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Pareto Distribution of Wealth Based on Overlapping Generation Model

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Abstract

In this paper, we shed light on the Pareto distribution of wealth on the basis of an overlapping generation model. We deduce in the model that the basic reason for a particular shape of Pareto distribution is attributed to the distribution of utility function among consumers. More specifically, we show that a formula relating the distribution of utility function to the distribution of wealth is analytically obtainable on the basis of the Cobb-Douglas utility function. By simulation, we recognize that the formula actually gives us a well approximation of a Pareto distribution.

Keywords

Pareto Distribution of Wealth, Overlapping Generation Model, Cobb-Douglas Utility Function, Steady State Price Equilibrium

1. Introduction

We have been for a long time without a decisive theory for a generating mechanism of Pareto distribution of wealth despite numerous efforts on it, where we interpret the Pareto distribution as a power-law distribution (for the survey, see e.g. Davies and Shorrocks [1]). Recently, new attempts have been made to derive a Pareto distribution of wealth or income in the field of econophysics. Their approach is based on the so called kinetic-wealth-exchange-model in which individuals are regarded as molecules in a gas, and the exchange of energy of two molecules by collusion is compared to the exchange of wealth between two individuals [2]-[7].

Their argument starts with the following exchanging process of wealth between two individuals i, j;

$$w_i(t+1) = w_i(t) + \Delta w, \quad w_j(t+1) = w_j(t) - \Delta w$$

where w_i, w_j indicate amounts of wealth for each individual and Δw designates a transferred wealth between them. Two agents are randomly chosen from a certain population at each time, thus this approach is called an agent based approach. This process is repeated infinitely, leading to a certain distribution of wealth. By additive considerations like savings involved in the process, a desirable distribution, *i.e.* the Pareto distribution, can be attained.

This theory is itself interesting, but from the viewpoint of economics, its approach has serious drawbacks. The first one is deficient explanation of incentives behind the exchange of wealth. In economics, an agent is thought to behave in accordance with a specific motivation; that is to say optimization of something. The approach does not refer to why and how much an individual is willing to exchange its wealth. In this relation, the approach disregards an important function of wealth that is store of value over time. Secondly, the approach reaches a goal without resort to any kind of equilibrium. That is, the resultant distribution of wealth is not considered as an equilibrium, which violates a fundamental idea of economics that the observable state is regarded as equilibrium. Actually there is some effort in econophysics that has been devoted to rationalize the process of the agent based approach [8]. But, as the authors admit, their argument depends on very artificial and ad hoc assumptions on utility function, production, and money holding.

We should also mention another problem with their approach, which is more fundamental. It is the fact that the actual distribution based on empirical data can only be approximated by the Pareto distribution. Therefore, we do not need to show that the distribution theoretically derived exactly fits the Pareto distribution. In this sense, the kinetic-wealth-exchange-model seems to hope too much.

In this note, we take into account all fundamental elements of economics and consider the possibility of generating a distribution of wealth that can be approximated by the Pareto distribution. We show its possibility by using an overlapping generation model with a Cobb-Douglas utility function, which actually provides us with a suitable framework for our purpose.

The overlapping generation model was first analytically developed by Samuelson [9] and thereafter this model has been used in various issues like national debt [10], bonds [11], monetary policy [12], bequest [13], generation accounting [14], social security [15], asset price [16] and so on (for the comprehensive textbook, see MaCandless and Wallace [17]). This model enables us to analyze an agent's intertemporal decision making by decomposing the time span into two periods, namely a young period and an old period. It is assumed that a certain number of people are born in every period, so that a young and an old generations are always overlapped in each period. Another important feature of this model is the working of price mechanism through time in which prices in different periods are interrelated, so that we have a general equilibrium as a result.

We give a brief perspective of our argument using the overlapping generation model in the following. We first assume that the population born at each period has a certain frequency distribution depending on different utility functions although all functions are based on the Cobb-Douglas one. Then, we deduce optimal savings of each type of agents through their intertemporal decision making, which consist of their wealth. We lastly solve the general equilibrium that makes a series of equilibrium prices of each period. In this note, however, we pay particular attention to the steady state price equilibrium in which prices are equal through different periods. Given this specific equilibrium, we are allowed to obtain a stable wealth distribution among agents close to the Pareto distribution.

In the next section, we introduce a basic overlapping generation model on which our argument is developed. One feature of our model is that we add money supply that is given by the authority so that every agent can retain its wealth as a form of money. In section III, first we show how a distribution of wealth is derived under the steady state price equilibrium and then discuss possibility that such distribution can be approximated by the Pareto distribution. In order to articulate its possibility, we make a numerical simulation to show how we can obtain a desirable distribution in section IV. Then we sum up our arguments in the last section.

2. Model

We adopt a simple overlapping generation model and adjust it to our purpose. The basic framework is as follows. Every agent lives for two periods; the first is called a young period and the second called an old period. The total number of agents born at the outset of a young period is normalized to unity, the meaning of which is later explained at some length. There exists one good that is perishable. All agents are endowed with e_1 of the good in the young period and e_2 of the good in the old period. We assume that (e_1, e_2) is common to all agents. This endowment is obviously viewed as an income for every agent. We may assume that both of e_1 and e_2 are strictly positive.

The good is traded in a perfectly competitive market in each period. Let p_t be a price of the good for agents born in the *t*-period and p_{t+1} be a price in their old period. We assume that every agent has a perfect foresight.

An agent can save a part of e_1 that turns out to be an addition to income in its old period. It is worth noting that savings of an agent are no more than a means for store of value, therefore are qualified as its wealth in our model. Since an agent is supposed to leave nothing at the end of its life, its wealth only consists of its savings in a young period. We focus on the distribution of agents' savings as the distribution of wealth.

Savings should be preserved in money, so that we assume that the authority issue a specific amount of money. Hence, we need to take into account not only a good market but also a money market in the following.

We view a diversity of agents through the difference of power of Cobb-Douglas utility function. Specifically, we use the following utility function to characterize an agent born in t-period $(t = 0,1,\cdots)$,

$$u = c_{t1}^{\alpha} c_{t2}^{1-\alpha}, \quad \alpha \in [0,1],$$

where c_{t1} implies a consumption in its young period, and c_{t2} a consumption in its old period. The preference of an agent is reflected in α of the function. For simplicity of notation, we call α a preference intensity. One agent has its own α, and the number of the agent having α is denoted by $f(\alpha)$. For simplicity of analysis, we assume that the whole set of agents constitutes a continuum and that $\int_0^1 f(\alpha)\mathrm{d}\alpha = 1$. Thus, $f(\alpha)$ is formally identified with a probability distribution function over $[0,1]$. We also assume that $f(\alpha)$ does not depend on t, that is, we have the same distribution in every t-period. Given the structure of our model, we start our analysis for the distribution of wealth in the next section.

3. Analysis and Results

Since we have the same distribution function $f(\alpha)$ at every period t and every agent has the same initial endowments (e_1, e_2), we have only to focus on the behavior of an agent at any given period t. An agent with a characteristic α adjusts its consumption allocation by solving the following optimization problem.

$$\max \quad u = c_{t1}^{\alpha} c_{t2}^{1-\alpha} \quad s.t. \begin{cases} p_t c_{t1} + p_t s_{t1} = p_t e_1 \\ p_{t+1} c_{t2} = p_t s_{t1} + p_{t+1} e_2 \end{cases}$$

We may simply replace c_{t1}, c_{t2}, and s_{t1} with c_1, c_2, and s without any confusion of notations. By calculation, we have that $c_1 = \alpha\left(\frac{e_2}{\theta} + e_1\right)$, where $\theta = \frac{p_t}{p_{t+1}}$. Thus, $s = e_1 - c_1 = e_1 - \alpha\left(\frac{e_2}{\theta} + e_1\right)$. We should note that s must be non negative.

We explicitly consider real savings s, which are supposed to be preserved in money. Thus, s is regarded as the demand for real money by α. On the other hand, money supply is provided by the authority. For simplicity, we normalize the money supply to unity. Given this framework, we have the following results leading to the distribution of wealth, which is, as we have stated, equivalent to the distribution of agents' savings.

Proposition 1. *If* $E(\alpha) < \frac{e_1}{e_1 + e_2}$, *we have the steady state price equilibrium at* $p_t^* = \frac{1}{(1 - E(\alpha))e_1 - E(\alpha)e_2}$, *where* $E(\alpha)$ *indicates the mean of* α, *i.e.*, $E(\alpha) = \int_0^1 f(\alpha)\alpha\mathrm{d}\alpha$.

Proof: *Since we have the distribution function* $f(\alpha)$, *the aggregate demand for real money (denoted by* M_d*) turns out to be the following.*

$$M_d = \int_0^1 f(\alpha)s(\alpha)\mathrm{d}\alpha = \int_0^1 f(\alpha)\left(e_1 - \alpha\left(\frac{e_2}{\theta} + e_1\right)\right)\mathrm{d}\alpha = e_1 - E(\alpha)\left(\frac{e_2}{\theta} + e_1\right)$$

Since the money supply is normalized to unity, we may set the real money supply (denoted by M_s) at $\frac{1}{p_t}$.

Given the supply and demand for money, we are allowed to consider equilibrium in the money market, which leads to the following equation.

$$M_d = e_1 - E(\alpha)\left(\frac{e_2}{\theta} + e_1\right) = \frac{1}{p_t} = M_s$$

A price sequence $\{p_t\}_{t=0,1,\cdots}$ satisfying the above condition constitutes an equilibrium. In particular, if $E(\alpha) < \frac{e_1}{e_1 + e_2}$, the steady state price equilibrium is obtainable, which is $p_t^* = \frac{1}{(1 - E(\alpha))e_1 - E(\alpha)e_2}$ since $\theta = 1$.

Remark: At $p_t^* = \frac{1}{(1 - E(\alpha))e_1 - E(\alpha)e_2}$, the market of the good is also cleared (see Appendix).

In the following, we use the symbol w instead of s to indicate wealth. Let $\phi(w)$ denote the distribution function of wealth.

Theorem 1. *At the steady state price equilibrium, the distribution of wealth* ($\phi(w)$) *is expressed as the following.*

$$\phi(w) = f(g(w)), \quad g(w) = \frac{e_1 - w}{e_1 + e_2}, \quad e_1 \geq w \geq 0$$

Proof: *Under the steady state price equilibrium, savings* (s) *of an agent αat t are that* $e_1 - \alpha(e_1 + e_2)$ *since* $\theta = 1$. *The distribution ofαis* $f(\alpha)$, *so that s has a specific distribution which is no more than a distribution of wealth. It is obvious that the distribution is described as* $\phi(w) = f(g(w)), g(w) = \frac{e_1 - w}{e_1 + e_2}$ *through the fact that* $s = w = e_1 - \alpha(e_1 + e_2)$. *It is worth noting that the distribution must be defined over* $[0, e_1]$.

By this theorem, we obtained the formula connecting distributions of wealth and utility function. Thus, through this formula, our desired distribution of wealth is expected to be gained, which is just what we will achieve in the next section. Before that, we should note some important points derived by this theorem.

First, w is no more than a negative affine transformation of *α*and the distribution of wealth is determined by the distribution of *α*only over a part of the interval $[0,1]$, that is $\left[0, \frac{e_1}{e_1 + e_2}\right]$, which implies that the ratio of e_1 to e_2 does not matter but the ratio of e_1 to the total income $e_1 + e_2$ does matter for the wealth distribution. Second, the distribution $f(\alpha)$ of*α*must have the mean less than $\frac{e_1}{e_1 + e_2}$ given the steady state price equilibrium.

In particular, the first point is suggestive. When we think of the occurrence of the Pareto distribution, we readily recognize the following fact. If e_2 is the same as e_1 or larger than e_1, the number of agents who need savings for their old period will be small. Thus, we naturally predict the occurrence of the Pareto distribution of wealth. On the other hand, if e_2 is less than e_1, many agents will need savings for their old period. Under these circumstances the distribution of wealth is not likely to represent the Pareto one. From the practical viewpoint, however, it is most plausible that e_2 is fairly less than e_1. Nevertheless we actually observe the distribution of wealth that can be approximated by the Pareto one, which seems to be paradoxical. By focusing on the first point above mentioned, we are able to cope with this paradox because we have that the factor playing the major role is not $\frac{e_1}{e_1}$ but $\frac{e_1}{e_1+e_2}$.

4. Simulation

In this section, we show through a numerical simulation the possibility of the occurrence of the wealth distribution that can be approximated by the Pareto Distribution. It is worth noting that necessary parameters for the model are only e_1 and e_2. Let e_1 be 5 and e_2 be 3, which seems to be a plausible setting from a realistic point of view.

In this case, we may think of the following log-normal distribution as a distribution $f(\alpha)$ causing a desirable distribution of wealth.

$$f(\alpha)=\frac{1}{\sqrt{2\pi}\sigma(1-\alpha)}\exp\left[-\frac{\left(\ln(1-\alpha)-\mu\right)^2}{2\sigma^2}\right],\quad \mu=-1,\quad \sigma=\frac{1}{3}.$$

The mean of this distribution is approximately 0.611, which is less than $\frac{e_1}{e_1+e_2}=0.625$. The steady state price equilibrium, therefore, does exist and is calculated as approximately 8.93 through the formula provided in Proposition 1.On the basis of this particular distribution of $f(\alpha)$ we are able to obtain the distribution of wealth that is algebraically expressed as

$$\phi(w)=\frac{1}{\sqrt{2\pi}\sigma\left(1-\frac{e_1-w}{e_1+e_2}\right)}\exp\left[-\frac{\left(\ln\left(1-\frac{e_1-w}{e_1+e_2}\right)-\mu\right)^2}{2\sigma^2}\right],\quad e_1\geq w\geq 0.$$

Since $e_1=5, e_2=3, \mu=-1$, and $\sigma=\frac{1}{3}$, we have a specific numerical distribution of wealth as follows.

$$\phi(w)=\frac{24}{\sqrt{2\pi}(3+w)}\exp\left[-\frac{9\left(\ln\frac{3+w}{8}+1\right)^2}{2}\right],\quad 5\geq w\geq 0.$$

This distribution is depicted in **Figure 1**, which can be approximated by a

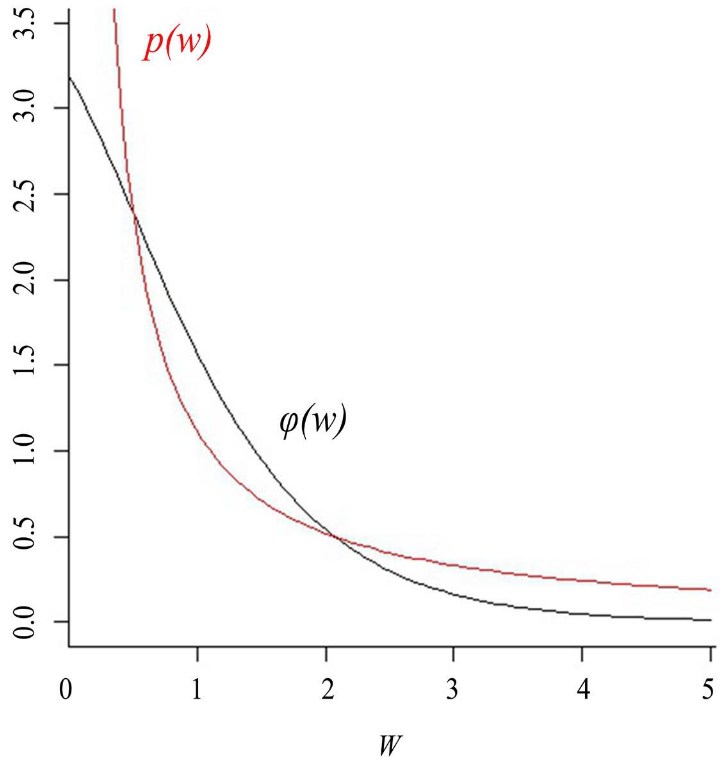

Figure 1. The resultant distribution and an approximating power-law distribution.

Pareto distribution. For reference, we superimpose a power-law distribution expressed as $p(w) = 1.11/w^{1.1}$ on the diagram.

$$\begin{cases} \phi(w) = \dfrac{24}{\sqrt{2\pi}(3+w)} \exp\left[-\dfrac{9\left(\ln\dfrac{3+w}{8}+1\right)^2}{2} \right], & 5 \geq w \geq 0 \\ p(w) = 1.11/w^{1.1} & \end{cases}$$

5. Concluding Remarks

In considering the distribution of wealth, the overlapping generation model has several merits. First, it involves the passage of time with the young and the old periods. Second, it is possible to separate an income as a flow and a wealth as a stock, the latter of which can be seen as a means of store of value. Third, it is easy to introduce money that enables savings to become wealth. Last, we are allowed to derive all the relevant consequences as equilibria. We are able to take advantage of these features to investigate the distribution of wealth while none of them are found in a kinetic-wealth-exchange-model. To put it in another way, the resultant distribution of wealth has the firm economic foundations unlike the agent based approach. It only remains to introduce the population of agents to the model and connect it to the distribution of wealth.

What we have obtained in our analysis is clear; that is, given the steady state price equilibrium, the distribution of wealth that can be approximated by a Pareto distribution is dependent on a specific distribution of a utility function among agents.

At the end, we should note that the usage of Cobb-Douglas function helps us derive the distribution of wealth. It really facilitates characterization of the distribution. On the contrary, it would not be easy to obtain a concrete functional form of the distribution with other types of utility function. But, no matter how complex it is, the fact that the distribution of the parameter of a utility function determines the distribution of wealth still remains. Thus, it is likely that we will be able to obtain a desirable distribution of wealth with any form of utility function as long as it is not pathological.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix

The proof of the market clearance of the good at $p_t^* = \dfrac{1}{\left(1-E(\alpha)\right)e_1 - E(\alpha)e_2}$.

For simplicity of notation, we use *α* to represent an agent who has a characteristic *α* in the following.

For an agent *α* born at *t*, its demand c_{t1} and c_{t2} are calculated as follows.

$$c_{t1} = \alpha\left(\frac{e_2}{\theta} + e_1\right), \quad c_{t2} = e_2 + \theta(e_1 - c_1) = (1-\alpha)(e_2 + \theta e_1), \quad \theta = \frac{p_t}{p_{t+1}}$$

Hence, an agent *α* born at $t-1$ declares its demand for the good at *t* that is $c_{t-1,2} = (1-\alpha)(e_2 + \theta_{-1}e_1)$ where $\theta_{-1} = \frac{p_{t-1}}{p_t}$. So the aggregate demand for the good at *t*, denoted as D_t, amounts to be

$$\int_0^1 f(\alpha)\alpha\left(\frac{e_2}{\theta} + e_2\right)\mathrm{d}\alpha + \int_0^1 f(\alpha)(1-\alpha)(e_2 + \theta_{-1}e_1)\mathrm{d}\alpha$$

It follows through calculation that

$$D_t = \left(E(\alpha) + \theta_1 - E(\alpha)\theta_{-1}\right)e_1 + \left(\frac{E(\alpha)}{\theta} + 1 - E(\alpha)\right)e_2$$

Since we think of the steady state price equilibrium, we have that $\theta = \theta_{-1} = 1$. Thus, by the above expression we obtain that $D_t = e_1 + e_2$.

On the other hand, the aggregate supply of the good at *t*, denoted as S_t, is expressed as follows.

$$S_t = \int_0^1 f(\alpha)e_1\mathrm{d}\alpha + \int_0^1 f(\alpha)e_2\mathrm{d}\alpha = e_1 + e_2.$$

Thus we have that $D_t = S_t$, which completes the proof.