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A Finite-Dimensional Integrable System Related to the Complex 3 × 3 Spectral Problem and the Coupled Nonlinear Schrödinger Equation

Lanxin Chen, Junxian Zhang

Shijiazhuang University, Shijiazhuang 050035, China
Email: LXCHENmath@163.com

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Abstract

The relation between the 3 × 3 complex spectral problem and the associated completely integrable system is generated. From the spectral problem, we derived the Lax pairs and the evolution equation hierarchy in which the coupled nonlinear Schrödinger equation is included. Then, with the constraints between the potential function and the eigenvalue function, using the nonlineared Lax pairs, a finite-dimensional complex Hamiltonian system is obtained. Furthermore, the representation of the solution to the evolution equations is generated by the commutable flows of the finite-dimensional completely integrable system.

Keywords

Integrable System, Evolution Equation, Spectral Problem, Commutable Flows

1. Introduction

As is well known, the technique of the nonlinearization of Lax pairs has been a powerful tool for the finding of integrable systems in the last two decades or so. With this technique, the representation of the solution to the systems can also be generated. A lot of researches have been made in this way [1]-[3]. So far higher order matrix spectral problem and complex spectral problem are all attractive to the mathematical and physical science. However, due to theoretical difficulty and the complexity of computation, the relevant research is relatively rare for the present.

In this paper, we present a 3 × 3 AKNS matrix spectral problem

$$\phi_x = M(\mathrm{u},\xi)\phi, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \quad M(\mathrm{u},\xi) = \begin{pmatrix} i\xi & u_1 & u_2 \\ v_1 & -i\xi & 0 \\ v_2 & 0 & -i\xi \end{pmatrix} \tag{1.1}$$

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where the potential $u=(u_1,v_1,u_2,v_2)^T$, $u_1=u_1(x,t), u_2=u_2(x,t), v_1=v_1(x,t), v_2=v_2(x,t)$ are complex-valued potential functions, ξ is a complex spectral parameter, $i=\sqrt{-1}$. The relation between this 3rd-order complex spectral problem and the associated completely integrable system is considered. We derived the related evolution equation hierarchy, one of which is often referred to on the literature as the coupled nonlinear Schrödinger equation:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t = \begin{pmatrix} -iu_{1_{xx}} + 2iu_1(|u_1|^2 + |u_2|^2) \\ -iu_{2_{xx}} + 2iu_2(|u_1|^2 + |u_2|^2) \end{pmatrix}, \tag{1.2}$$

which is used by Manakov for studying the propagation of the electric field in a waveguide [4]. Each equation governs the evolution of one of the components of the field transverse to the direction of propagation. Also it can be derived as a model for wave propagation under conditions similar to those where nonlinear Schrödinger equation applies and there are two wavetrains moving with nearly the same group velocity [5]. In recent years, this system is widely studied [6] [7] and used as a key model in the field of optical solitons in fibers [8] [9] to explain how the solitons waves transmit in optical fiber, what happens when the interaction among optical solitons influences directly the capacity and quality of communication and so on [10]-[12].

2. The Evolution Equations and Their Lax Pairs

Now, suppose Ω is the basic interval, if u,ϕ and their derivatives on x are all decay at infinity, then $\Omega=(-\infty,+\infty)$; if they are all periodic T functions, then $\Omega=[0,2\pi]$. In order to get the evolution equations, we first solve the stationary zero-curvature equation:

$$V_x - [M,V] = 0, \tag{2.1}$$

where

$$V = (V_{lk})_{3\times 3,} \qquad V_{lk} = \sum_{j=0}^{\infty} a_{lk}^{(j-1)} \xi^{-j}. \tag{2.2}$$

The auxiliary problem of the spectral problem is set as follows:

$$\phi_{t_m} = V_m(u,\xi)\phi, \quad V_m(u,\xi) = (\xi^m V)_+ = \sum_{j=0}^{m} a_{lk}^{(j-1)} \xi^{m-j} \tag{2.3}$$

Let $\partial\partial^{-1} = \partial^{-1}\partial = 1$ and the initial value

$$a_{11}^{(-1)} = 2i,\ a_{22}^{(-1)} = a_{33}^{(-1)} = -2i,\ a_{21}^{(-1)} = a_{12}^{(-1)} = a_{31}^{(-1)} = a_{13}^{(-1)} = a_{23}^{(-1)} = a_{32}^{(-1)} = 0. \tag{2.4}$$

So the Lenard recursive sequence

$$g_j = (a_{21}^{(j)}, a_{12}^{(j)}, a_{31}^{(j)}, a_{13}^{(j)})^T,\ j=-1,0,1,2,\cdots$$

is obtained and the Lenard recursive equation

$$Jg_{j+1} = Kg_j, \qquad j=0,1,2,3,\ldots \tag{2.5}$$

is given, where K and J are two bi-Hamiltonian operators [13]

$$J = \begin{pmatrix} 0 & 2i & 0 & 0 \\ -2i & 0 & 0 & 0 \\ 0 & 0 & 0 & 2i \\ 0 & 0 & -2i & 0 \end{pmatrix}, \quad K = \left(K_{lm}\right)_{4\times 4}$$

$$K_{11} = 2u_1\partial^{-1}u_1, K_{12} = \partial - 2u_1\partial^{-1}v_1 - u_2\partial^{-1}v_2, K_{13} = u_1\partial^{-1}u_2 + u_2\partial^{-1}u_1, K_{14} = -u_1\partial^{-1}v_2,$$

$$K_{21} = \partial - 2v_1\partial^{-1}u_1 - v_2\partial^{-1}u_2, K_{22} = 2v_1\partial^{-1}v_1, K_{23} = -v_1\partial^{-1}u_2, K_{24} = v_2\partial^{-1}v_1 + v_1\partial^{-1}v_2,$$

$$K_{31} = u_1\partial^{-1}u_2 + u_2\partial^{-1}u_1, K_{32} = -u_2\partial^{-1}v_1, K_{33} = 2u_2\partial^{-1}u_2, K_{34} = \partial - 2u_2\partial^{-1}v_2 - u_1\partial^{-1}v_1,$$

$$K_{41} = -v_2\partial^{-1}u_1, K_{42} = v_1\partial^{-1}v_2 + v_2\partial^{-1}v_1, K_{43} = \partial - v_1\partial^{-1}u_1 - 2v_2\partial^{-1}u_2, K_{44} = 2v_2\partial^{-1}v_2.$$

and

$$Jg_{-1} = 0$$

The isospectral evolution equations are

$$u_{t_m} = Jg_m = Kg_{m-1}, \quad \mathrm{m} = 1,2,3,... \tag{2.6}$$

By (2.1)-(2.5), we have

Theorem 2.1.

$$\begin{cases} \phi_x = M(u,\xi)\phi \\ \phi_{t_m} = V_m(u,\xi)\phi, \end{cases}$$

is the Lax forms of evolution Equation (2.5). In other words, the hierarchy of solition Equation (2.5) is a isospectral compatible condition of (2.6).

Especially, if we take

$$a_{11}^{(0)} = a_{22}^{(0)} = a_{33}^{(0)} = a_{23}^{(0)} = a_{32}^{(0)} = 0, \tag{2.7}$$

$$g_0 = (a_{21}^{(0)}, a_{12}^{(0)}, a_{31}^{(0)}, a_{13}^{(0)})^T = (2v_1, 2u_1, 2v_2, 2u_2)^T, \tag{2.8}$$

$$g_1 = (a_{21}^{(1)}, a_{12}^{(1)}, a_{31}^{(1)}, a_{13}^{(1)})^T = (iv_{1x}, -iu_{1x}, iv_{2x}, -iu_{2x})^T, \tag{2.9}$$

By (2.5),

$$u_{t_1} = \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix}_{t_1} = \begin{pmatrix} 2u_{1_x} \\ 2v_{1_x} \\ 2u_{2_x} \\ 2v_{2_x} \end{pmatrix} \qquad (a \ \ trivial \ \ case).$$

$$u_{t_2} = \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix}_{t_2} = \begin{pmatrix} -iu_{1_{xx}} + 2iu_1(u_1v_1 + u_2v_2) \\ iv_{1_{xx}} - 2iv_1(u_1v_1 + u_2v_2) \\ -iu_{2_{xx}} + 2iu_2(u_1v_1 + u_2v_2) \\ iv_{2_{xx}} - 2iv_2(u_1v_1 + u_2v_2) \end{pmatrix},$$

if $u_1 = v_1^*$, it is exactly the coupled nonlinear Schrödinger Equation (1.2) which is a well-known equation and is of great value in physics (where the symbol * denotes the complex conjugate).

3. A Finite-Dimensional Hamiltonian System

In order to give the constraints between the potential and the eigenfunction, First, the complex representation of the Poisson bracket is discussed.

The Poisson bracket of the real-valued function $\bar{F}, \bar{H}$ in the symplectic space $(\varpi = \sum_{j=1}^{3} dq_j \wedge dp_j, R^{12N})$ is defined as follows:

$$(\bar{F}, \bar{H}) = \sum_{j=1}^{3}\sum_{k=1}^{2N}\left(\frac{\partial \bar{F}}{\partial q_{jk}}\frac{\partial \bar{H}}{\partial p_{jk}} - \frac{\partial \bar{F}}{\partial p_{jk}}\frac{\partial \bar{H}}{\partial q_{jk}}\right) = \sum_{j=1}^{3}(\langle \bar{F}_{qj}, \bar{H}_{pj}\rangle - \langle \bar{F}_{pj}, \bar{H}_{qj}\rangle).$$

The Poisson bracket of the complex-valued function F, H in the symplectic space $(\omega = \sum_{j=1}^{3} dy_j \wedge dz_j, C^{6N})$ is defined as follows:

$$\{F, H\} = \sum_{j=1}^{3}\sum_{k=1}^{2N}\left(\frac{\partial F}{\partial y_{jk}}\frac{\partial H}{\partial z_{jk}} - \frac{\partial F}{\partial z_{jk}}\frac{\partial H}{\partial y_{jk}}\right) = \sum_{j=1}^{3}(\langle F_{yj}, H_{zj}\rangle - \langle F_{zj}, H_{yj}\rangle).$$

Lemma 3.1. [15] Let

$$\begin{cases} y_1 = \frac{1}{\sqrt{2}}(p_1 + iq_3), & y_2 = \frac{1}{\sqrt{2}}(p_2 - iq_2), & y_3 = \frac{1}{\sqrt{2}}(p_3 + iq_1), \\ z_1 = \frac{1}{\sqrt{2}}(p_3 - iq_1), & z_2 = -\frac{1}{\sqrt{2}}(p_2 + iq_2), & z_3 = \frac{1}{\sqrt{2}}(p_1 - iq_3). \end{cases}$$

then the symplectic form $\varpi = \sum_{j=1}^{3} dq_j \wedge dp_j$ can be written $\omega = \sum_{j=1}^{3} dy_j \wedge dz_j$. the complex Poisson bracket is equivalent to the real Poisson bracket when the Hamiltonian functions H and F are all real-valued functions, namely,

$$\{F,H\} = \sum_{j=1}^{3}(\langle F_{\psi_j}, H_{\phi_j}\rangle - \langle F_{\phi_j}, H_{\psi_j}\rangle) = \sum_{j=1}^{3}(\langle F_{q_j}, H_{p_j}\rangle - \langle F_{p_j}, H_{q_j}\rangle) = (F,H).$$

Especially, if $h = ih^I, h^I \in R$, the complex Hamiltonian canonical equation

$$y_{j,t} = \{y_j, h\} = \frac{\partial h}{\partial z_j} \qquad z_{j,t} = \{z_j, h\} = -\frac{\partial h}{\partial y_j} \qquad j = 1,2,3$$

are equivalent to the real Hamiltonian canonical equation

$$q_{j,t} = (q_j, h^I) = \frac{\partial h^I}{\partial p_j} \qquad p_{j,t} = (p_j, h^I) = -\frac{\partial h^I}{\partial q_j} \qquad j = 1,2,3$$

Which plays an important role in the generation of the completely integrable system in the Liouville sense.

Consider the spectral problem (1.1) and it's adjoint spectral problem

$$\psi_x = -M^T(u, \xi\psi), \qquad \psi = (\psi_1, \psi_2, \psi_3)^T, \tag{3.1}$$

A direct calculation shows that

$$\int_\Omega \psi^T \dot{M} \phi dx = 0$$

where $\dot{M} = \frac{d}{d\epsilon}|_{\epsilon=0}[M(u + \epsilon\delta u)]$. Then [14]

$$\Delta\xi = \left(\frac{\delta\xi}{\delta u_1}, \frac{\delta\xi}{\delta v_1}, \frac{\delta\xi}{\delta u_2}, \frac{\delta\xi}{\delta v_2}\right)^T = \left(i\int_\Omega (\phi_1\psi_1 - \phi_2\psi_2 - \phi_3\psi_3)dx\right)^{-1}(\phi_2\psi_1, \phi_1\psi_2, \phi_3\psi_1, \phi_1\psi_3)^T \tag{3.2}$$

and

$$K\Delta\xi = \xi J \Delta\xi, \tag{3.3}$$

Now, suppose $\xi_k, k = 1,2,\cdots,N$ is an eigenvalue of (1.1) and (3.1), $(\phi_{1k}, \phi_{2k}, \phi_{3k})^T$, $(\psi_{1k}, \psi_{2k}, \psi_{3k})^T$ are the eigenfunctions for ξ_k, $\Lambda = diag(\xi_1, \xi_2, \cdots, \xi_N)$. Then, the spectral problem (1.1) and it's adjoint spectral problem (3.1) can be rewritten as follows:

$$\begin{pmatrix} \phi_{1k} \\ \phi_{2k} \\ \phi_{3k} \end{pmatrix}_x = \begin{pmatrix} i\xi_k & u_1 & u_2 \\ v_1 & -i\xi_k & 0 \\ v_2 & 0 & -i\xi_k \end{pmatrix} \begin{pmatrix} \phi_{1k} \\ \phi_{2k} \\ \phi_{3k} \end{pmatrix} = M(u, \xi_k) \begin{pmatrix} \phi_{1k} \\ \phi_{2k} \\ \phi_{3k} \end{pmatrix}, \tag{3.4}$$

$$\begin{pmatrix} \psi_{1k} \\ \psi_{2k} \\ \psi_{3k} \end{pmatrix}_x = -M^T(u, \xi_k) \begin{pmatrix} \psi_{1k} \\ \psi_{2k} \\ \psi_{3k} \end{pmatrix}. \tag{3.5}$$

Set $\Phi_j = (\phi_{j1}, \phi_{j2}, \cdots, \phi_{jN})^T, \Psi_j = (\psi_{j1}, \psi_{j2}, \cdots, \psi_{jN})^T, j = 1,2,3$. We consider the following constraint:

$$u=(u_1,v_1,u_2,v_2)^T=(i\langle\Phi_1,\Psi_2\rangle,i\langle\Phi_2,\Psi_1\rangle,i\langle\Phi_1,\Psi_3\rangle,i\langle\Phi_3,\Psi_1\rangle)^T \tag{3.6}$$

Substituting (3.6) into (3.4), (3.5), we can get the
Hamiltonian function F

$$F=f+f^* \tag{3.7}$$

where

$$f=i\langle\Lambda\Phi_1,\Psi_1\rangle-i\langle\Lambda\Phi_2,\Psi_2\rangle-i\langle\Lambda\Phi_3,\Psi_3\rangle+i\langle\Lambda\Phi_1,\Psi_2\rangle\langle\Lambda\Phi_2,\Psi_1\rangle+i\langle\Lambda\Phi_1,\Psi_3\rangle\langle\Lambda\Phi_3,\Psi_1\rangle$$

If the coordinates are as follows:

$$\begin{cases} y_1=(\Phi_1,\Phi_1^{*T}), \quad y_2=(\Phi_2,\Phi_2^*)^T, \quad y_3=(\Phi_3,\Phi_3^*)^T, \\ z_1=(\Psi_1,\Psi_1^*)^T, \quad z_2=(\Psi_2,\Psi_2^*)^T, \quad z_3=(\Psi_3,\Psi_3^*)^T. \end{cases} \tag{3.8}$$

The following theorem immediately holds.

Theorem 3.2. On the constraint (3.6), (3.4) and (3.5) with their conjugate representations are equal to the Hamiltonian canonical system

$$\begin{cases} y_{jx}=\dfrac{\partial F}{\partial z_j}, \\ z_{jx}=-\dfrac{\partial F}{\partial y_j}, \end{cases} \quad j=1,2,3. \tag{3.9}$$

Define the Hamiltonian function as follows:

$$F_0=f_0+f_0^*, \quad F_1=f_1+f_1^*, \quad F_m=f_m+f_m^*, \; m=2,3,\ldots \tag{3.10}$$

where

$$f_0=2i\langle\Phi_1,\Psi_1\rangle-2i\langle\Phi_2,\Psi_2\rangle-2i\langle\Phi_3,\Psi_3\rangle,$$

$$f_1=2i\langle\Lambda\Phi_1,\Psi_1\rangle-2i\langle\Lambda\Phi_2,\Psi_2\rangle-2i\langle\Lambda\Phi_3,\Psi_3\rangle+2i\langle\Phi_1,\Psi_2\rangle\langle\Phi_2,\Psi_1\rangle+2i\langle\Phi_1,\Psi_3\rangle\langle\Phi_3,\Psi_1\rangle$$

$$\begin{aligned} f_m={}&2i\langle\Lambda^m\Phi_1,\Psi_1\rangle-2i\langle\Lambda^m\Phi_2,\Psi_2\rangle-2i\langle\Lambda^m\Phi_3,\Psi_3\rangle \\ &+\sum_{j=1}^{m}2i(\langle\Lambda^{j-1}\Phi_1,\Psi_2\rangle\langle\Lambda^{m-j}\Phi_2,\Psi_1\rangle+\langle\Lambda^{j-1}\Phi_1,\Psi_3\rangle\langle\Lambda^{m-j}\Phi_3,\Psi_1\rangle) \\ &+\sum_{j=2}^{m}i(\langle\Lambda^{j-1}\Phi_1,\Psi_1\rangle\langle\Lambda^{m-j}\Phi_1,\Psi_1\rangle+\langle\Lambda^{j-1}\Phi_2,\Psi_2\rangle\langle\Lambda^{m-j}\Phi_2,\Psi_2\rangle \\ &+\langle\Lambda^{j-1}\Phi_3,\Psi_3\rangle\langle\Lambda^{m-j}\Phi_3,\Psi_3\rangle+2\langle\Lambda^{j-1}\Phi_2,\Psi_3\rangle\langle\Lambda^{m-j}\Phi_3,\Psi_2\rangle). \end{aligned}$$

By (3.13)-(3.15), the following theorem hold.

Theorem 3.2. On the constraint (3.6), (3.4) and (3.5) with their conjugate representations are equal to the Hamiltonian canonical system

$$\begin{cases} y_{j,t_m}=\dfrac{\partial F_m}{\partial z_j}, \\ z_{j,t_m}=-\dfrac{\partial F_m}{\partial y_j}, \end{cases} \quad m=0,1,2,\ldots, \; j=1,2,3. \tag{3.11}$$

Theorem 3.4. [16] [17] Suppose $(y_1,y_2,y_3,z_1,z_2,z_3)$ is an involutive solution of the Hamiltonian canonical equation systems (3.9) (3.11), then

$$\begin{cases} u_1=i\langle\Phi_1,\Psi_2\rangle, \\ v_1=i\langle\Phi_2,\Psi_1\rangle, \\ u_2=i\langle\Phi_1,\Psi_3\rangle, \\ v_2=i\langle\Phi_3,\Psi_1\rangle \end{cases}$$

satisfies the evolution Equation (2.5).

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