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On a New Equation for Critical Current Density Directly in Terms of the BCS Interaction Parameter, Debye Temperature and the Fermi Energy of the Superconductor

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ABSTRACT

Recasting the BCS theory in the larger framework of the Bethe-Salpeter equation, a new equation is derived for the temperature-dependent critical current density $j_c(T)$ of an elemental superconductor (SC) directly in terms of the basic parameters of the theory, namely the dimensionless coupling constant [N(0)V], the Debye temperature θ_D and, additionally, the Fermi energy E_F—unlike earlier such equations based on diverse, indirect criteria. Our approach provides an *ab initio* theoretical justification for one of the latter, text book equations invoked at $T = 0$ which involves Fermi momentum; additionally, it relates j_c with the relevant parameters of the problem at $T \neq 0$. Noting that the numerical value of E_F of a high-T_c SC is a necessary input for the construction of its Fermi surface—which sheds light on its gap-structure, we also briefly discuss extension of our approach for such SCs.

Keywords: Critical Current Density; BCS Parameters; Fermi Energy; Elemental/Non-Elemental Superconductors

1. Introduction

The critical current density (j_c) of a superconductor (SC) is the maximum current density that it can carry beyond which it loses the characteristic of superconductivity. It is an important parameter because greater its value, greater is the practical use to which the SC can be put. The *basic* relation between j_c and the critical velocity (v_c) of Cooper pairs (CPs) at any temperature T and an applied field H is:

$$j_c(T,H) = n_s(T,H)e^*v_c(T,H),\ (v_c = P_c/2m^*) \quad (1)$$

where n_s is the number of CPs, e^*, P_c and $(2m^*)$ are, respectively, the charge, the critical momentum, and the effective mass of a CP. We note that, since formation of CPs in the BCS theory is synonymous with the formation of their condensate [e.g., 1], P_c in (1) may also be defined as the minimum momentum that causes dissociation of the condensate.

As alternatives to (1), several *derived* relations for j_c can be found in the literature [2-5], some of which have been reproduced in **Table 1**. Salient features of these relations are: 1) They are obtained via indirect approaches based on diverse criteria such as the type of SC being dealt with (type I or II) and its geometry; 2) They lead to values of j_cs that are generally much greater than the experimental values; and 3) Only one of them involves the Fermi energy E_F (via Fermi momentum) of the SC—this will be further discussed below.

E_F of an SC is an important parameter too because, as has been remarked [6], "There is every evidence that the remarkable low value of E_F (<100 meV) and the strong coupling of carriers with high-frequency phonons is the cause of high T_c in all newly discovered superconductors." Furthermore, the input of the numerical value of E_F is essential to construct the Fermi surface $E_j(k)$ of an SC via $E_j(k) = E_F$, from which it is seen that [7; p. 117] the whole process of determining theoretically the shape of the Fermi surface involves calculating $E_j(k)$ over the entire Brillouin zone and then constructing the particular constant energy surface that corresponds to E_F. However, this assumes that the actual numerical value of E_F is available, which may well not be the case. The importance of the Fermi surface stems from the fact it sheds light on the gap-structure of the SC since it marks the boundary between the occupied and the unoccupied parts of the band j. This explains the considerable experimental effort that has been expended on constructing the

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Table 1. Some of the relations in the literature for calculating the critical current densities (j_cs) of different types of superconductors obtained via diverse, indirect methods.

S. No.	SC	j_c	Remark	Ref.
1	Type I; wire of radius a in the absence of external field	$H_c ac/2$ H_c : critical magnetic field c : velocity of light	j_c is the current that generates a field = H_c	
2	Type I; thin film or wire	$cH_c/4\pi\lambda$ λ : penetration depth	London theory; kinetic energy density is equated to condensation energy density	[2] p. 118
3	Type I; thin film or wire	$cH_c(T)/3\sqrt{6}\pi\lambda(T)$	Ginsburg-Landau theory	[2] p. 117
4	Type I	$en_s\Delta/\hbar k_F$ e : electronic charge n_s : density of superconducting electrons Δ : gap; k_F : Fermi wave vector	BCS theory	[3] p. 248
5	Type I; cylinder of radius a	$30\Delta M/a$ ΔM : width of magnetization loop at a given field and temperature	Bean's critical state model	[4]
6	Type I; slab of thickness d	$40\Delta M/d$	Bean's critical state model	[4]
7	Type I; hollow cylinder	$\alpha/(B+B_0)$ α and B_0 (thermodynamic critical field) are obtained from experiment	Kim *et al.* model	[5]

Fermi surfaces of a variety of high-T_c SCs as reported in [e.g., 8,9] and, more recently, in [10-12]. In particular, in the latter of these references, the gaps observed in iron-pnictide SCs as nodes or line nodes on the Fermi surface have evinced considerable interest. For a quantitative account of the T_c and the *multiple* gaps of a prominent member of the iron-pnictide family, namely $Ba_{0.6}K_{0.4}Fe_2As_2$, in the framework of the generalized-BCS equations (GBCSEs) [13]—which will be further discussed below, we draw attention to [14].

The purpose of this note is to present an approach in which $P_c(T)$—defined as the momentum at which the binding energy of the CPs vanishes (this is equivalent to the vanishing of the gap [13])—is calculated via the dynamics of CPs. As will be seen, we are then led via (1) to an equation for $j_c(T)$ directly in terms of the familiar BCS parameters, namely the dimensionless coupling constant [N(0)V], the Debye temperature θ_D and, additionally, E_F of the SC. The framework employed by us is that of the Bethe-Salpeter equation (BSE) for reasons to be spelled out shortly.

The paper is organized as follows. In the next section, we obtain equations for $P_c(T, H = 0)$ and $P_c(T = 0, H = 0)$ for a simple SC. The solutions of these equations for Sn are obtained in Section 3 and compared with similar results obtained by a different method. Extension of our approach for non-elemental SCs is presented in Section 4. In Section 5 we make four brief comments. The final section sums up our conclusions.

2. Equations for $P_c(T, H = 0)$ and $P_c(T = 0, H = 0)$ for a Simple SC

Our starting point is the $T = 0$, $H = 0$ BSE [15] for the bound states of particles a, b bound via the interaction kernel $I^c_{a,b}$ in the ladder and instantaneous approximations:

$$F_a F_b \psi(p_\mu) = (2\pi i)^{-1} \int \mathrm{d}^4 q_\mu \psi(p_\mu + q_\mu) I^c_{a,b}(\boldsymbol{q}). \quad (2)$$

Customization of this equation for CPs requires th at a, b should be electrons. We then have

$$\begin{aligned} F_a^{-1} &= 1/\left[\gamma^a_\mu P_\mu/2 + \gamma^a_\mu p_\mu - m + i\varepsilon\right] \\ F_b^{-1} &= 1/\left[\gamma^b_\mu P_\mu/2 - \gamma^b_\mu p_\mu - m + i\varepsilon\right], \end{aligned} \quad (3)$$

where m is the electron mass, $\gamma^{a,b}_\mu$ are the Dirac matrices, $\pm p_\mu$ are 4-momenta of the two electrons in the centre of mass (c.m.) frame, and P_μ is the 4-momentum of the c.m. of the CP in the laboratory frame.

In our earlier work [13] based on (2), it sufficed to set

$$P_\mu = (E, 0); (E = 2E_F + W) \quad (4)$$

where E is the total energy of a CP; it then turned out that $|W| \cong \Delta$. The BCS interaction kernel in (2) then was

$$\begin{aligned} I^c_{a,b}(\boldsymbol{q} - \boldsymbol{p}) &= \frac{-V}{(2\pi)^3}(V > 0) \text{ for} \\ E_F - \omega_D &\le \frac{\boldsymbol{p}^2}{2m}, \frac{\boldsymbol{q}^2}{2m} \le E_F + \omega_D \end{aligned} \quad (5)$$

[Note: We use natural units: mass, momentum, energy, etc. in eV, $c=\hbar=1$].

The role of the 4th dimension in (2) is simply to provide the means to temperature-generalize the theory at the outset via the Matsubara recipe. Thus, following the steps that have been detailed in [13,16] we obtain from (2) the 3-dimensional equation

$$S(\boldsymbol{p})=(-2\pi i)^{-1}\int \mathrm{d}^3 q I_{a,b}^{c}{}'(\boldsymbol{q}-\boldsymbol{p})S(\boldsymbol{q})J(\boldsymbol{q}), \qquad (6)$$

where

$$\begin{aligned} &S(\boldsymbol{p})=\left[p_4+A(\boldsymbol{p})\right]\left[B(\boldsymbol{p})-p_4\right]\psi(p_\mu),\\ &A(\boldsymbol{p})=E/2-\boldsymbol{p}^2/2m,\\ &B(\boldsymbol{p})=A(\boldsymbol{p}),\\ &I_{ab}^{c}{}'(\boldsymbol{q}-\boldsymbol{p})=\gamma_4^a\ \gamma_4^b\ I_{ab}^{c}(\boldsymbol{q}-\boldsymbol{p}), \end{aligned} \qquad (7)$$

and

$$J(\boldsymbol{q})=\int\frac{\mathrm{d}q_4}{\left[q_4+A(\boldsymbol{q})\right]\left[B(\boldsymbol{q})-q_4\right]} \qquad (8)$$

If we simply carry out the integration in (8), we obtain the usual $T=0$ theory; subjecting it to the Matsubara recipe, however, we obtain an equation valid at any temperature—causing the theory to incorporate many-body effects. With the aid of the Matsubara recipe, (8) yields [13,16]:

$$J(\vec{q})=i\pi\frac{\left[\tanh\left\{\frac{\beta}{2}A(\vec{q})\right\}+\tanh\left\{\frac{\beta}{2}B(\vec{q})\right\}\right]}{D(\vec{q})}, \qquad (9)$$

where

$$D(\boldsymbol{q})\equiv A(\boldsymbol{q})+B(\boldsymbol{q})=\left(E-\boldsymbol{q}^2/m\right), \qquad (9)$$

and $\beta=1/k_BT$, k_B being the Boltzmann constant.

Since critical velocity is defined as the velocity of CPs at which $W=0$, we now need to consider (2) for the case of moving CPs. Hence (4) is replaced by

$$P_\mu=(E,\boldsymbol{p}) \qquad (11)$$

where $\boldsymbol{p}$ is the 3-momentum of the c.m. of a CP. It i s pertinent at this stage to draw attention to the intera ction Hamiltonian corresponding to (2), which is actua lly apparent from the structure of the equation:

$$H_{\mathrm{int},BSE}=g\bar{\psi}\psi\varphi$$

where ψ is the electron field and φ the phonon field; exchanges of the latter field between the electrons with coupling strength g being responsible for pairing. Both for elemental and non-elemental SCs, one is now enabled to calculate not only the Tcs and Δs—as has been shown [17,18], but also Pc(T)s of the pairs as will be seen below. *Because the BSE formalism accommodates CPs having non-zero c.m. momentum, it constitutes a larger framework than the original BCS formalism which restricts the Hamiltonian at the outset to comprise of terms corresponding to pairs having zero c.m. momentum.*

Since energies of the electrons forming a CP now take on the values $\left(\boldsymbol{P}/2\pm\boldsymbol{p}\text{ or }\pm\boldsymbol{q}\right)^2/2m$, the BCS model interaction given in (5) gets replaced by

$$\begin{aligned} &I_{ab}^{c}(\boldsymbol{q}-\boldsymbol{p})=\frac{-V}{(2\pi)^3}(V>0)\ \text{for}\ E_F-\hbar\omega_D\\ &\le\frac{(\boldsymbol{P}/2+\boldsymbol{p}\text{ or }\boldsymbol{q})^2}{2m},\frac{(\boldsymbol{P}/2-\boldsymbol{p}\text{ or }-\boldsymbol{q})^2}{2m}\le E_F+\hbar\omega_D \qquad (12)\\ &=0(\text{otherwise}). \end{aligned}$$

Substituting (9)-(12) into (6), we obtain

$$S(\boldsymbol{p})=\frac{V}{2}\int_L^U\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{T_1(\boldsymbol{q})+T_2(\boldsymbol{q})}{De(\boldsymbol{q})}S(\boldsymbol{q}) \qquad (13)$$

where

$$\begin{aligned} &T_1(\boldsymbol{q})=\tanh\left\{\beta\left[E/2-(\boldsymbol{P}/2+\boldsymbol{q})^2\right]/2m\right\},\\ &T_2(\boldsymbol{q})=\tanh\left\{\beta\left[E/2-(\boldsymbol{P}/2-\boldsymbol{q})^2\right]/2m\right\}, \qquad (13a)\\ &De(\boldsymbol{q})=\left[E-\frac{(\boldsymbol{P}/2+\boldsymbol{q})^2}{2m}-\frac{(\boldsymbol{P}/2-\boldsymbol{q})^2}{2m}\right] \end{aligned}$$

and it is seen that, as is well known for a constant kernel, the wave function for the pair is a constant; the limits (L, U) will be dealt with shortly. Putting

$$\frac{T_1(\boldsymbol{q})+T_2(\boldsymbol{q})}{De(\boldsymbol{q})}=\varphi(\boldsymbol{q})$$

in (13), multiplying the resulting equation with

$$\int_L^U \mathrm{d}^3p/(2\pi)^3$$

and simplifying, we obtain

$$1=\frac{V}{2}\int_L^U\frac{\mathrm{d}^3p}{(2\pi)^3}\cdot\frac{T_1(\boldsymbol{p})+T_2(\boldsymbol{p})}{De(\boldsymbol{p})} \qquad (14)$$

with

$$d^3p=p^2dp\sin\theta d\theta d\varphi,\xi=p^2/2m-E_F,$$

so that, since the integration range for ξ □ E_F,

$$p\ \square\ (2mE_F)^{1/2},p^2dp\ \square\ m(2mE_F)^{1/2}d\xi, \qquad (15)$$

we obtain from (14) the equation

$$1=\frac{V}{8}\left[\frac{(2m)^{3/2}E_F^{1/2}}{4\pi^2}\right]\int_{-1}^{1}\mathrm{d}x\int_L^U\mathrm{d}\xi\frac{X}{Y} \qquad (16)$$

where

$$X = A_1 + A_2$$

$$A_1 = \tanh\left[\frac{\beta}{2}\left(\xi + P\alpha x + \frac{P^2}{8m} - \frac{W}{2}\right)\right]$$

$$A_2 = \tanh\left[\frac{\beta}{2}\left(\xi - P\alpha x + \frac{P^2}{8m} - \frac{W}{2}\right)\right]$$

$$Y = \xi + \frac{P^2}{8m} - \frac{W}{2}$$

$$\alpha = \left(\frac{E_F}{2m}\right)^{1/2},\ x = \cos(\boldsymbol{P}, \boldsymbol{p})$$

and (13a) and the definition of E in (4) have been used. In the natural units employed by us, both m and E_F are in eVs; the second pre-factor within the square brackets on the RHS of (16) is therefore recognized as the 3-dimensional density of states at the Fermi surface (with the dimensions of $(eV^{-1}\cdot cm^{-3})$ in the units customarily employed in the BCS theory). Henceforth we denote this factor by N(0). Note that the term corresponding to $P\ p\ x/2m$ in the expansion of $(\boldsymbol{P}/2 \pm \boldsymbol{p})^2/2m$ has been written as $P\alpha x$ by using (15) and the definitions of α and x that follow (16).

We now specify the limit L. It follows from (12) that

$$E_F - \omega_D \le \frac{p^2}{8m} + \frac{p^2}{2m} + P\alpha x,\ E_F - \omega_D \le \frac{p^2}{8m} + \frac{p^2}{2m} - P\alpha x$$

where (15) has been used. These relations may be written as

$$-\omega_D - \frac{p^2}{8m} - P\alpha x \le \frac{p^2}{2m} - E_F \equiv \xi,$$

$$-\omega_D - \frac{p^2}{8m} + P\alpha x \le \frac{p^2}{2m} - E_F \equiv \xi.$$

If we denote $\left(-\omega_D - \frac{p^2}{8m} - P\alpha x\right)$ by point A, and $\left(-\omega_D - \frac{p^2}{8m} + P\alpha x\right)$ by point B on the energy axis, then it follows that ξ should always lie to the right of *both* A, and B. Thus L is fixed as

$$L = -\omega_D - \frac{p^2}{8m} + P\alpha x \tag{17}$$

Similarly,

$$U = \omega_D - \frac{p^2}{8m} - P\alpha x. \tag{18}$$

We now put $W = 0$ in (16) in order to determine the critical momentum $P_c(T)$ at any temperature. Simultaneously, we neglect $P_c^2/8m$ everywhere—*a posteriori* justification to follow, excepting in the denominator of the integrand where it must be retained so as to avoid the singularity at $\xi = 0$. It is then seen that it is an excellent approximation to write (16) as:

$$1 = \frac{\left[N(0)V\right]}{2}\int_0^1 dx \int_0^{U'} d\xi \frac{\psi(x,\xi,P_c)}{\xi + P_c^2/(8m)} \tag{19}$$

where

$$\psi(x,\xi,P_c) = \tanh\left[\frac{\beta}{2}(\xi + P_c\alpha x)\right] + \tanh\left[\frac{\beta}{2}(\xi - P_c\alpha x)\right]$$
$$\equiv f_1(x,\xi,P_c) + f_2(x,\xi,P_c)$$
$$U' = \omega_D - P_c\alpha x. \tag{20}$$

Equation (19) *affords a consistency check of our procedure so far*: *Putting* $P_c = 0$ *causes the x-integral to yield unity*, *and the two tanh-functions to add up*, l*eading to the correct BCS equation for* T_c. Note that when $T = 0$ ($\beta = \infty$), $f_1(P_c, \xi, x) = 1$,whereas the value of $f_2(P_c, \xi, x)$ depends on whether ξ is less or greater than $P_c\alpha x$. Therefore, when $T = 0$, we can write (19) as

$$1 - \frac{\lambda}{2}(I_1 + I_2) = 0, \tag{21}$$

where

$$I_1 = \int_0^1 dx \int_0^{E_1 x - E_2} \frac{d\xi}{\xi + E_3}, \tag{22}$$

$$I_2 = \int_0^1 dx \left[\int_0^{E_2 x} \frac{-d\xi}{\xi + E_3} + \int_{E_2 x}^{E_1 - E_2 x} \frac{d\xi}{\xi + E_3}\right], \tag{23}$$

$$\lambda = \left[N(0)V\right], E_1 = \omega_D, E_2 = P_0\alpha,$$

$$P_0 = P_c(T=0), E_3 = P_0^2/8m.$$

Carrying out the elementary integrations in (22) and (23), (21) yields

$$1 - \lambda\left[\frac{E_1 + E_3}{E_2}\ln\left(\frac{E_1 + E_3}{E_2}\right) + \frac{E_3}{E_2}\ln\left(\frac{E_3}{E_2 + E_3}\right) + \ln\left(\frac{E_1 - E_2 + E_3}{E_2 + E_3}\right)\right] = 0. \tag{24}$$

Since, as will be seen below, $E_3 \ll E_1, E_2$, we may write it more compactly as

$$1 - \lambda\left[y\ln\left(\frac{y}{y-1}\right) + \ln(y-1)\right] = 0, \tag{25}$$

where the dimensionless parameter

$$y = E_1/E_2 = \omega_D/P_0\alpha.$$

3. Solutions of (25) and (19) for Sn and Comparison of Results for j_c via (1) with Those Obtained via an Alternate, Indirect Approach

We deal with Sn because superconducting properties based on its j_c have been discussed in standard texts such

as [3; p. 248] and [19; p. 138]. The equation invoked for j_c at $T = 0$ in these texts is:

$$j_c = \frac{en_s\Delta}{p_F}, \tag{26}$$

where n_s is the number of electrons (*not pairs*) and p_F the Fermi momentum. Indeed this equation is equivalent to using (1) since Δ/p_F has the dimensions of velocity. With $\Delta = 1.80\ k_BT_c$ ($T_c = 3.72$ K), $m^* = 1.26\ x$ free electron mass [20, p. 254] and v_F *taken at the Fermi edge* to be 6.97×10^7 cm/sec [3, p. 248], we have

$$\begin{aligned} v_c &= \Delta/p_F = \Delta/m^*v_F = 1.46\times10^4\ \text{cm}\cdot\text{sec}^{-1}; \\ \text{E}_F &= (1/2)m^*v_F^2 = 1.74\ \text{eV}, \end{aligned} \tag{27}$$

Using (27) and the experimental value of j_c for Sn (~2 $\times 10^7$ Ampere cm^{-2}), (26) is invoked to calculate n_s since it is the most uncertain quantity in the equation. It is thus found that

$$n_s = 8.50 \times 10^{21}\ \text{cm}^{-3}, \tag{28}$$

which, it has been remarked [19], is appreciably less than one electron per atom, but not unreasonable in view of the complicated band structure of tin, which has been discussed in [7, p. 294].

In our approach, we first need the value of λ to solve (25). Substituting the experimental value of T_c quoted above and θ_D = 195 K in the BCS equation for T_c: $\lambda = -1/\ln(T_c/1.14\theta_D)$, we obtain $\lambda = 0.2445$, whence (25) yields

$$y = \frac{\omega_D}{P_0\alpha} = \frac{k_B\theta_D}{P_0}\sqrt{\frac{2m^*}{E_F}} = 22.48, \tag{29}$$

where the definition of α given after (16) has been used. Using (29) and (1), we have

$$\begin{aligned} v_0 &= \frac{P_0}{2m^*} = A\sqrt{\frac{B}{E_F}}; \\ \text{A} &= \left(\frac{k_B\theta_D}{2m^*cy}\right) = 17.402\ \text{cm sec}^{-1}, \\ \text{B} &= 2\times1.26\times0.5110\times10^6\ \text{eV} \end{aligned} \tag{30}$$

and

$$j_0 = n_se^*A\sqrt{\frac{B}{E_F}}, \tag{31}$$

where $2m^*$ is the effective mass of a CP and m^* has been taken to be 1.26 times the free electron mass as before, e^* is twice the electronic charge and the value of E_F is in eV.

Since we have already determined A via dynamics of the problem, e^* and B are known constants and j_0 is known from experiment, (31) involves two unknowns: n_s and E_F; knowledge of either of them enables one to calculate the other. Guided by text book wisdom, if we use the values of j_0 (2×10^7 Amp cm^{-2}) and E_F (given in (27)), we obtain from (30) and (31) the following results

$$\begin{aligned} &\alpha = 1.162\times10^{-3},\ \text{P}_0 = 0.643\ \text{eV} \\ &\text{E}_1 = 1.68\times10^{-3}\ \text{eV},\ \text{E}_2 = 7.476\times10^{-4}\ \text{eV}, \\ &\text{E}_3 = 8.029\times10^{-8}\ \text{eV} \end{aligned} \tag{32}$$

$$v_0 = 1.50 \times 10^4\ \text{cm sec}^{-1} \tag{33}$$

$$n_s(\text{CPs}) = 4.17 \times 10^{21}\ \text{cm}^{-3}. \tag{34}$$

The values of E_1, E_2 and E_3 in (32) justify the approximation made in reducing (24) to (25). The result in (33) is almost identical with the value obtained via (26) and quoted in (27), while the result in (34) translates into 8.34×10^{21} cm^{-3} for the number of super *electrons* which, again, is in excellent agreement with the value quoted in (28). It is thus established that the approach followed in this paper provides an *ab initio* theoretical justifi- cation for the text book equation (26) valid at $T = 0$; *additionally*: 1) it relates j_c with the relevant parameters of the problem at $T \neq 0$ via (19) and 2) it can easily be extended to bring non-elemental high-T_c SCs under its purview as will be discussed in the next section.

With P_0 known, it is convenient to solve (19) in terms of the reduced (or normalized) variables defined as $t = T/T_c$ and $p(t) = P_c(t)/P_0$. **Figure 1** gives the results of this exercise for $0 \le t \le 1$. We have also studied the variation of p with t for five other elements: Pb, Hg, In, Tl, and Nb—by taking for their E_Fs the values given by the free electron model [20, p. 248], and found it to be similar to that of Sn.

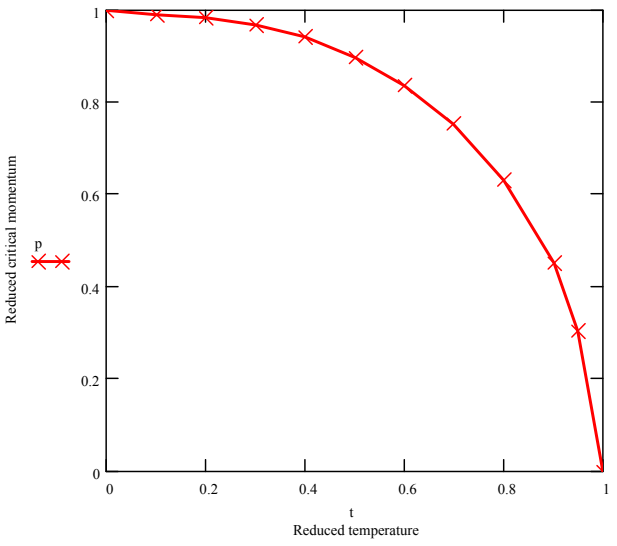

Figure 1. Variation of reduced critical momentum with reduced temperature for Sn obtained via (19) with the input of λ = 0.2445, θ_D = 195 K and E_F = 1.74 eV.

4. Equations for $Pc(T, H = 0)$ and $Pc(T = 0, H = 0)$ for a Non-Elemental SC

The T_cs and the multiple gaps of several non-elemental high-T_c SCs (other than iron-pnictide SCs) have been dealt with in [17,18] via GBCSEs. We recall from [13,16] that these equations constitute a generalization of the BCS equations because: 1) they incorporate the mechanism of multi-phonon exchanges for the formation of Cooper pairs besides the usual one-phonon exchange mechanism; and 2) they invoke more than one Debye temperature—which is another way to specify the mass-dependent Debye *frequency* of an ion species—to characterize the SC.

In order to calculate P_c in the scenario in which CPs are bound via say, two-phonon exchange mechanism in a CS A_xB_{1-x}, we need to generalize (19) and (24). This is accomplished by replacing the propagator in (12) by a superpropagator [13]:

$$\begin{aligned} & I_{ab}^{c}\left(\boldsymbol{q}-\boldsymbol{p}\right)=-\frac{\left(V_1^c+V_2^c\right)}{\left(2\pi\right)^3}, \\ & \text{range of } \mathrm{V}_{1,2}^{c}: \quad E_F-\left(\omega_D\right)_{1,2}^{c} \\ & \leq \frac{\left(\frac{\boldsymbol{P}}{2}+\boldsymbol{p} \text{ or } \boldsymbol{q}\right)^2}{2m}, \frac{\left(\frac{\boldsymbol{P}}{2}-\boldsymbol{p} \text{ or } -\boldsymbol{q}\right)^2}{2m} \\ & \leq E_F+\left(\omega_D\right)_{1,2}^{c} \\ & =0,\left(\text{otherwise}\right) \end{aligned} \tag{35}$$

where $V_{1,2}^{c}>0$ are the BCS model interactions for the species of phonons belonging to A, B in the combined state of the constituents A and B, to be distinguished from $V_{1,2}^{c}$, which are the free state interactions of A, B; $\left(\omega_D\right)_{1,2}^{c}$ are to be similarly distinguished from $\left(\omega_D\right)_{1,2}^{c}$. Following now the sequence of steps between (8) and (24), we obtain the generalized version of (19) as:

$$1=\int_0^1 \mathrm{d}x\ \left[J_1\left(x\right)+J_2\left(x\right)\right], \tag{36}$$

where

$$\begin{aligned} & J_1\left(x\right)=\frac{\lambda_1^c}{2}\int_0^{U_1}\mathrm{d}\xi\ \frac{\psi\left(x,\xi,P_c\right)}{\xi+p_c^2/8m} \\ & J_2\left(x\right)=\frac{\lambda_2^c}{2}\int_0^{U_2}\mathrm{d}\xi\ \frac{\psi\left(x,\xi,P_c\right)}{\xi+p_c^2/8m} \\ & \psi\left(x,\xi,P_c\right)=\tanh\left[\frac{\beta}{2}\left(\xi+P_c\alpha x\right)\right]+\tanh\left[\frac{\beta}{2}\left(\xi-P_c\alpha x\right)\right] \end{aligned} \tag{37}$$

$$\lambda_{1,2}^{c}=\left[N\left(0\right)V\right]_{1,2}^{c}, \mathrm{U}_{1,2}=\left(\omega_D\right)_{1,2}^{c}-P_c\alpha x. \tag{38}$$

Equation (24) now goes over to

$$\lambda_1^c\phi_1\left(\left(\omega_{\mathrm{D}}\right)_1^c,\alpha,p_0\right)-\lambda_2^c\phi_2\left(\left(\omega_{\mathrm{D}}\right)_2^c,\alpha,p_0\right)=0, \tag{39}$$

where

$$\begin{aligned} \phi_1=\Bigg[&\frac{E_1}{E_2}\ln\left(\frac{E_1}{E_1-E_2}\right)+\frac{P_0}{8m\alpha}\ln\left(\frac{P_0}{8m\alpha}\right)+ \\ &+\ln\left(\frac{E_1-E_2}{E_2}\right)\Bigg] \\ & E_1=\left(\omega_D\right)_1^c, E_2=P_0\alpha; \end{aligned} \tag{40}$$

ϕ_2 is obtained from ϕ_1 by putting $E_1=\left(\omega_D\right)_2^c$.

The solution of (39) for MgB_2, for example, requires the inputs of $\lambda_{1,2}^c$ and the two Debye temperatures: $\left(k_B\theta_D\right)_{1,2}^c=\left(\omega_D\right)_{1,2}^c$; in addition, we require E_F of the CS. Such solutions will be addressed else where n puts of $\lambda_{1,2}^c$ and the two Debye temperatures: $\left(k_B\theta_D\right)_{1,2}^c=\left(\omega_D\right)_{1,2}^c$; in addition, we require E_F of the CS. Such solutions will be addressed elsewhere.

5. Discussion

We have dealt above with equations that were obtained via positive energy projection operators (PEPOs). This suffices for the problem addressed because P_c corresponds to the situation when $W = 0$; in this limit, it has been shown in [21] that the equation obtained via the negative energy projection operators is identical with the one obtained via the PEPOs; also that: 1) CPs formed via electron-electron and hole-hole scatterings make equal contributions to the BS amplitude; and 2) the amplitudes for the formation of CPs corresponding to the mixed energy projection operators are zero.

Note that if we concern ourselves with the ratios of j_cs at different temperatures, which seems to be a realistic application of our equations, then the choice of the effective mass of the electron in (1) becomes immaterial.

Even a cursory survey of the literature shows that j_c of an elemental/non-elemental SC can vary between wide limits—depending upon the shape, size and alloying materials of the sample. The study presented here suggests that this variation comes about because each sample has its own set of intrinsic parameters: T_c, θ_D, and E_F. Substituting these into the equation for j_c (which is known from experiment) leads to a relation involving n_s and E_F. Knowledge of either of them then determines the other.

We finally note that the equations for $j_c(T)$ presented in this paper can be generalized to include an external magnetic field via the Landau quantization scheme—as has been done to obtain dynamics-based equations for critical magnetic fields for both elemental and non-elemental

SCs in [22].

6. Conclusions

Equation (19) for an elemental SC and (36) for a non-elemental SC are the new results of this paper: they enable one to calculate the critical momentum P_c of the SC at any T in zero-external magnetic field directly in terms of the familiar parameters [N(0)V], θ_D and E_F that characterize it. Substitution of these values of P_c into (1) then constitutes a direct approach for the calculation of the critical current densities.

$T = 0$ limits of both (19) and (36) were obtained—leading to (24) and (40), respectively. While it was further shown that (24) can justifiably be reduced to (25), we note that caution needs to be exercised should one seek to carry out a similar reduction of (40).

A necessary input for the calculation of P_c (and hence j_c) of an SC is its E_F, which is a parameter that is seldom available for the high-T_c SCs. It therefore seems to us that an immediate and realistic application of the approach presented here is to calculate the E_F of such SCs via the input of their j_cs which are readily available in the literature. The importance of E_F of the high-T_c SCs is borne out by the studies reported in [6] and [8-12].

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