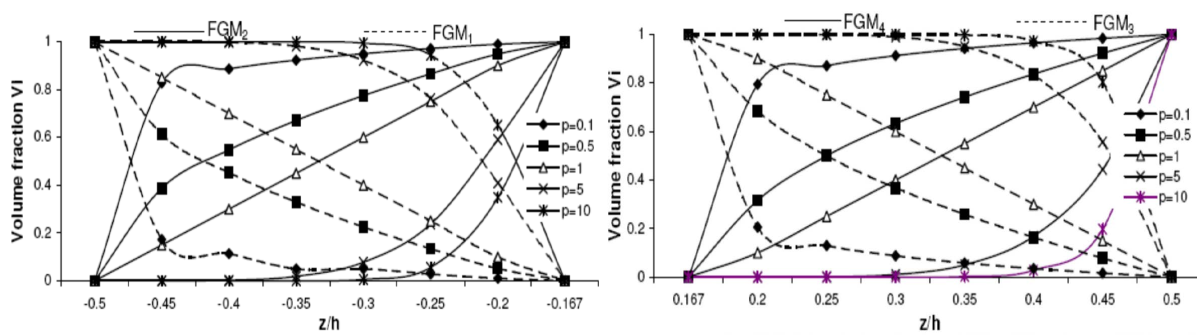
Material properties for inner and outer FGM layers of the cylindrical shell vary from $-h/2$ to $-h/6$ and from $+h/6$ to $+h/2$ respectively. From these relations, one can conclude that at $z=-h/2$, the effective material properties become $E=E_1$, $\nu=\nu_1$, $\rho=\rho_1$ whereas for $z=-h/6$, material properties become $E=E_2$, $\nu=\nu_2$ $\rho=\rho_2$ at the inner FGM layer and at $z=+h/6$, the material properties turn into $E=E_3$, $\nu=\nu_3$, $\rho=\rho_3$ while at $z=+h/2$, the material properties turn into $E=E_4$, $\nu=\nu_4$, $\rho=\rho_4$ for functionally graded outer layer of the cylindrical shell.

These results conclude that the material properties vary smoothly and continuously of constituent materials M_1 and M_3 from the inner surface to M_2 and M_4 to the outer surface of both the FGM layers respectively. Similar response of the material properties is seen in the inverse direction. Variation of volume fractions V_1, V_3 and V_2, V_4 of constituent materials M_1, M_3 and M_2, M_4 placed at the inner and the outer shell surfaces at the inner and outer FGM layers respectively of the shell are sketched in **Figure 2(a)**, **Figure 2(b)** of the three-layered cylindrical shells. In **Figure 2(a)**, variation of volume fractions V_1 and V_2 of the constituent materials M_1 and M_2 are sketched for the shell inner FGM layer. In this layer, the volume fraction V_1 of constituent material M_1 declines from its highest value 1 to its lowest value 0 whereas the volume fraction V_2 of material M_2 rises from 0 to 1 in the thickness interval $-0.5 \le z/h \le -0.167$. Similar behaviour of volume fractions V_3 and V_4 of the FGM constituents M_3 and M_4 is shown in **Figure 2(b)** for the shell outer FGM layer. Volume fraction V_3 of material M_3 reduces from 1 to 0 and volume fraction V_3 of constituent material M_3 advances from 0 to its maximum value 1 in the thickness variable interval $+0.167 \le z/h \le +0.50$ respectively. The middle layer is made up with some isotropic material, whose thickness span over the interval $-0.167 \le z/h \le +0.167$.

Now to study the influence of power law exponent p on the volume fraction of the constituents in the FGM layers, M_1 and M_3 are pure while M_2 and M_4 have zero concentration at the inner surface of the inner

Table 6. Configuration of types of FGM cylindrical shells.

Type of Shell	Inner FGM layer	Isotropic Layer	Outer FGM layer
Shell I	Nickel-Zarconia	Stainless Steel	Nickel-Zarconia
Shell II	Nickel-Zarconia	Stainless Steel	Zarconia-Nickel
Shell III	Zarconia-Nickel	Stainless Steel	Nickel-Zarconia
Shell IV	Zarconia-Nickel	Stainless Steel	Zarconia-Nickel



(a) (b)

Figure 2. (a) Variation of volume fractions FGM_1 & FGM_2 of Materials M_1 and M_2 at the inner FGM layer of the three layered cylindrical shells; (b) Variation of volume fractions FGM_3 & FGM_4 of Materials M_3 and M_4 at the outer FGM layer of the three layered cylindrical shells.

and outer FGM layers. Similar but opposite behaviour of the FGM constituents is observed at outer surface of both the layers. For the thickness interval $-0.50 \le z/h \le -0.45$ and $p < 1$, V_1 decreases while V_2 increase rapidly whereas for $-0.45 \le z/h \le -0.167$ and $p < 1$, V_1 decreases while V_2 increases slowly and constantly. For the thickness interval $-0.50 \le z/h \le -0.25$ and $p > 1$, V_1 decreases while V_2 increases gradually but for $-0.25 \le z/h \le -0.167$ and $p > 1$, V_1 decreases and V_2 increases swiftly. Eventually, V_1 and V_2 approach to their minimum and maximum values 0 and 1 respectively. Similar behaviour of volume fractions V_3 and V_4 of the constituent materials M_3 and M_4 are observed in the outer FGM layer of the three layered cylindrical shell containing middle layer of isotropic material but in this case thickness interval changes from $+0.167 \le z/h \le +0.50$.

3.5. Frequency Analysis of FGM Cylindrical Shell

In this section variation of natural frequencies (Hz) for four types of shell described in the above table will be discussed. The boundary conditions are taken to be simply supported-simply supported (SS-SS), clamped-clamed (C-C) and clamped-free (C-F).

3.5.1. Variation of Natural Frequencies (Hz) against Circumferential Wave Number for SS-SS, C-C and C-F Boundary Conditions

In **Table 7**, the variation of natural frequencies (Hz) for three sets of boundary conditions *i.e.* simply supported-simply supported (SS-SS), clamped-clamped (C-C) and clamped-free(C-F) are studied against the circumferential wave number n. In these types of shells the inner and outer layers are composed of the constituents nickel and zirconia while the middle isotropic layer is made of stainless steel. The shell parameters are $L/R = 20$, $h/R = 0.002$, $m = 1$ and $p = 0.5$. It is seen that the natural frequency first decreases and after at-

Table 7. Variation of natural frequencies (Hz) against circumferential wave number n for shells I, II, III, IV $(m = 1,\ R = 1,\ h = 0.002,\ L = 20,\ p = 0.5)$.

n	SS-SS	C-C	C-F
		Type I Shell	
1	16.1932	34.6722	6.1921
2	5.4819	12.1477	2.5145
3	5.0470	7.2610	4.4855
4	8.5343	9.0512	8.4259
5	13.6324	13.7689	13.6005
		Type II Shell	
1	16.0239	34.3095	6.1273
2	5.4174	12.0174	2.4726
3	4.9318	7.1420	4.3684
4	8.3093	8.8288	8.2003
5	13.2676	13.4052	13.2358
		Type III Shell	
1	16.0239	34.3095	6.1273
2	5.4275	12.0219	2.4946
3	5.0199	7.2029	4.4673
4	8.5006	9.0089	8.3940
5	13.5807	13.7149	13.5494
		Type IV Shell	
1	15.8527	33.9430	6.0619
2	5.3624	11.8903	2.4524
3	4.9040	7.0828	4.3497
4	8.2747	8.7854	8.1676
5	13.2145	13.3496	13.1832

taining its minimum value it begins to increase with the circumferential wave number n. It is also noticed that the natural frequencies for three boundary conditions become closer and closer with the increase of circumferential wave number *n*. The frequencies associated with the clamped-free boundary condition are the lowest among those with the clamped-clamped and simply supported-simply supported boundary conditions. The variation of the frequency is similar to that of an isotropic shell.

3.5.2. Variation of Natural Frequencies (Hz) against Length-to-Radius Ratio

In Table 8, the variation of natural frequencies (Hz) against length-to-radius ratio L/R of the four types of three-layered cylindrical shells are studied for three boundary conditions simply supported-simply supported, clamped-clamped and clamped-free. The shell parameters are $h/R = 0.002$ with circumferential wave number $n = 4$ and the fundamental axial mode $m = 1$.

From tables it is observed that the natural frequency (Hz) for all three boundary conditions decreases with the increase of length to radius ratio. It is also seen that the frequencies related to the clamped-clamped boundary conditions are greater than the simply supported-simply supported and clamped-free boundary conditions and the difference between the frequencies for all boundary conditions becomes very negligible at $L/R > 20$.

3.5.3. Variation of Natural Frequencies (Hz) against Thickness-to-Radius Ratio

In Table 9, variation of natural frequencies of the three-layered cylindrical shell is calculated against the thickness-to-radius ratio under the three selected boundary conditions *i.e.* simply supported-simply supported, clamped-clamped and clamped-free. In this observation the shell geometrical parameters are $L/R = 20$, the fundamental axial mode is $m = 1$, and the circumferential wave number $n = 4$. From tables it is seen that the

Table 8. Variation of natural frequencies (Hz) against L/R ratios for shell I, II, III, IV $(n = 4,\ m = 1,\ h/R = 0.002,\ p = 0.5)$.

L/R	SS-SS	C-C	C-F
		Type I Shell	
0.5	676.7750	720.7210	442.8185
1.0	356.1130	438.4916	171.9393
5.0	23.9693	48.6353	12.0134
10	10.1874	15.5551	8.6910
50	8.4088	8.4222	8.4054
		Type II Shell	
0.5	669.6757	713.1185	438.1774
1.0	352.3808	433.8956	170.1311
5.0	23.6691	48.1026	11.7902
10	9.9667	15.3182	8.4666
50	8.1833	8.1968	8.1799
		Type III Shell	
0.5	669.7055	713.2100	438.1912
1.0	352.3917	433.9097	170.1452
5.0	23.7391	48.1366	11.9279
10	10.1277	15.4230	8.6548
50	8.3771	8.3903	8.3738
		Type IV Shell	
0.5	662.5305	705.5265	433.5006
1.0	348.6196	429.2646	168.3177
5.0	23.4359	47.5983	11.7031
10	9.9057	15.1841	8.4294
50	8.1507	8.1640	8.1474

Table 9. Variation of natural frequencies (Hz) against h/R ratios for shell I. $(n=4,\ m=1,\ R=1,\ L=20,\ p=0.5)$.

h/R	SS-SS	C-C	C-F
		Type I Shell	
0.001	4.4440	5.3762	4.2386
0.005	21.0814	21.2855	21.0286
0.01	42.0892	42.1729	42.0464
0.03	126.1878	126.1485	126.1154
0.05	210.2552	210.1509	210.1423
		Type II Shell	
0.001	4.3324	5.2663	4.1260
0.005	20.5174	20.7230	20.4646
0.01	40.9607	41.0458	40.9183
0.03	122.8027	122.7657	122.7320
0.05	204.6163	204.5155	204.5063
		Type III Shell	
0.001	4.4242	5.3420	4.2222
0.005	21.0015	21.2019	20.9494
0.01	41.9306	42.0126	41.8882
0.03	125.7130	125.6734	125.6409
0.05	209.4636	209.3594	209.3512
		Type IV Shell	
0.001	4.3121	5.2314	4.1091
0.005	20.4352	20.6372	20.3832
0.01	40.7977	40.8810	40.7558
0.03	122.3149	122.2775	122.2445
0.05	203.8029	203.7022	203.6933

natural frequencies for all three boundary conditions are very close to each other. For every boundary condition, it is also observed from the tables that the natural frequencies increase as the thickness-to-radius ratio increase.

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Appendix

$$C_{11} = A_{11}I_1 + \frac{n^2}{R^2}A_{66}I_2$$

$$C_{12} = \frac{n}{R}\left(A_{12} + \frac{1}{R}B_{12}\right)I_3 - \frac{n}{R}\left(A_{66} + \frac{2}{R}B_{66}\right)I_2$$

$$C_{13} = \frac{1}{R}\left(A_{12} + \frac{n^2}{R}B_{12}\right)I_3 - B_{11}I_1 - \frac{2n^2}{R^2}B_{66}I_2$$

$$C_{22} = \frac{n^2}{R^2}\left(A_{22} + \frac{2}{R}B_{22} + \frac{1}{R^2}D_{22}\right)I_4 + \left(A_{66} + \frac{4}{R}B_{66} + \frac{4}{R^2}D_{66}\right)I_2$$

$$C_{23} = \frac{n}{R^2}\left(A_{22} + \frac{1+n^2}{R}B_{22} + \frac{n^2}{R^2}D_{22}\right)I_4 - \frac{n}{R}\left(B_{12} + \frac{1}{R}D_{12}\right)I_3 + \frac{2n}{R}\left(B_{66} + \frac{2}{R}D_{66}\right)I_2$$

$$C_{33} = \frac{1}{R^2}\left(A_{22} + \frac{2n^2}{R}B_{22} + \frac{n^4}{R^2}D_{22}\right)I_4 - \frac{2}{R}\left(B_{12} + \frac{n}{R}D_{12}\right)I_3 + D_{11}I_1 + \frac{4n^2}{R^2}D_{66}I_2$$

where

$$I_1 = \frac{m^4\pi^4}{2L^3}, \quad I_2 = \frac{m^2\pi^2}{2L}$$

$$I_3 = -\frac{m^2\pi^2}{2L}, \quad I_4 = \frac{L}{2}$$

$$A_{11} = A_{22} = \frac{h}{3(1-\nu^2)}\left(\frac{E_1 - E_2}{p+1} + E_2\right) + \frac{hE}{3(1-\nu^2)} + \frac{h}{3(1-\nu^2)}\left(\frac{E_3 - E_4}{p+1} + E_4\right)$$

$$A_{12} = \frac{h\nu}{3(1-\nu^2)}\left(\frac{E_1 - E_2}{p+1} + E_2\right) + \frac{h\nu E}{3(1-\nu^2)} + \frac{h\nu}{3(1-\nu^2)}\left(\frac{E_3 - E_4}{p+1} + E_4\right)$$

$$A_{66} = \frac{h}{6(1+\nu)}\left(\frac{E_1 - E_2}{p+1} + E_2\right) + \frac{hE}{6(1+\nu)} + \frac{h}{6(1+\nu)}\left(\frac{E_3 - E_4}{p+1} + E_4\right)$$

$$B_{11} = B_{22} = \frac{-h^2}{9(1-\nu^2)}\left(\frac{(E_1 - E_2)(p+4)}{2(p^2+3p+2)} + E_2\right) + \frac{h^2}{9(1-\nu^2)}\left(\frac{(E_3 - E_4)(3p+4)}{2(p^2+3p+2)} + E_4\right)$$

$$B_{12} = \frac{-h^2\nu}{9(1-\nu^2)}\left(\frac{(E_1 - E_2)(p+4)}{2(p^2+3p+2)} + E_2\right) + \frac{h^2\nu}{9(1-\nu^2)}\left(\frac{(E_3 - E_4)(3p+4)}{2(p^2+3p+2)} + E_4\right)$$

$$B_{66} = \frac{-h^2}{18(1+\nu)}\left(\frac{(E_1 - E_2)(p+4)}{2(p^2+3p+2)} + E_2\right) + \frac{h^2}{18(1+\nu)}\left(\frac{(E_3 - E_4)(3p+4)}{2(p^2+3p+2)} + E_4\right)$$

$$D_{11} = D_{22} = \frac{h^3}{108(1-\nu^2)}\left(\frac{(E_1 - E_2)(p^2+9p+26)}{p^3+6p^2+11p+6} + \frac{13}{3}E_2\right) + \frac{h^3E}{324(1-\nu^2)} + \frac{h^3}{108(1-\nu^2)}\left(\frac{(E_3 - E_4)(p^2+33p+26)}{p^3+6p^2+11p+6} + \frac{13}{3}E_4\right).$$

$$D_{12} = \frac{h^3\nu}{108\left(1-\nu^2\right)}\left(\frac{\left(E_1 - E_2\right)\left(p^2 + 9p + 26\right)}{p^3 + 6p^2 + 11p + 6} + \frac{13}{3}E_2\right) + \frac{h^3 E\nu}{324\left(1-\nu^2\right)}$$

$$+ \frac{h^3\nu}{108\left(1-\nu^2\right)}\left(\frac{\left(E_3 - E_4\right)\left(p^2 + 33p + 26\right)}{p^3 + 6p^2 + 11p + 6} + \frac{13}{3}E_4\right).$$

$$D_{66} = \frac{h^3}{216\left(1+\nu\right)}\left(\frac{\left(E_1 - E_2\right)\left(p^2 + 9p + 26\right)}{p^3 + 6p^2 + 11p + 6} + \frac{13}{3}E_2\right) + \frac{h^3 E}{648\left(1+\nu\right)}$$

$$+ \frac{h^3}{216\left(1+\nu\right)}\left(\frac{\left(E_3 - E_4\right)\left(9p^2 + 33p + 26\right)}{p^3 + 6p^2 + 11p + 6} + \frac{13}{3}E_4\right).$$