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# Interplay between Carrier Polarization, Spin-Orbit Coupling and Exchange Field on Anomalous Hall Conductivity in the Presence of Magnetic Impurity in Mn Doped GaAs

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## Abstract

We develop a model Hamiltonian to treat anomalous Hall conductivity in dilute magnetic semiconductor (DMS) of type (III, Mn, V) considering the impurity potentials (potential due to interaction of spin of carriers with localized spin of dopant (Mn) and coulomb like potential). Using equation of motion in Green function together with Quantum Kubo-formula of conductivity, the anomalous Hall conductivity is calculated as function of spin-orbit coupling, exchange field and carrier polarization. The calculated result shows that at low impurity concentration, the interplay between spin polarization of carriers, spin-orbit coupling and exchange fields is crucial for existence of anomalous Hall conductivity. The monotonic increment of anomalous Hall conductivity with exchange field is observed for strong spin-orbit coupling limit. In weak spin-orbit coupling limit, the magnitude of anomalous Hall conductivity increases parabolically with the spin-orbit coupling. Our results provide an important basis for understanding the interplay between the spin polarization, spin-orbit coupling, and exchange field on anomalous Hall conductivity at low impurity concentration. The findings are also a key step to realize dissipationless quantum transport without external magnetic field.

## Keywords

Conductivity, Magnetic Impurity, Spin-Obit Coupling, Exchange Field, Carrier Polarization

## 1. Introduction

Transport properties of spin-polarized electrons receive considerable interest for their importance in basic science and for their potential in technological applications [1]. One of the peculiar phenomena of the conduction of spin-polarized electrons in magnetic metals that contains rich physics is the anomalous Hall effect (AHE). The anomalous Hall (AH) effect measurement has emerged as a powerful tool to gain deep insights into magnetic materials, ferromagnetism in diluted magnetic semiconductors (DMS), the materials with the best potential for spintronic devices [2] [3] ferromagnetic metals and magnetic topological insulators. Not only this, but also anomalous Hall (AH) effect is also one of the most fundamental transport properties of magnetic materials, in which the interplay between magnetism and spin-orbit coupling produces a transverse Hall voltage perpendicular to the applied current and the magnetization [4]. Although AHE was discovered more than a century ago, its mechanism is not yet well understood [1]. Many different mechanisms that contribute to the AHE were proposed by different scholars. Among them, one well-studied mechanism is the intrinsic mechanism that is related to the Berry curvature of electronic bands [5] and many experimental results are ascribed to this mechanism [6]-[11]. Moreover, S. Mekonnen and P. Singh have shown that the intrinsic version of AHC is quantized [12], later, the extrinsic scattering mechanisms such as skew-scattering (SS) [13] and side-jumps (SJ) [14]. The skew-scattering model predicts that there is a linear dependence of anomalous Hall resistivity ( $\rho_{AHE}$ ) on the longitudinal resistivity ( $\rho_{xx}$ ), *i.e.* $\rho_{AHE} \sim \rho_{XX}$ [13]; whereas the side-jump model predicts a quadratic dependence, $\rho_{AHE} \sim \rho_{xx}^2$ [14]. On the best of our knowledge, the detail investigation of interplay between carrier polarization, spin-orbit coupling and exchange field in the presence of impurity potential in DMS like $Ga_{1-x}Mn_x$ is not yet studied in detail. Hence, the goal of this paper is to examine the interplay between carrier polarization, spin-orbit coupling and dopant induced exchange field on anomalous Hall conductivity in the presence of magnetic impurity in $Ga_{1-x}Mn_xAs$ DMS. The rest of the paper is organized as follows. Section 2 the model Hamiltonian is developed. In Section 3 some mathematical steps are highlighted. In Section 4, numerical estimation is made using some experimental parameters. Section 5, main findings of the results were concluded and some of mathematical steps used in the Appendix Section.

The goal of this paper is to examine the interplay between spin-orbit coupling and dopant induced exchange field on anomalous Hall conductivity in the presence of magnetic impurity in $Ga_{1-x}Mn_xAs$ dilute magnetic semiconductor.

The rest of the paper is organized as follows. In Section 2, the model Hamiltonian is developed. In Section 3, we make numerical estimation using some experimental parameters a brief review of maximum likelihood estimate and its properties. We propose an alternative least square estimator in Section 4. We made conclusion of main findings of the result and some of mathematical steps used in Appendix Section.

## 2. Theoretical Model

We consider two dimensional hole gas (2 DhG) in the presence of Spin-orbit coupling considering the form of the usual Rashba term, exchange field, kinetic energy of itinerant holes and magnetic impurity resulting from disorder. Our Hamiltonian is in the form of

$$\hat{H} = \hat{H}_0 + \hat{H}_{Imp} \tag{1}$$

$$\hat{H}_0 = \frac{k^2\hbar^2}{2m^*} + \alpha_R\left(\hat{\sigma}_x k_y - \hat{\sigma}_y k_x\right) - h_{ex}\hat{\sigma}_z \tag{2}$$

whereas the perturbing term of Hamiltonian is given by

$$\hat{H}_{Imp} = \sum_j \left(JS_J \cdot \sigma + V\right)\delta\left(r - R_j\right), \tag{3}$$

where $\sigma_{x,y,z}$ is the Pauli matrix along $x, y, z$, $\alpha_R$ is Rashba spin-orbit coupling, $k_{x,y}$ is the wave vector along *x* and *y*, *J* is exchange coupling constant, $S_J$ is spin magnetic moment resulting from impurity atom located at site *J*. Here we are assuming that randomly distributed impurity atom ($Mn^{2+}$) interact with itinerant carriers (holes) with its spin and results exchange field $JS_j\sigma$ and it also affects with some potential (like coloumb) potential $V_i$, $S_i$ is impurity spin located at site *i*, and $\sigma$ is Pauli spin operator for holes, here the korncker delta $\delta\left(r - R_j\right)$ assures that the impurity atoms ($Mn^{2+}$) affects the hole if and only if it is located at ($r = R$). Of course, we don't know exactly position of impurities. Therefore, we need to perform configuration average (position average). The average can be done for only lower order of scattering. Employing change of variables, the unperturbed Hamiltonian in Equation (2) can be rewritten in second quantized form

$$\begin{aligned}\hat{H}_0 = &\sum_{k\sigma}\varepsilon(k)a^+_{k\sigma}a_{k\sigma} + \sum_k \frac{\alpha_R}{2}\left(\left(k_y + ik_x\right)a^+_{k\uparrow}a_{k\downarrow} + \left(k_y - ik_x\right)a^+_{k\downarrow}a_{k\uparrow}\right)\\ &- h_{ex}\sum_k\left(a^+_{k\uparrow}a_{k\uparrow} - a^+_{k\downarrow}a_{k\downarrow}\right)\end{aligned} \tag{4}$$

Now we use equation of motion in Green function to determine spin resolved Green function associated with unperturbed Hamiltonian in Equation (4)

$$\omega \ll \hat{A}, \hat{B} \gg_\omega = <[\hat{A}, \hat{B}]> + \ll [\hat{A}, \hat{H}], \hat{B} \gg_\omega \tag{5}$$

After detail mathematical manipulation

$$\hat{G}^{\uparrow\uparrow}_{kk} = \frac{\omega - \varepsilon(k) - h_{ex}}{\left(\omega - \varepsilon(k) - h_{ex}\right)\left(\omega - \varepsilon(k) + \nabla_{ex}\right) - \frac{\alpha_R^2 k^2}{4}} \tag{6}$$

$$\hat{G}^{\downarrow\uparrow}_{kk} = \frac{\alpha_R\left(k_y - ik_x\right)}{2\left(\omega - \varepsilon(k) - h_{ex}\right)}\left(\frac{\omega - \xi(k) - h_{ex}}{\left(\omega - \varepsilon(k) - \nabla_{ex}\right)\left(\omega - \xi(k) + h_{ex}\right) - \frac{\alpha_R^2 k^2}{4}}\right) \tag{7}$$

$$\hat{G}_{kk}^{\downarrow\downarrow} = \frac{\omega - \varepsilon(k) + h_{ex}}{(\omega - \varepsilon(k) + h_{ex})(\omega - \varepsilon(k) - h_{ex}) - \frac{\alpha_R^2 k^2}{4}} \tag{8}$$

$$\hat{G}_{kk}^{\uparrow\downarrow} = \frac{\frac{\alpha_R}{2}(k_y + ik_x)}{2\left((\omega - \varepsilon(k) - h_{ex})(\omega - \varepsilon(k) + h_{ex}) - \frac{\alpha_R^2 k^2}{4}\right)} \tag{9}$$

which can be written in matrix form in spin space as

$$\hat{G}_0^R = \begin{pmatrix} \hat{G}_{o\uparrow\uparrow}^R & \hat{G}_{o\uparrow\downarrow}^R \\ \hat{G}_{o\downarrow\uparrow}^R & \hat{G}_{o\downarrow\downarrow}^R \end{pmatrix} \tag{10}$$

Using Equations (6)-(9) into Equation (10), we have

$$\hat{G}_0^R = \begin{pmatrix} \frac{\omega - \varepsilon(k) - h_{ex}}{(\omega - \varepsilon(k) - h_{ex})(\omega - \varepsilon(k) + h_{ex}) - \frac{\alpha_R^2 k^2}{4}} & \frac{\frac{\alpha_R}{2}(k_y + ik_x)}{2\left((\omega - \varepsilon(k) - h_{ex})(\omega - \varepsilon(k) + h_{ex}) - \frac{\alpha_R^2 k^2}{4}\right)} \\ \frac{\frac{\alpha_R}{\hbar}(k_y - ik_x)}{2\left((\omega - \varepsilon(k) - h_{ex})(\omega - \varepsilon(k) + h_{ex}) - \frac{\alpha_R^2 k^2}{4}\right)} & \frac{\omega - \varepsilon(k) + h_{ex}}{(\omega - \varepsilon(k) + h_{ex})(\omega - \varepsilon(k) - h_{ex}) - \frac{\alpha_R^2 k^2}{4}} \end{pmatrix} \tag{11}$$

It is convenient to introduced the reference Green function $g_\pm^0$ to write Equation (11) more elegant form

$$g_\pm^0 = \frac{1}{\omega - \varepsilon(k) \pm \sqrt{\frac{\alpha_R^2 k^2}{4} + h_{ex}^2}} = \frac{1}{\omega - \varepsilon(k) \pm \zeta(k)} = \frac{1}{\omega - E_{k\pm}} \tag{12}$$

where $E_\pm(k) = \varepsilon(k) + \zeta_{sk} = \varepsilon(k) + \pm\sqrt{\alpha_R^2 k^2 + h_{ex}^2}$ is the eigenvalue of the bar (unperturbed) Hamiltonian. On the view of Equation (11), Equation (12) becomes

$$G_0^R = \frac{1}{2}(g_+^0 + g_-^0)\hat{I} + \frac{\alpha_R k_y}{2\lambda(k)}(g_+^0 - g_-^0)\hat{\sigma}_x - \frac{\alpha_R k_x}{2\lambda(k)}(g_+^0 - g_-^0)\hat{\sigma}_y - \frac{h_{ex}}{2\zeta(k)}(g_+^0 - g_-^0)\hat{\sigma}_z \tag{13}$$

Hence, the Green function along $\hat{\sigma}_i$ where $i$ designates identity matrix ($I$) and $x$, $y$ and $z$ components of Pauli spin operators is given by expression.

$$G_R^0 = G_I^{0R}\hat{I} + G_x^{0R}\hat{\sigma}_x + G_y^{0R}\hat{\sigma}_y + G_z^{0R}\hat{\sigma}_z \tag{14}$$

where,

$$G_I^{0R} = \frac{1}{2}(g_+^0 + g_-^0) \tag{15}$$

$$G_x^{0R} = \frac{\alpha_R k_y}{2\lambda(k)}(g_+^0 - g_-^0) \tag{16}$$

$$G_y^{0R} = -\frac{\alpha_R k_x}{2\lambda(k)}\left(g_+^0 - g_-^0\right) \quad (17)$$

$$G_z^{0R} = -\frac{h_{ex}}{2\lambda(k)}\left(g_+^0 - g_-^0\right) \quad (18)$$

## 3. Self Energy and Life Time

To treat the impurity part of the perturbed Hamiltonian we use Dyson series. After some mathematical simplification self energy can be found,

$$\begin{aligned}\Sigma = &\left\langle V_{imp}\right\rangle_c + \left\langle V_{imp}(r) G_0(z) V_{imp}(r)\right\rangle_c \\ &+ \left\langle V_{imp}(r) G_0(z) V_{imp}(r) G_0(z) V_{imp}(r)\right\rangle_c + \cdots\end{aligned} \quad (19)$$

This expression can be also obtained using diagrammatic rule. Since we are considering dilute limit (the concentration of impurity (Mn) is low). Therefore, disorder potential is considered to be weak, it is common to take only two terms (terms with linear with impurity ( $n_i$ ) from series of iterative equation, in this limit Born approximation is valid). The self energy in Equation (19) can be written in bases of $k$ and $k'$ (Fourier transform) as

$$\begin{aligned}\Sigma_{kk'} = &\overline{\left\langle k\left|V_{imp}\right|k'\right\rangle} + \sum_{kk'}\overline{\left\langle k\left|V_{imp}(r) G_0(z) V_{imp}(r)\right|k'\right\rangle} \\ &+ \sum_{kk'}\overline{\left\langle k V_{imp}(r) G_0(z) V_{imp}(r) G_0(z) V_{imp}(r) k'\right\rangle} + \cdots\end{aligned} \quad (20)$$

The first two leading order of self energy on the right side in Equation (20) become

$$\Sigma_{kk'} = \overline{\left\langle k\left|V_{imp}\right|k\right\rangle} + \sum_{kk'}\overline{\left\langle k\left|V_{imp}(r) G_0(z) V_{imp}(r)\right|k'\right\rangle} \quad (21)$$

From Equation (21), the first term gives only constant shift in energy spectrum and it has no effect on disorder boarding (life time which is proportional to imaginary part of self energy). Hence in born approximation the only remaining term,

$$\Sigma_{kk'} = \sum_{kk'}\overline{\left\langle k\left|V_{imp}(r) G_0(z) V_{imp}(r)\right|k'\right\rangle} \quad (22)$$

which can be written in compact form

$$\Sigma_{kk'} = \sum_{kk'} G_k^0 \overline{\left|V_{kk'}\right|^2} \quad (23)$$

From Equation (23), we plagout $G_k^0$ from configuration average since it is corresponding to non perturbative part of Hamiltonian (it is free from disorder). After impurity averaging together with detail mathematical manipulation one can obtain the impurity potential as

$$\overline{\left|V_{kk'}\right|^2} = \frac{1}{V^2}\sum_j^{N_{imp}}\sum_l^{N_{imp}}\overline{\left(V_j + JS_j\cdot\sigma\right)\exp\left(-i(k'-k)\cdot R_j\right)\left(V_l + JS_l\cdot\sigma\right)\exp\left(i(k'-k)\cdot R_l\right)} \quad (24)$$

For $j = l$, Equation (24) results

$$\overline{\left|V_{kk'}\right|^2} = \frac{n_{imp}}{V}\left(\overline{V + JS\cdot\sigma}\right)^2 \quad (25)$$

In Equation (24) we have used $\frac{N_{imp}}{V} = n_{imp}$ and substitution of Equation (25) into Equation (23) yields

$$\Sigma_{kk'} = \frac{n_{imp}}{V}\left(\overline{V + JS \cdot \sigma}\right)^2 \sum_{kk'} G_k^0 \tag{26}$$

On account of Equations (14)-(18) Equation (26) becomes

$$\begin{aligned}\Sigma_{kk'} = \frac{n_{imp}}{V}\left(\overline{V + JS \cdot \sigma}\right)^2 \sum_{kk'} \bigg( &\frac{1}{2}\left(g_+^0 + g_-^0\right)\hat{I} + \frac{\alpha_R k_y}{2\lambda(k)}\left(g_+^0 - g_-^0\right)\hat{\sigma}_x \\ &- \frac{\alpha_R k_x}{2\lambda(k)}\left(g_+^0 - g_-^0\right)\hat{\sigma}_y - \frac{\nabla_{ex}}{2\lambda(k)}\left(g_+^0 - g_-^0\right)\hat{\sigma}_z \bigg)\end{aligned} \tag{27}$$

Due to symmetry ( $k_x \leftrightarrow k_y$ ) together and angular dependency of $k_x$ and $k_y$, the two terms, $\hat{\sigma}_x$ and $\hat{\sigma}_y$, components vanishes. Therefore, the only surviving terms of self energy,

$$\Sigma_{kk'} = \frac{n_{imp}}{V}\left(\overline{V + JS \cdot \sigma}\right)^2 \sum_{kk'} \left( \frac{1}{2}\left(g_+^0 + g_-^0\right)\hat{I} - \frac{\nabla_{ex}}{2\lambda(k)}\left(g_+^0 - g_-^0\right)\hat{\sigma}_z \right) \tag{28}$$

On account of Equation (12)

$$\begin{aligned}\Sigma_{kk'} = \frac{n_{imp}}{V}\left(\overline{V + JS \cdot \sigma}\right)^2 \sum_{kk'} \Bigg( &\frac{1}{2}\left( \frac{1}{\omega - E_{k_+} + i\epsilon} + \frac{1}{\omega - E_{k_-} + i\epsilon} \right)\hat{I} \\ &- \frac{\nabla_{ex}}{2\lambda(k)}\left( \frac{1}{\omega - E_{k_+} + i\epsilon} - \frac{1}{\omega - E_{k_-} + i\epsilon} \right)\hat{\sigma}_z \Bigg)\end{aligned} \tag{29}$$

here we introduced the small parameter $\epsilon$ in the denominator to account singularity of this expression. Since the self energy is complex function which can be decomposed into real and imaginary part as

$$\Sigma_{ks} = Rel\Sigma + iIm\Sigma \tag{30}$$

Using Dirac identity,

$$\frac{1}{x \pm i\epsilon} = \wp\frac{1}{x} \mp i\pi\delta(x) \tag{31}$$

Using Equation (31), Equation (29) becomes

$$\begin{aligned}\overline{\Sigma^R}_{kk'} = \frac{n_{imp}}{V}\left(\overline{V + JS \cdot \sigma}\right)^2 \sum_{kk'} \Bigg( &\frac{1}{2}\Bigg( \wp\left(\frac{1}{\omega - E_{k_+}}\right) - i\pi\delta\left(\omega - E_{k_+}\right) \\ &+ \wp\left(\frac{1}{\omega - E_{k_-}}\right) - i\pi\delta\left(\omega - E_{k_-}\right)\Bigg)\hat{I}\end{aligned} \tag{32}$$

$$\begin{aligned}&- \frac{\nabla_{ex}}{2\lambda(k)}\Bigg( \frac{1}{\wp}\left(\frac{1}{\omega - E_{k_+}}\right) - i\pi\delta\left(\omega - E_{k_+}\right) \\ &- \left( \wp\left(\frac{1}{\omega - E_{k_-}}\right) - i\pi\delta\left(\omega - E_{k_-}\right)\right)\Bigg)\hat{\sigma}_z \Bigg)\end{aligned} \tag{33}$$

Since the real part has only effect on of quasi particles and it has no effect on life time of quasiparticles, we shall take only imaginary part (it is common approximation if the disorder effect is weak). Therefore, after removing the real part we left only,

$$\overline{\Sigma^{R}}_{kk'} = -i\pi n_{imp}\left(\overline{V+JS\cdot\sigma}\right)^{2}\left(\frac{1}{2}\left(\frac{1}{\mho}\sum_{k}\delta\left(\omega-E_{k_{+}}\right)+\frac{1}{\mho}\sum_{k}\delta\left(\omega-E_{k_{-}}\right)\right)\hat{I}\right.$$
$$\left.-\frac{\nabla_{ex}}{2\lambda(k)}\left(\frac{1}{\mho}\sum_{k}\delta\left(\omega-E_{k_{+}}\right)-\frac{1}{\mho}\sum_{k}\delta\left(\omega-E_{k_{-}}\right)\right)\hat{\sigma}_{z}\right) \tag{34}$$

In Equation (34) we have used $\mho$ instead of V which designates volume in order to save ourselves from confusion of impurity potential (V) and volume. Now defining spin split density of states (holes in DMS) as

$$D_{\pm}(\omega)=\frac{1}{\mho}\sum_{k}\delta\left(\omega-E_{k_{\pm}}\right). \tag{35}$$

Using Equation (35) into Equation (34), it becomes

$$\overline{\Sigma^{R}}_{kk'} = -i\pi n_{imp}\overline{V_{T}^{2}}\left(\frac{1}{2}\left(D_{+}(\omega)+D_{-}(\omega)\right)\hat{I}-\frac{\nabla_{ex}}{2\lambda(k)}\left(D_{+}(\omega)-D_{-}(\omega)\right)\hat{\sigma}_{z}\right) \tag{36}$$

where we have introduced notation $\overline{V_{T}^{2}}=\left(\overline{V+JS\cdot\sigma}\right)^{2}$ for the seek of simplify. However, most commonly notation $\Gamma$ is used instead of $Im\Sigma$ or ( $Im\Sigma=\Gamma$ ), thus,

$$\Gamma = -i\pi n_{imp}\overline{V_{T}^{2}}\left(\frac{1}{2}\left(D_{+}(\omega)+D_{-}(\omega)\right)\hat{I}-\frac{\nabla_{ex}}{2\lambda(k)}\left(D_{+}(\omega)-D_{-}(\omega)\right)\hat{\sigma}_{z}\right) \tag{37}$$

$$\Gamma=-i\left(\Gamma_{I}\hat{I}+\Gamma_{Z}\hat{\sigma}_{z}\right) \tag{38}$$

where, $\Gamma_{I}=\pi n_{imp}\overline{V_{T}^{2}}\frac{1}{2}\left(D_{+}(\omega)+D_{-}(\omega)\right)$ and

$\Gamma_{Z}=-\pi n_{imp}\frac{\nabla_{ex}}{2\lambda(k)}\left(D_{+}(\omega)-D_{-}(\omega)\right)$

where indices ( $\pm$ ) indicates spin up and spin down components of density of holes associated to system under consideration. The single particle relaxation rate $\tau_{\sigma}$ is given by the imaginary part of the self-energy,

$$\frac{1}{2\tau_{\sigma}}=-s\Sigma_{k\sigma}(\omega) \tag{39}$$

Using Equation (39) together with assumption that spin dependent density of states (DOS) is evaluated at the Fermi level ( $\varepsilon_{F}$ ) *i.e.* ( $\omega\hbar\to\varepsilon_{F}$ ), we shall write spin split life time as

$$\frac{1}{\tau_{+}}=i\pi n_{imp}\overline{V_{T}^{2}}\left(\left(D_{+}+D_{-}\right)\hat{I}-\frac{h_{ex}}{\zeta(k)}\left(D_{+}-D_{-}\right)\hat{\sigma}_{z}\right) \tag{40}$$

$$\frac{1}{\tau_{-}}=i\pi n_{imp}\overline{V_{T}^{2}}\left(\left(D_{+}+D_{-}\right)\hat{I}+\frac{h_{ex}}{\zeta(k)}\left(D_{+}-D_{-}\right)\hat{\sigma}_{z}\right) \tag{41}$$

which can be written as in compact form in component wise

$$\frac{1}{\tau_+} = \frac{1}{\tau_I}\hat{I} - \frac{h_{ex}}{\zeta(k)}\frac{1}{\tau_z}\hat{\sigma}_z \tag{42}$$

$$\frac{1}{\tau_-} = \frac{1}{\tau_I}\hat{I} + \frac{h_{ex}}{\zeta(k)}\frac{1}{\tau_z}\hat{\sigma}_z \tag{43}$$

$$\Rightarrow \frac{1}{\tau_\pm} = \frac{1}{\tau_I}\hat{I} \pm \frac{h_{ex}}{\zeta(k)}\frac{1}{\tau_z}\hat{\sigma}_z \tag{44}$$

where, $\frac{1}{\tau_I} = i\pi n_{imp}\left(\overline{V + JS\cdot\sigma}\right)^2\left(D_+ + D_-\right)$,

$\frac{1}{\tau_z} = i\pi n_{imp}\left(\overline{V + JS\cdot\sigma}\right)^2\frac{h_{ex}}{\zeta(k)}\left(D_+ - D_-\right)$ where $\tau_+$ and $\tau_-$ are relaxation times holes in different sub-bands.

To introduce the contribution impurities (disorder) into bar reference Green functions $g_\pm^{R/A}$, we use Dyson series for reference Green function as

$$\hat{g}_\pm^{R/A} = \frac{\hat{g}_{0\pm}^{r/a}}{1 - \hat{g}_{0\pm}^{R/A}(z)\Sigma_\pm^{R/A}} \tag{45}$$

where

$$g_{0\pm}^{R/A} = \frac{1}{\omega - \xi(k) \pm \sqrt{\frac{\alpha_R^2 k^2}{4} + \nabla_{ex}^2}} = \frac{1}{\omega - \xi(k) \pm \zeta(k)} = \frac{1}{\omega - E_{k\pm} \pm i\epsilon} \tag{46}$$

After plugging Equation (46) into Equation (45) and after some mathematical algebra it yields

$$g_\pm^{R/A} = \frac{1}{\omega - E_{k\pm} \pm i\Sigma_\pm^{R/A}} \tag{47}$$

Then relating the life time and imaginary part of self energy using Equation (39)

$$g_\pm^{R/A} = \frac{1}{\omega - E_{k\pm} \pm i\frac{1}{2\tau_\pm}} \tag{48}$$

where $E_\pm(k) = \varepsilon(k) + \zeta_{sk} = \varepsilon(k) \pm \sqrt{\alpha_R^2 k^2 + h_{ex}^2}$. Therefore, after impurity correction the Retarded Green function along $\hat{\sigma}_i$ can obtained as

$$G_I^R = \frac{1}{2}\left(g_+^R + g_-^R\right) \tag{49}$$

$$G_x^R = \frac{k\frac{\alpha_R}{\hbar}\sin(\phi)}{2\zeta(k)}\left(g_+^R - g_-^R\right) \tag{50}$$

$$G_y^R = -\frac{k\frac{\alpha_R}{\hbar}\cos(\phi)}{2\zeta(k)}\left(g_+^R - g_-^R\right) \tag{51}$$

$$G_z^R = -\frac{h_{ex} + i\frac{1}{2\tau_z}}{2\zeta(k)}\left(g_+^R - g_-^R\right) \tag{52}$$

After detail mathematical manipulation, the impurity averaged retarded and corresponding advanced Green function respectively along $\hat{\sigma}_i$ reads

$$G^R = G_I^R\hat{I} + G_x^R\hat{\sigma}_x + G_y^R\hat{\sigma}_y + G_z^R\hat{\sigma}_z \tag{53}$$

$$G^A = G_I^A\hat{I} + G_x^A\hat{\sigma}_x + G_y^A\hat{\sigma}_y + G_z^A\hat{\sigma}_z \tag{54}$$

## 4. Anomalous Hall Conductivity

The Anomalous Hall Conductivity $\sigma_{yx}$ which corresponds to the non vertex correction at zero temperature based on the Kubo's formula for Fermi surface contribution is given by

$$\sigma_{xy} = \frac{\hbar q^2}{2\pi V}\sum_k Tr\left(G^R(k)v_x G^A(k)v_y\right). \tag{55}$$

The $x$, $y$ components of velocity will be calculated from unperturbed part of Hamiltonian in Equation (2)

$$\hat{v}_x = \frac{k_x}{m^*} - \frac{\alpha_R}{\hbar}\sigma_y \tag{56}$$

$$\hat{v}_y = \frac{k_y}{m^*} + \frac{\alpha_R}{\hbar}\sigma_x \tag{57}$$

Upon substitution of Equation (56) and Equation (57) into Equation (55), we have

$$\sigma_{xy} = \frac{\hbar q^2}{2\pi V}\sum_k Tr\left(\left(G_I^R\hat{I} + G_x^R\hat{\sigma}_x + G_y^R\hat{\sigma}_y + G_z^R\hat{\sigma}_z\right)\times\left(\frac{k_x}{m^*} - \frac{\alpha_R}{\hbar}\sigma_y\right)\right.$$
$$\left.\times\left(G_I^A\hat{I} + G_x^A\hat{\sigma}_x + G_y^A\hat{\sigma}_y + G_z^A\hat{\sigma}_z\right)\times\left(\frac{k_y}{m^*} + \frac{\alpha_R}{\hbar}\sigma_x\right)\right). \tag{58}$$

After changing Summation into integration using $\sum_k \to V^D\int\frac{\mathrm{d}^D k}{(2\pi)^D}$. For 2D case $\sum_k \to V^2\int\frac{\mathrm{d}^2 k}{(2\pi)^2}$ some terms get vanishes due to angular integration (integration of $\sin(\phi)\cos(\phi)$) and after detail mathematical algebra we shall obtain

$$\sigma_{xy} = \frac{\hbar q^2}{2\pi}\iint \mathrm{d}\phi\frac{k\mathrm{d}k}{(2\pi)^2}2\frac{\frac{\alpha_R}{\hbar}k\cos(\phi)}{m}\left(G_x^R G_I^A + G_I^R G_x^A + i\left(G_y^R G_z^A - G_z^R G_y^A\right)\right)$$
$$-\frac{\hbar q^2}{2\pi}\iint \mathrm{d}\phi\frac{k\mathrm{d}k}{(2\pi)^2}2\frac{\frac{\alpha_R}{\hbar}k\sin(\phi)}{m}\left(G_y^R G_I^A + G_I^R G_y^A + i\left(G_x^R G_z^A - G_z^R G_x^A\right)\right) \tag{59}$$
$$-\frac{\hbar q^2}{2\pi}\iint \mathrm{d}\phi\frac{k\mathrm{d}k}{(2\pi)^2}\frac{\alpha_R^2}{\hbar^2}\left(G_y^R G_x^A + G_x^R G_y^A + i\left(G_I^R G_z^A - G_z^R G_I^A\right)\right)$$

The final expression for Hall Conductivity in the presence of random magnetic impurity were obtained by substitution of Equations (49)-(52) into Equation (59), the detail mathematical step are indicated in Appendix Section.

$$\sigma_{xy}^{im} = -\frac{i\alpha_R q^2}{\pi^2}\left(-i\frac{\pi\alpha_R}{2\hbar}\left(\left(\frac{D_+\varepsilon_F}{\zeta_+^2}+\frac{D_+}{\lambda_+}\right)\left(\frac{\tau_+}{\tau_z}\right)+\left(\frac{D_-\varepsilon_F}{\lambda_-^2}-\frac{D_-}{\zeta_-}\right)\left(\frac{\tau_-}{\tau_z}\right)\right)\right.$$
$$\left.+\frac{i\pi\alpha_R}{2\hbar}\left(\frac{D_+}{\zeta_+}\left(\frac{\tau_+}{\tau_z}\right)-\frac{D_-}{\zeta_-}\left(\frac{\tau_-}{\tau_z}\right)\right)\right)$$

Hence some terms are canceled and we left with

$$\sigma_{xy}^{im} = -\frac{\alpha_R^2 q^2 \varepsilon_F}{h}\left(\frac{D_+}{\zeta_+^2}\left(\frac{\tau_+}{\tau_z}\right)+\frac{D_-}{\zeta_-^2}\left(\frac{\tau_-}{\tau_z}\right)\right) \tag{60}$$

where

$$\frac{1}{\tau_z} = i\pi n_{imp}\left(\overline{V+JS\cdot\sigma}\right)^2\frac{h_{ex}}{\zeta(k)}\left(D_+ - D_-\right) \tag{61}$$

$$\frac{1}{\tau_+} = i\pi n_{imp}\left(\overline{V+JS\cdot\sigma}\right)^2\left(\left(D_+ + D_-\right)\hat{I}-\frac{h_{ex}}{\zeta(k)}\left(D_+ - D_-\right)\hat{\sigma}_z\right) \tag{62}$$

$$\frac{1}{\tau_-} = i\pi n_{imp}\left(\overline{V+JS\cdot\sigma}\right)^2\left(\left(D_+ + D_-\right)\hat{I}+\frac{h_{ex}}{\zeta(k)}\left(D_+ - D_-\right)\hat{\sigma}_z\right) \tag{63}$$

$$\zeta_\pm = \pm\sqrt{\alpha_R^2 k^2 + h_{ex}^2} \tag{64}$$

and $D_\pm$ is the density of state in 2D system given by expression

$$D_\pm = \frac{m^*\zeta(k)}{\left|\hbar^2\zeta(k)\pm m^*\alpha_R^2\right|} \tag{65}$$

## Numerical Estimation and Discussion

From Equation (60), as we see it the Fermi surface contribution of Anomalous Hall conductivity in the presence of magnetic impurity does not depend on impurity concentration at all, but the ratio between $\frac{\tau_+}{\tau_z}$ and $\frac{\tau_-}{\tau_z}$ which are related to polarization of carriers. Now it is convenient to approximate $\frac{\tau_+}{\tau_z}$ and $\frac{\tau_-}{\tau_z}$ to rewritten Equation (60) more elegant form

$$\frac{\tau_+}{\tau_z} = \frac{\frac{h_{ex}}{\zeta(k)}\left(D_+ - D_-\right)}{\left(D_+ + D_-\right)\hat{I}-\frac{h_{ex}}{\zeta(k)}\left(D_+ - D_-\right)\hat{\sigma}_z} \tag{66}$$

Using valid approximation, *i.e.* $D_+ + D_- \gg \frac{h_{ex}}{\zeta(k)}\left(D_+ - D_-\right)$

$$\frac{\tau_+}{\tau_z} \approx \frac{h_{ex}}{\zeta(k)}\frac{D_+ - D_-}{D_+ + D_-} \tag{67}$$

But $\frac{D_+ - D_-}{D_+ + D_-}$ is polarization of carrier, in our particular case the carriers are holes and let it be $P_h$, hence

$$\frac{\tau_+}{\tau_z} = \frac{h_{ex}}{\zeta(k)} P_h \tag{68}$$

Similarly

$$\frac{\tau_-}{\tau_z} = \frac{\frac{h_{ex}}{\zeta(k)}(D_+ - D_-)}{(D_+ + D_-)\hat{I} + \frac{h_{ex}}{\zeta(k)}(D_+ - D_-)\hat{\sigma}_z} \tag{69}$$

Using similar approximation $D_+ + D_- \gg \frac{h_{ex}}{\zeta(k)}(D_+ - D_-)$

$$\frac{\tau_-}{\tau_z} = \frac{h_{ex}}{\zeta(k)} P_h \tag{70}$$

making use of Equation (68) and Equation (70) into Equation (60), anomalous Hall conductivity in the presence of magnetic impurity is related with spin polarization of carrier as

$$\sigma_{xy}^{im} = -\frac{\alpha_R^2 q^2 \varepsilon_F}{h}\left(\frac{D_+}{\zeta_+^2}\frac{h_{ex}}{\zeta(k)}P_h + \frac{D_-}{\zeta_-^2}\frac{h_{ex}}{\zeta(k)}P_h\right) \tag{71}$$

Equation (71) shows that Anomalous Hall conductivity in the presence of magnetic and non magnetic impurity, resulting from Fermi surface is independent of lifetimes $\tau$ and depends only on its spin-dependence (polarization of carriers). The life time independent of anomalous Hall conductivity is characteristic of side jump mechanism [15]. Therefore, anomalous hall conductivity in the presence of weak impurity potential is more likely mediated by skew scattering. Moreover, Equation (71) clearly reveals that if both bands (spin up and spin down) are occupied, the value of spin polarization ($p_h$) get vanishes which yields zero ($\sigma_{x,y}^{im} = 0$). And also increasing carriers polarization results increasing anomalous Hall conductivity. Therefore, the polarization of carriers resulting from exchange interaction between localized spin of dopant ($Mn^{2+}$) and itinerant holes are crucial for existence of Skew type of anomalous Hall conductivity low impurity limit. Hence it is formal improcedure to assume only majority band contribute for $\sigma_{x,y}^{im}$ and switching off the minority band, which simplifies Equation (71) to

$$\sigma_{xy}^{im} = -\frac{\alpha_R^2 q^2 \varepsilon_F h_{ex}}{h}\left(\frac{D_+}{\zeta_+^3}\right)P \tag{72}$$

To investigate the role of energy splitting due to spin-orbit coupling $\Delta_{so}$ and exchange field ($h_{ex}$), we plot $\beta = \frac{h_{ex}}{\alpha_R k_F}$ and $\sigma_{x,y}^{im}$ versus $\alpha = \frac{\alpha_R k_F}{h_{ex}}$ which associated with strong and weak limit of spin-orbit coupling respectively (see **Figure 1** and **Figure 2**). The parameters used in our calculations are taken from

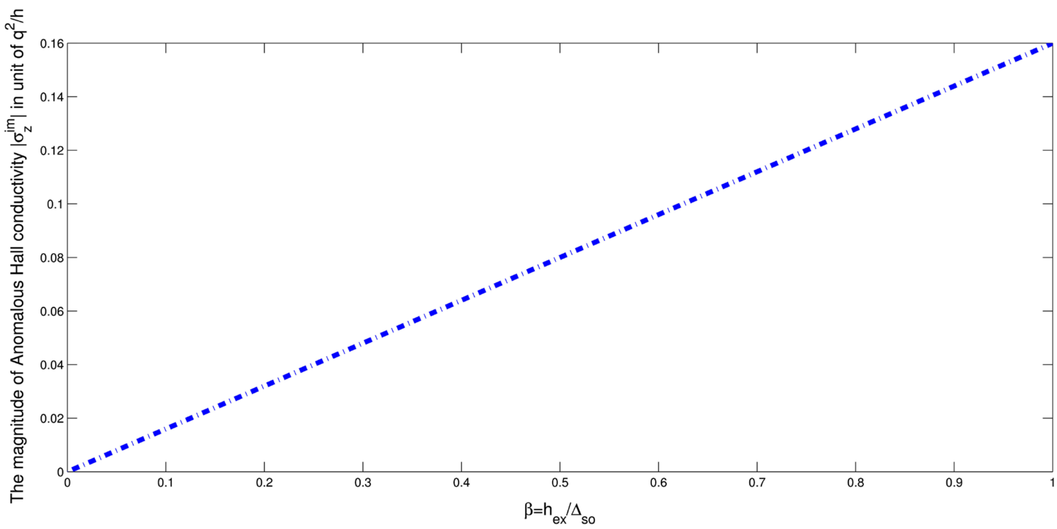


**Figure 1.** The magnitude of Anomalous Hall conductivity in the presence of magnetic impurity Vs. the ratio of exchange field and energy splitting due to spin-orbit coupling ( $\beta=\frac{h_{ex}}{\alpha_R k_F}$ ) for strong spin-orbit coupling limit.

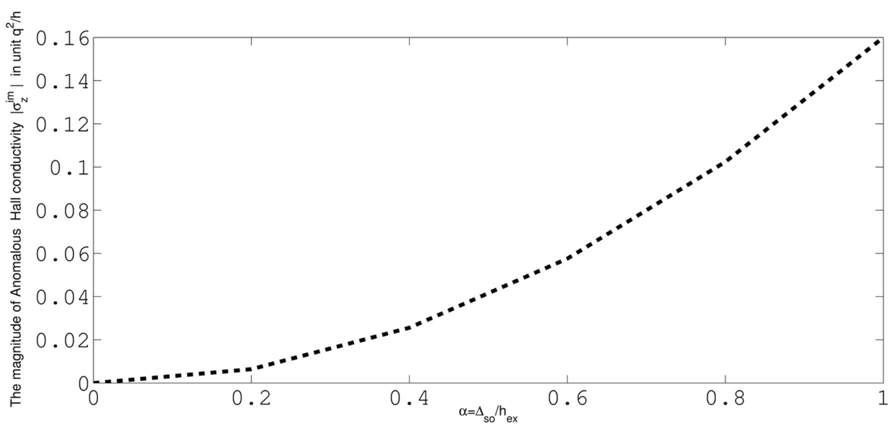


**Figure 2.** The magnitude of Anomalous Hall conductivity in the presence of magnetic impurity Vs. the ratio of energy splitting due to spin-orbit coupling and exchange field ( $\beta=\frac{\alpha_R k_F}{h_{ex}}$ ) for weak spin-orbit coupling limit.

[16]. Experimentally, observed values of Rashba SOC parameter lie in the range $1 \times 10^{12}$ eVm - $6.3 \times 10^{11}$ eVm for a large variety of systems [16], the values of exchange field can vary from (0 - 200) meV [2]. From **Figure 1**, we can see that as $\beta$ increases the magnitude of anomalous Hall conductivity increases monotonically, which in turn shows that as magnitude of exchange field ( $h_{ex}$ resulting from interaction of localized spin of Mn atoms and itinerant holes spin) increases the anomalous Hall conductivity for strong-spin orbit coupling limit our results are in good agreement with experimental trend [4]. On the other hand, as $\alpha=\frac{\alpha_R k_F}{h_{ex}}$ increases, magnitude of $\left|\sigma_z^{im}\right|$ increases in parabolic way (see **Figure 2**). Since $\alpha$ proportional to $\Delta_{so}$. Therefore we can conclude that

that magnitude of anomalougs Hall conductivity increases approximately half parabolic way as energy splitting due spin-orbit coupling increases the result also agree with recent experimental findings reported on anomalous Hall conductivity in $Mn_{1-x}Fe_xGe$ [17]. And hence, the interplay between spin-orbit coupling and Zeeman like exchange field are crucial for existence of finite values of anomalous Hall conductivity in the presence of weak impurity potential.

## 5. Conclusion

In conclusion, we have studied anomalous Hall conductivity in $Ga_{1-x}Mn_xAs$ DMS in low impurity potential limit. The calculated result revealed that, the interplay between carrier polarization, spin-orbit coupling and Zeeman like exchange field is vital for existence of finite values of anomalous Hall conductivity in dilute limit. Our results are in agreement with latest experimental trends. This result shows opportunity associated to control, enhance and create anomalous Hall conductivity by controlling the density of spin-polarized density of electrons, spin-orbit coupling and exchange field, which also platform to realize dissipationless conductivity in low impurity limit.

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## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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# Appendix

## Steps to Evaluate Trace

After messy and detail mathematical algebra, the trace part (expression in big bracket of Equation (58)) yields

$$
\begin{aligned}
& Tr\left(\left(G_I^R\hat{I}+G_x^R\hat{\sigma}_x+G_y^R\hat{\sigma}_y+G_z^R\hat{\sigma}_z\right)\times\left(\frac{k_x}{m^*}-\frac{\alpha_R}{\hbar}\sigma_y\right)\times\left(G_I^A\hat{I}+G_x^A\hat{\sigma}_x+G_y^A\hat{\sigma}_y+G_z^A\hat{\sigma}_z\right)\times\left(\frac{k_y}{m^*}+\frac{\alpha_R}{\hbar}\sigma_x\right)\right) \\
&=\frac{k^2\cos(\phi)\sin(\phi)}{m^2}\left(G_I^RG_I^ATr\left(\hat{I}\hat{I}\right)+G_x^RG_I^ATr\left(\hat{\sigma}_x\hat{I}\right)+G_y^RG_I^ATr\left(\hat{\sigma}_y\hat{I}\right)+G_z^RG_I^ATr\left(\hat{\sigma}_z\hat{I}\right)\right) \\
&+\frac{k^2\cos(\phi)\sin(\phi)}{m^2}\left(G_I^RG_x^ATr\left(\hat{I}\sigma_x\right)+G_x^RG_x^ATr\left(\hat{\sigma}_x\sigma_x\right)+G_y^RG_x^ATr\left(\hat{\sigma}_y\sigma_x\right)+G_z^RG_x^ATr\left(\hat{\sigma}_z\hat{\sigma}_x\right)\right) \\
&+\frac{k^2\cos(\phi)\sin(\phi)}{m^2}\left(G_I^RG_y^ATr\left(\hat{I}\sigma_y\right)+G_x^RG_y^ATr\left(\hat{\sigma}_x\hat{\sigma}_y\right)+G_y^RG_y^ATr\left(\hat{\sigma}_y\hat{\sigma}_y\right)+G_z^RG_I^yTr\left(\hat{\sigma}_z\hat{\sigma}_y\right)\right) \\
&+\frac{k^2\cos(\phi)\sin(\phi)}{m^2}\left(G_I^RG_z^ATr\left(\hat{I}\hat{\sigma}_z\right)+G_x^RG_z^ATr\left(\hat{\sigma}_x\hat{\sigma}_z\right)+G_y^RG_z^ATr\left(\hat{\sigma}_y\hat{\sigma}_z\right)+G_z^RG_z^ATr\left(\hat{\sigma}_z\hat{\sigma}_z\right)\right) \\
&+\frac{\frac{\alpha_R}{\hbar}k\cos(\phi)}{m}\left(G_I^RG_I^ATr\left(\hat{I}\hat{I}\hat{\sigma}_x\right)+G_x^RG_I^ATr\left(\hat{\sigma}_x\hat{I}\hat{\sigma}_x\right)+G_y^RG_I^ATr\left(\hat{\sigma}_y\hat{I}\hat{\sigma}_x\right)+G_z^RG_I^ATr\left(\hat{\sigma}_z\hat{I}\hat{\sigma}_x\right)\right) \\
&+\frac{\frac{\alpha_R}{\hbar}k\cos(\phi)}{m}\left(G_I^RG_x^ATr\left(\hat{I}\hat{\sigma}_x\sigma_x\right)+G_x^RG_x^ATr\left(\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\right)+G_y^RG_x^ATr\left(\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_x\right)+G_z^RG_x^ATr\left(\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_x\right)\right) \\
&+\frac{\frac{\alpha_R}{\hbar}k\cos(\phi)}{m}\left(G_I^RG_y^ATr\left(\hat{I}\hat{\sigma}_y\sigma_x\right)+G_x^RG_y^ATr\left(\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_x\right)+G_y^RG_y^ATr\left(\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_x\right)+G_z^RG_y^ATr\left(\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_x\right)\right) \\
&+\frac{\frac{\alpha_R}{\hbar}k\cos(\phi)}{m}\left(G_I^RG_z^ATr\left(\hat{I}\hat{\sigma}_z\sigma_x\right)+G_x^RG_z^ATr\left(\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_x\right)+G_y^RG_z^ATr\left(\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_x\right)+G_z^RG_z^ATr\left(\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_x\right)\right) \\
&-\frac{\frac{\alpha_R}{\hbar}k\sin(\phi)}{m}\left(G_I^RG_I^ATr\left(\hat{I}\hat{\sigma}_y\right)+G_x^RG_I^ATr\left(\hat{I}\hat{\sigma}_x\hat{\sigma}_y\right)+G_y^RG_I^ATr\left(\hat{\sigma}_y\hat{\sigma}_I\hat{I}\right)+G_z^RG_z^ATr\left(\hat{I}\hat{\sigma}_z\hat{\sigma}_y\right)\right) \\
&-\frac{\frac{\alpha_R}{\hbar}k\sin(\phi)}{m}\left(G_I^RG_x^ATr\left(\hat{I}\hat{\sigma}_y\hat{\sigma}_x\right)+G_x^RG_x^ATr\left(\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_x\right)+G_y^RG_x^ATr\left(\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_x\right)+G_z^RG_x^ATr\left(\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_x\right)\right) \\
&-\frac{\frac{\alpha_R}{\hbar}k\sin(\phi)}{m}\left(G_I^RG_y^ATr\left(\hat{I}\hat{\sigma}_y\hat{\sigma}_y\right)+G_x^RG_y^ATr\left(\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_y\right)+G_y^RG_y^ATr\left(\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y\right)+G_z^RG_y^ATr\left(\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_y\right)\right) \\
&-\frac{\frac{\alpha_R}{\hbar}k\sin(\phi)}{m}\left(G_I^RG_z^ATr\left(\hat{I}\hat{\sigma}_y\hat{\sigma}_z\right)+G_x^RG_z^ATr\left(\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_z\right)+G_y^RG_z^ATr\left(\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_z\right)+G_z^RG_z^ATr\left(\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_z\right)\right) \\
&-\frac{\alpha_R}{\hbar}\left(G_I^RG_I^ATr\left(\hat{I}\hat{\sigma}_y\hat{\sigma}_x\right)+G_x^RG_I^ATr\left(\hat{I}\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_x\right)+G_y^RG_I^ATr\left(\hat{\sigma}_y\hat{\sigma}_y\hat{I}\hat{\sigma}_x\right)+G_z^RG_I^ATr\left(\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_x\right)\right) \\
&-\frac{\alpha_R}{\hbar}\left(G_I^RG_x^ATr\left(\hat{I}\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_x\right)+G_x^RG_x^ATr\left(\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_x\right)+G_y^RG_x^ATr\left(\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_x\right)+G_z^RG_x^ATr\left(\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_x\right)\right) \\
&-\frac{\alpha_R}{\hbar}\left(G_I^RG_y^ATr\left(\hat{I}\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_x\right)+G_x^RG_y^ATr\left(\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_x\right)+G_y^RG_y^ATr\left(\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_x\right)+G_z^RG_y^ATr\left(\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_x\right)\right) \\
&-\frac{\alpha_R}{\hbar}\left(G_I^RG_z^ATr\left(\hat{I}\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_x\right)+G_x^RG_z^ATr\left(\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_x\right)+G_y^RG_y^ATr\left(\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_x\right)+G_z^RG_z^ATr\left(\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_x\right)\right)
\end{aligned}
\tag{73}
$$

To solve Equation (73), we use the following properties of Pauli matrices:

- All of Pauli matrices are orthogonal to each other and the product of any two Pauli matrices, up to a factor of $\pm i$, is another Pauli matrices

$$\hat{\sigma}_x\hat{\sigma}_y = i\hat{\sigma}_z, \hat{\sigma}_y\hat{\sigma}_x = -i\hat{\sigma}_z$$
$$\hat{\sigma}_z\hat{\sigma}_x = i\hat{\sigma}_y, \hat{\sigma}_x\hat{\sigma}_z = -i\hat{\sigma}_y,$$
$$\hat{\sigma}_y\hat{\sigma}_z = i\hat{\sigma}_x, \hat{\sigma}_z\hat{\sigma}_y = -i\hat{\sigma}_x \tag{74}$$

- All Pauli matrices have zero trace,

$$Tr\left(\hat{\sigma}_x\right) = Tr\left(\hat{\sigma}_y\right) = Tr\left(\hat{\sigma}_z\right) = 0 \tag{75}$$

- Square of any Pauli matrix is identity and whose trace value is two *i.e.*

$$Tr\left(\hat{\sigma}_x^2\right) = Tr\left(\hat{\sigma}_y^2\right) = Tr\left(\hat{\sigma}_z^2\right) = Tr\left(\hat{I}\right) = 2 \tag{76}$$

After applying Equations (74)-(76) into Equation (73) we have

$$\begin{aligned} Tr\left(G^R\hat{v}_xG^A\hat{v}_y\right) = {} & 2\frac{k^2\cos(\phi)\sin(\phi)}{m^2}\left(G_I^RG_I^A + G_x^RG_x^A + G_y^RG_y^A + G_z^RG_z^A\right) \\ & + 2\frac{\frac{\alpha_R}{\hbar}k\cos(\phi)}{m}\left(G_x^RG_I^A + G_I^RG_x^A + i\left(G_y^RG_z^A - G_z^RG_y^A\right)\right) \\ & - 2\frac{\frac{\alpha_R}{\hbar}k\sin(\phi)}{m}\left(G_y^RG_I^A + G_I^RG_y^A + i\left(G_x^RG_z^A - G_z^RG_x^A\right)\right) \\ & - 2\frac{\alpha_R}{\hbar}\left(G_y^RG_x^A + G_x^RG_y^A + i\left(G_I^RG_z^A - G_z^RG_I^A\right)\right) \end{aligned} \tag{77}$$