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# The Origin of Piketty's Inequality *r* > *g* Considered in a General Framework

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## Abstract

This paper studies the origin of Piketty's inequality between the profit rate ($r$) and the growth rate of the national income ($g$) by focusing on the growth rate ($\gamma$) of the $\frac{r}{g}$ ratio in an economy that grows gradually along a succession of production cycles. It is shown that, given a succession of three production cycles, the value of $\gamma$ in the last cycle is determined by the equation $1+\gamma=(1+\nu)(1+\kappa)$ where $\nu$ is the growth rate of the profit share ($\alpha$) in the last cycle while $\kappa$ is a function of three variables: the income/capital ratio of the last cycle, the values of the savings rate in the first two cycles and those of the growth rate of the income/capital ratio in the last two. The equation just presented is also relevant for a succession of more than three production cycles for which the yearly values of $r$, $g$ and $\alpha$ are known. Indeed, in this case it is possible to calculate the average values of $\gamma$ and $\nu$ from the empirical data, which then can be used in the equation to determine the average value of $\kappa$. Once the three variables are known, it is possible to calculate the parts attributable respectively to the average values of $\nu$ and $\kappa$ in the determination of the average value of $\gamma$. A similar result is obtained regarding the part attributable to the average changes in the savings rate and in the growth rate of the income/capital ratio, taken together, in the determination of the average value of $\kappa$. The paper also identifies those configurations of the relevant variables where $\gamma>0$, out of which, when the succession of production cycles is long enough, results the inequality $r>g$.

## Keywords

Growth Rate, Income/Capital Ratio, Piketty, Profit Rate

## 1. Introduction

Piketty [1] documents the presence in modern economies of a tendency towards the concentration of income and wealth in the highest socioeconomic strata (see [1], Fig. 9.8 p. 324 and Fig. 10.6 p. 349). This does not include the period 1910-1950 when, due to the world wars, the tendency was reversed, nor does it include the period 1950-1970, which was characterized by shares of income and wealth relatively stable for the top strata. According to this analysis (see [1], pp. 25-27), the main force driving this development is the fact that, as a general rule, the profit rate ($r$) is greater than the growth rate of the national income ($g$). The importance of Piketty's work has been widely recognized by the specialized literature [2] [3] [4] [5] [6] and there have also been critical comments [7] [8] [9].

For the purposes of this study, it is important to mention Benítez [10] and [11] where both the origin and consequences of the inequality $r > g$ are discussed using a linear production model. Indeed, as indicated ahead, in Section 7, the first paper exposes certain results closely related to those that are established here by simpler ways within a more general framework. The second paper, which contains some errors corrected in Benítez [12], explores the effects of the inequality $r > g$ on the reproduction of economic elites. In this respect, it considers the existence of quasi-feudal renters, which constitute an important social category not previously discussed in the specialized literature, as far as I know. It also underscores the fact that the effects just mentioned can take place even when the profit rate of a fraction of the national capital in a country, but not that of the national capital, is greater than the growth rate of the national income.

This paper studies the origin of the inequality $r > g$ by focusing on the growth rate ($\gamma$) of the $\frac{r}{g}$ ratio in an economy that grows gradually along a succession of production cycles. It establishes the following eight main results of which the odd-numbered and even-numbered results refer to successions of three and more than three production cycles, respectively.

First, the value of $\gamma$ in the last cycle is determined by the equation $1+\gamma=(1+\nu)(1+\kappa)$ where $\nu$ is the growth rate of the profit share in the last cycle while $\kappa$ is a function of the income/capital ratio of the last cycle and of two other variables that, to simplify, will be referred to as the two $\kappa$ variables: on the one hand, the values of the savings rate in the first two cycles and, on the other hand, those of the growth rate of the income/capital ratio in the last two. Each one of the variables $\nu$ and $\kappa$ can be greater than, equal to or less than zero with independence of the value of the other variable and, as a consequence of this, $\gamma$ can adopt the same values.

Second, if the yearly values of the profit rate, the growth rate, and the profit share are known, they can be used to calculate the averages values of $\gamma$ and $\nu$ and, once they are obtained, substituting by these values the corresponding variables in the equation presented in the precedent paragraph, the average value of $\kappa$ can be determined. In turn, this set of results allows us to calculate the part of the change in the $\frac{r}{g}$ ratio attributable, respectively, to the average values of $\nu$ and $\kappa$.

Third, if the values of the savings rate in the first two cycles (*s*) and those of the growth rate of the income/capital ratio in the last two (*f*) are constant, then

$$\kappa = \frac{f^2}{sE + f}$$

where $E$ is the income/capital ratio of the last cycle while the sum $sE + f$ is equal to the growth rate of the national income in the same cycle. Because this rate is, by hypothesis, greater than zero, $\kappa > 0$ whether the income/capital ratio increases or decreases.

Fourth, the average values of the variables involved can be used in the equation just presented to calculate the average value of $\kappa$. Then, comparing the outcome with the estimation of $\kappa$ indicated in the second result, it becomes possible to have an idea of the part of this estimation due to changes in the two $\kappa$ variables taken together.

Fifth, if the two $\kappa$ variables are not constant, which is generally the case, $\kappa$ is a monotonous decreasing function of the increase in any of the two $\kappa$ variables with respect to its corresponding initial value within the succession.

Sixth, it follows from the last result that, if the two $\kappa$ variables fluctuate around their corresponding average values, the increasing and decreasing effects of these changes over the average value of $\kappa$ tend to cancel out reciprocally. In this case, if the income/capital ratio changes steadily, whether it increases or decreases, the average value of $\kappa$ is greater than zero.

Seventh, due to the continuity of the variables involved, the third and the fifth results, taken together, imply that $\kappa > 0$ provided that none of the two $\kappa$ variables increase relatively to its corresponding initial value above certain limits. Under these conditions, $\gamma \geq \nu$. Thus, $\gamma$ is greater than zero if the profit share does not decrease and it is less than zero only if the profit share decreases enough to reduce sufficiently the product of the two factors on the right-hand side of the equation presented in the first result.

Eighth, the third, sixth, and seventh results present some arguments explaining those cases where $\gamma > 0$, to which it can be added the possibility of an increase in the profit share to compensate the effect on $\gamma$ of a negative value of $\kappa$. In all these cases, the inequality $r > g$ results as a consequence of the growth of the $\frac{r}{g}$ ratio if the succession of production cycles is long enough.

Including this introduction, the paper has 10 sections. Section 2 presents the model studied here while Sections 3 and 4 define certain relations between some of its main variables. Sections 5 and 6 establish respectively the first and the second of the main results and Section 7 establishes the next two. Section 8 establishes the fifth main result and the remaining three results are discussed in Section 9. Some final comments are presented in the last section.

## 2. The Model

I consider a country undergoing a succession of annual production cycles starting at dates $t = 0,1,2,\cdots$ and, to identify each one of them, I refer to the date corresponding to the end of the production year. For this reason, a variable indexed with a date *t* is defined for $t > 0$, unless otherwise indicated. The notations $Y_t$ and $K_t$ represent respectively the national income obtained and the national capital used in cycle *t* while $E_t$ is the corresponding income/capital ratio. Thus,

$$E_t = \frac{Y_t}{K_t}. \tag{1}$$

For each $t > 1$, $g_t$, $m_t$, and $f_t$ denote respectively the growth rates of national income, of national capital and of the income/capital ratio from cycle $t-1$ to cycle *t*. Therefore,

$$1 + g_t = \frac{Y_t}{Y_{t-1}}, \tag{2}$$

$$1 + m_t = \frac{K_t}{K_{t-1}}, \tag{3}$$

$$1 + f_t = \frac{E_t}{E_{t-1}}. \tag{4}$$

Equation (1) implies that:

$$K_t E_t = Y_t, \tag{5}$$

and Equation (5) corresponding to cycle $t-1$ is:

$$K_{t-1} E_{t-1} = Y_{t-1}. \tag{6}$$

Substituting the numerator and denominator on the right-hand side of Equation (2) respectively by the left-hand side of Equations (5) and (6) gives:

$$1 + g_t = \frac{K_t E_t}{K_{t-1} E_{t-1}} \tag{7}$$

$$= \left(\frac{K_t}{K_{t-1}}\right)\left(\frac{E_t}{E_{t-1}}\right). \tag{8}$$

Now, substituting on the right-hand side of this equation the first and the second factor respectively by the left-hand side of Equations (3) and (4) yields:

$$1 + g_t = (1 + m_t)(1 + f_t). \tag{9}$$

From this equation it follows that:

$$g_t = m_t(1 + f_t) + f_t. \tag{10}$$

This equation will be used in Section 4.

## 3. Income Distribution

For each *t*, $\alpha_t$ denotes the profit share of national income and $r_t$ represents the profit rate while, for each $t > 1$, $v_t$ is the growth rate of the profit share. Therefore,

$$1+v_t=\frac{\alpha_t}{\alpha_{t-1}}, \tag{11}$$

and

$$r_t=\frac{\alpha_t Y_t}{K_t}, \tag{12}$$

$$=\alpha_t\left(\frac{Y_t}{K_t}\right). \tag{13}$$

The last result and Equation (1), taken together, imply that:

$$r_t=\alpha_t E_t, \tag{14}$$

and Equation (14) for cycle $t-1$ is:

$$r_{t-1}=\alpha_{t-1}E_{t-1}. \tag{15}$$

Dividing term by term Equation (14) by Equation (15) yields:

$$\frac{r_t}{r_{t-1}}=\frac{\alpha_t E_t}{\alpha_{t-1}E_{t-1}} \tag{16}$$

$$=\left(\frac{\alpha_t}{\alpha_{t-1}}\right)\left(\frac{E_t}{E_{t-1}}\right). \tag{17}$$

Substituting the first and second factor on the right-hand side of Equation (17) respectively by the left-hand side of Equations (11) and (4) gives:

$$\frac{r_t}{r_{t-1}}=(1+v_t)(1+f_t). \tag{18}$$

This equation will be used in Section 5.

## 4. Savings and Investments

The national income is partitioned between the fractions $s_t$ and $1-s_t$ destined respectively to savings and consumption. For this reason, for each $t>1$, the national capital of cycle $t$ increases in the amount $s_{t-1}Y_{t-1}$ with respect to that employed in the previous cycle. Thus, the growth rate of national capital is determined by the formula:

$$m_t=\frac{s_{t-1}Y_{t-1}}{K_{t-1}} \tag{19}$$

$$=s_{t-1}\left(\frac{Y_{t-1}}{K_{t-1}}\right). \tag{20}$$

This result and Equation (1) corresponding to cycle $t-1$, taken together, imply that:

$$m_t=s_{t-1}E_{t-1}. \tag{21}$$

Substituting $m_t$ in Equation (10) by the right-hand side of Equation (21) results in:

$$g_t=s_{t-1}E_{t-1}(1+f_t)+f_t. \tag{22}$$

Equation (4) implies that $E_{t-1}\left(1+f_t\right)=E_t$. Substituting $E_{t-1}\left(1+f_t\right)$ in Equation (22) by $E_t$ yields:

$$g_t = s_{t-1}E_t + f_t. \tag{23}$$

I assume that, for every $t>1$, the national income increases with respect to its value in the previous cycle. This assumption and the last equation, taken together, imply that:

$$s_{t-1}E_t + f_t > 0. \tag{24}$$

Furthermore, Equation (23) corresponding to cycle $t-1$ is:

$$g_{t-1} = s_{t-2}E_{t-1} + f_{t-1}. \tag{25}$$

Equation (4) implies that $E_{t-1}=\dfrac{E_t}{1+f_t}$. Substituting $E_{t-1}$ in Equation (25) by $\dfrac{E_t}{1+f_t}$ yields:

$$g_{t-1} = s_{t-2}\left(\frac{E_t}{1+f_t}\right) + f_{t-1}. \tag{26}$$

Multiplying and dividing the second term on the right-hand side of this equation by the denominator of the first term and simplifying, results in:

$$g_{t-1} = \frac{s_{t-2}E_t + f_{t-1}\left(1+f_t\right)}{1+f_t}. \tag{27}$$

Dividing term by term Equation (27) by Equation (23) yields:

$$\frac{g_{t-1}}{g_t} = \frac{\left[\dfrac{s_{t-2}E_t + f_{t-1}\left(1+f_t\right)}{1+f_t}\right]}{s_{t-1}E_t + f_t} \tag{28}$$

$$= \left(\frac{\left[\dfrac{s_{t-2}E_t + f_{t-1}\left(1+f_t\right)}{1+f_t}\right]}{s_{t-1}E_t + f_t}\right)\left(\frac{\dfrac{1}{s_{t-1}E_t + f_t}}{\dfrac{1}{s_{t-1}E_t + f_t}}\right) \tag{29}$$

$$= \frac{s_{t-2}E_t + f_{t-1}\left(1+f_t\right)}{\left(1+f_t\right)\left(s_{t-1}E_t + f_t\right)}. \tag{30}$$

This equation will be used in the next Section.

## 5. The $\frac{r}{g}$ Ratio in a Succession of Three Production Cycles

For each $t>2$, let $\gamma_t$ be the growth rate of the $\dfrac{r}{g}$ ratio from cycle $t-1$ to cycle *t*. Then,

$$1+\gamma_t = \frac{\dfrac{r_t}{g_t}}{\dfrac{r_{t-1}}{g_{t-1}}} \tag{31}$$

$$= \left(\frac{r_t}{g_t}\right)\left(\frac{g_{t-1}}{r_{t-1}}\right) \tag{32}$$

$$= \left(\frac{r_t}{r_{t-1}}\right)\left(\frac{g_{t-1}}{g_t}\right). \tag{33}$$

Substituting on the right-hand side of this equation the first and the second factor respectively by the right-hand side of Equations (18) and (30) yields:

$$1+\gamma_t = \left[(1+v_t)(1+f_t)\right]\left[\frac{s_{t-2}E_t + f_{t-1}(1+f_t)}{(1+f_t)(s_{t-1}E_t + f_t)}\right]. \tag{34}$$

Eliminating the factor $1+f_t$ from both the first factor between square brackets and the denominator of the second one, we obtain:

$$1+\gamma_t = (1+v_t)\left(\frac{s_{t-2}E_t + f_{t-1} + f_{t-1}f_t}{s_{t-1}E_t + f_t}\right). \tag{35}$$

For any given succession of three production cycles $t-2$, $t-1$ and *t*, this equation determines the growth rate of the $\frac{r}{g}$ ratio from cycle $t-1$ to cycle *t*. It expresses this rate as the product of two factors of which the first one is the quotient of the profit shares and the second one is the quotient of the income/capital ratios divided by the quotient of the growth rates, in each quotient the value corresponding to cycle $t-1$ divides that of cycle *t*.

Furthermore, for each $t>2$, the residue $\kappa_t$ is defined by:

$$\kappa_t = \frac{s_{t-2}E_t + f_{t-1} + f_{t-1}f_t}{s_{t-1}E_t + f_t} - 1. \tag{36}$$

This variable indicates the increase in the second factor on the right-hand side of Equation (35) due to changes in the income/capital ratio and in the savings rate from cycle $t-2$ to cycle *t*. With this notation, it is possible to write Equation (35) as follows:

$$1+\gamma_t = (1+v_t)(1+\kappa_t). \tag{37}$$

This result implies that:

$$\gamma_t = v_t + \kappa_t + v_t\kappa_t. \tag{38}$$

Therefore, we can formulate the following conclusion.

**Proposition 1**. The growth rate of the $\frac{r}{g}$ ratio is equal to the sum plus the product of the growth rate of the profit share and the $\kappa_t$ residue.

## 6. The $\frac{r}{g}$ Ratio in a Succession of More than Three Production Cycles

Given two dates $t_0$ and $t_1$ such that $1<t_0<t_1-2$, the period $t_0-t_1$ is the succession of production cycles starting with $t_0$ and ending with $t_1$. Let $\gamma_{t_0-t_1}$, $v_{t_0-t_1}$, $f_{t_0-t_1}$, and $\kappa_{t_0-t_1}$ be the values that would correspond respectively to the

variables $\gamma_t, v_t, f_t$ and $\kappa_t$ if they had been constant during the period $t_0 - t_1$. For short, I will refer to each one of these constant values as the average value of the variable concerned.

The set of Equations (31) corresponding to the period $t_0 - t_1$ can be written as follows:

$$\left(\frac{r_{t_1-1}}{g_{t_1-1}}\right)\left(1+\gamma_{t_1}\right)=\frac{r_{t_1}}{g_{t_1}} \quad (39)$$

$$\left(\frac{r_{t_1-2}}{g_{t_1-2}}\right)\left(1+\gamma_{t_1-1}\right)=\frac{r_{t_1-1}}{g_{t_1-1}} \quad (40)$$

$$\cdots$$

$$\left(\frac{r_{t_0+1}}{g_{t_0+1}}\right)\left(1+\gamma_{t_0+2}\right)=\frac{r_{t_0+2}}{g_{t_0+2}} \quad (41)$$

$$\left(\frac{r_{t_0}}{g_{t_0}}\right)\left(1+\gamma_{t_0+1}\right)=\frac{r_{t_0+1}}{g_{t_0+1}}. \quad (42)$$

Substituting in the penultimate equation the quotient $\frac{r_{t_0+1}}{g_{t_0+1}}$ by the left-hand side of the last equation, then substituting in the antepenultimate equation $\frac{r_{t_0+2}}{g_{t_0+2}}$ by the left-hand side of the equation resulting of the first replacement and so on, yields:

$$\left(\frac{r_{t_0}}{g_{t_0}}\right)\left(1+\gamma_{t_0+1}\right)\cdots\left(1+\gamma_{t_1-1}\right)\left(1+\gamma_{t_1}\right)=\frac{r_{t_1}}{g_{t_1}}. \quad (43)$$

Substituting each one of the variables $\gamma_{t_0+1},\cdots,\gamma_{t_1-1},\gamma_{t_1}$ by $\gamma_{t_0-t_1}$ and simplifying, we can write:

$$\left(\frac{r_{t_0}}{g_{t_0}}\right)\left(1+\gamma_{t_0-t_1}\right)^{t_1-t_0}=\frac{r_{t_1}}{g_{t_1}} \quad (44)$$

$\Rightarrow$

$$\gamma_{t_0-t_1}=\left(\frac{r_{t_1}\times g_{t_0}}{r_{t_0}\times g_{t_1}}\right)^{\frac{1}{t_1-t_0}}-1. \quad (45)$$

Similar procedures permit to obtain the following equations:

$$\alpha_{t_0}\left(1+v_{t_0-t_1}\right)^{t_1-t_0}=\alpha_{t_1} \quad (46)$$

$\Rightarrow$

$$v_{t_0-t_1}=\left(\frac{\alpha_{t_1}}{\alpha_{t_0}}\right)^{\frac{1}{t_1-t_0}}-1, \quad (47)$$

and

$$E_{t_0}\left(1+f_{t_0-t_1}\right)^{t_1-t_0}=E_{t_1} \tag{48}$$

$\Rightarrow$

$$f_{t_0-t_1}=\left(\frac{E_{t_1}}{E_{t_0}}\right)^{\frac{1}{t_1-t_0}}-1. \tag{49}$$

Moreover, Equations (43) and (44), taken together, imply that:

$$\left(1+\gamma_{t_0-t_1}\right)^{t_1-t_0}=\left(1+\gamma_{t_0+1}\right)\cdots\left(1+\gamma_{t_1-1}\right)\left(1+\gamma_{t_1}\right). \tag{50}$$

Substituting for each $t$ the factor $\left(1+\gamma_t\right)$ by the right-hand side of Equation (37) corresponding to $t$ we get:

$$\left(1+\gamma_{t_0-t_1}\right)^{t_1-t_0}=\left(1+v_{t_0+1}\right)\left(1+\kappa_{t_0+1}\right)\cdots\left(1+v_{t_1-1}\right)\left(1+\kappa_{t_1-1}\right)\left(1+v_{t_1}\right)\left(1+\kappa_{t_1}\right) \tag{51}$$

$$=\left[\left(1+v_{t_0+1}\right)\cdots\left(1+v_{t_1-1}\right)\left(1+v_{t_1}\right)\right]\left[\left(1+\kappa_{t_0+1}\right)\cdots\left(1+\kappa_{t_1-1}\right)\left(1+\kappa_{t_1}\right)\right]. \tag{52}$$

Now, substituting in Equation (52) each one of the variables $v_{t_0+1},\cdots,v_{t_1-1},v_{t_1}$ and $\kappa_{t_0+1},\cdots,\kappa_{t_1-1},\kappa_{t_1}$ respectively by $v_{t_0-t_1}$ and $\kappa_{t_0-t_1}$ we get, after simplifying:

$$\left(1+\gamma_{t_0-t_1}\right)^{t_1-t_0}=\left(1+v_{t_0-t_1}\right)^{t_1-t_0}\left(1+\kappa_{t_0-t_1}\right)^{t_1-t_0}. \tag{53}$$

The left-hand side of this equation is the factor that multiplies the $\frac{r}{g}$ ratio in the passage from date $t_0$ to date $t_1$ (see Equation (44)). Its right-hand side shows that this factor is a composite of two other factors the first of which is determined by the average growth rate of the profit share and the second one by the average $\kappa_t$ residue.

It follows from the first conclusion presented in the prior paragraph that, when we consider two successive periods $t_0-t_1$ and $t_1-t_2$, the $\frac{r}{g}$ ratio is multiplied by $\left(1+\gamma_{t_0-t_1}\right)^{t_1-t_0}$ in the first period and by $\left(1+\gamma_{t_1-t_2}\right)^{t_2-t_1}$ in the second one. Then, in the passage from date $t_0$ to date $t_2$, the $\frac{r}{g}$ ratio is multiplied by the factor:

$$\left(1+\gamma_{t_0-t_2}\right)^{t_2-t_0}=\left(1+\gamma_{t_0-t_1}\right)^{t_1-t_0}\left(1+\gamma_{t_1-t_2}\right)^{t_2-t_1}. \tag{54}$$

Substituting each one of the factors on the right-hand side by its corresponding equivalent according to Equation (53) we get:

$$\left(1+\gamma_{t_0-t_2}\right)^{t_2-t_0}=\left[\left(1+v_{t_0-t_1}\right)^{t_1-t_0}\left(1+\kappa_{t_0-t_1}\right)^{t_1-t_0}\right]\left[\left(1+v_{t_1-t_2}\right)^{t_2-t_1}\left(1+\kappa_{t_1-t_2}\right)^{t_2-t_1}\right] \tag{55}$$

$$=\left[\left(1+v_{t_0-t_1}\right)^{t_1-t_0}\left(1+v_{t_1-t_2}\right)^{t_2-t_1}\right]\left[\left(1+\kappa_{t_0-t_1}\right)^{t_1-t_0}\left(1+\kappa_{t_1-t_2}\right)^{t_2-t_1}\right]. \tag{56}$$

This equation shows that the factor multiplying the $\frac{r}{g}$ ratio in the passage from date $t_0$ to date $t_2$ is itself a composite of two other factors the first of which is determined by the average growth rates of the profit share and the second one by the average values of the $\kappa_t$ residues corresponding to the two periods.

It also must be mentioned that Equation (53) implies:

$$1+\gamma_{t_0-t_1}=\left(1+v_{t_0-t_1}\right)\left(1+\kappa_{t_0-t_1}\right) \tag{57}$$

$\Rightarrow$

$$\gamma_{t_0-t_1}=v_{t_0-t_1}+\kappa_{t_0-t_1}+v_{t_0-t_1}\kappa_{t_0-t_1} \tag{58}$$

$\Rightarrow$

$$\kappa_{t_0-t_1}=\frac{\gamma_{t_0-t_1}-v_{t_0-t_1}}{1+v_{t_0-t_1}}. \tag{59}$$

If in a given a period $t_0-t_1$ of more than three production cycles the yearly values of the profit rate, the growth rate and the profit share are known, then $\gamma_{t_0-t_1}$ and $v_{t_0-t_1}$ can be calculated substituting in Equations (45) and (47) each variable by its value in the period considered. Then, proceeding in a similar manner in Equation (59), $\kappa_{t_0-t_1}$ can be calculated. Once the three average values are known, it is possible to establish the proportion of the change in the growth rate of the $\frac{r}{g}$ ratio due to $v_{t_0-t_1}$ and $\kappa_{t_0-t_1}$, respectively.

In this regard, because in the Examples presented next the product $v_{t_0-t_1}\kappa_{t_0-t_1}$ is very small relative to the sum of the two factors (less than 0.5% of the sum), it will be neglected in the calculation of the proportions just mentioned in order to simplify. Under this condition, when the sign of both factors is the same, it follows from Equation (58) that the part of $\gamma_{t_0-t_1}$ attributable to each variable $v_{t_0-t_1}$ and $\kappa_{t_0-t_1}$ is proportional to the magnitude of the corresponding variable. Thus, they can be calculated respectively by the formulas:

$$\frac{v_{t_0-t_1}}{v_{t_0-t_1}+\kappa_{t_0-t_1}}, \tag{60}$$

and

$$\frac{\kappa_{t_0-t_1}}{v_{t_0-t_1}+\kappa_{t_0-t_1}}. \tag{61}$$

It is also important to indicate that, in the Examples, the choice made of decennial averages instead of yearly values for some variables has the purpose of reducing the effects of the volatility of these variables. In these cases, the value of each decennial average is assigned to the date at the middle of the corresponding decennial period. For this reason, the duration of the period considered in certain calculations has been reduced in nine years, for instance in Equations (62) and (64) below. Finally, in the numerical computations, no more than six decimals are taken.

**Example 1**. Columns [1] and [6] of Table FR.3c by Piketty and Zucman [13] present the first and the second data of the following pairs of decennial averages: $g_{1900\text{-}1909}=0.004$ , $r_{1900\text{-}1909}=0.039$ ; $g_{1940\text{-}1949}=0.014$ , $r_{1940\text{-}1949}=0.052$ ; $g_{1950\text{-}1959}=0.045$ , $r_{1950\text{-}1959}=0.107$ ; $g_{2000\text{-}2009}=0.011$ , $r_{2000\text{-}2009}=0.057$ . Substituting each variable on Equation (45) by its value in the period 1900-1949 yields:

$$\gamma_{1900\text{-}1949} = \left(\frac{0.052 \times 0.004}{0.014 \times 0.039}\right)^{\frac{1}{40}} - 1 \tag{62}$$

$$= -0.023838 \tag{63}$$

Proceeding similarly with the data from the period 1950-2009 results in:

$$\gamma_{1950\text{-}2009} = \left(\frac{0.057 \times 0.045}{0.011 \times 0.107}\right)^{\frac{1}{50}} - 1 \tag{64}$$

$$= 0.015701 \tag{65}$$

In turn, Column [5] of the same Table presents the following decennial averages $\alpha_{1900\text{-}1909} = 0.28$, $\alpha_{1940\text{-}1949} = 0.14$, $\alpha_{1950\text{-}1959} = 0.23$ and $\alpha_{2000\text{-}2009} = 0.26$. Substituting each variable on Equation (47) by its value in the period 1900-1949 yields $v_{1900\text{-}1949} = \left(\frac{0.14}{0.28}\right)^{\frac{1}{40}} - 1 = -0.017179$. Proceeding in the same way with the data from the period 1950-2009 gives $v_{1950\text{-}2009} = \left(\frac{0.26}{0.23}\right)^{\frac{1}{50}} - 1 = 0.002455$.

Now, substituting each variable on Equation (59) by its value in the period 1900-1949 we get:

$$\kappa_{1900\text{-}1949} = \frac{-0.023838 + 0.017179}{1 - 0.017179} \tag{66}$$

$$= -0.006774 \tag{67}$$

Proceeding in a similar manner with the data from the period 1950-2009 yields:

$$\kappa_{1950\text{-}2009} = \frac{0.015701 - 0.002455}{1 + 0.002455} \tag{68}$$

$$= 0.013213 \tag{69}$$

Furthermore, substituting each variable in formula (60) by its value in the period 1900-1949 yields $\frac{-0.017179}{-0.017179 - 0.006774} = 0.717171$. Proceeding in the same way in formula (61) with the data from the period 1950-2009 yields $\frac{0.013213}{0.002455 + 0.013213} = 0.843331$.

Therefore, in the period 1900-1949 the $\frac{r}{g}$ ratio decreased at an average rate of 2.3838% per year of which 71.7171% was due to the decrease in the profit share and 28.28287% to changes in the income/capital ratio and in the savings rate. Afterwards, in the period 1950-2009 the $\frac{r}{g}$ ratio increased at an average rate of 1.5701% per year of which 84.3316% was due to changes in the income/capital ratio and in the savings rate and 15.6683% was due to the increase in the profit share.

Moreover, Equation (63) implies that:

$$\left(1 + \gamma_{1900\text{-}1949}\right)^{49} = \left(1 - 0.023838\right)^{49} \tag{70}$$

$$= 0.3066 \tag{71}$$

Hence, during the period 1900-1949, the $\frac{r}{g}$ ratio was multiplied by the factor 0.3066 which means a loss of about two thirds of its initial value. In turn, Equation (65) implies that:

$$\left(1+\gamma_{1950\text{-}2009}\right)^{59} = \left(1+0.015701\right)^{59} \tag{72}$$

$$= 2.507188 \tag{73}$$

Thus, during the period 1950-2009 the $\frac{r}{g}$ ratio was multiplied by the factor 2.507188. This constituted a substantial recovery but was insufficient to compensate the diminution that took place during the previous period. Indeed, substituting the first and the second factor on the right-hand side of Equation (54) respectively by the right-hand side of Equations (71) and (73) we get:

$$\left(1+\gamma_{1900\text{-}2009}\right)^{109} = \left(0.3066\right)\left(2.507188\right) \tag{74}$$

$$= 0.7687 \tag{75}$$

This calculation, as well as the next two, does not include the data corresponding to the year 1950, an omission that simplified the task although it may have altered slightly the results. Hence, during the period 1900-2009 the $\frac{r}{g}$ ratio was multiplied by the factor 0.7687 which means a diminution of about one fourth of its initial value. In this regard, Equation (56) can be used to further study the role played by the variables $v_t$ and $\kappa_t$ in this evolution. Indeed, substituting each variable in the first factor on the right-hand side of this formula by its value results in:

$$\left(1+v_{1900\text{-}1949}\right)^{49}\left(1+v_{1950\text{-}2009}\right)^{59} = \left(1-0.017179\right)^{49}\left(1+0.002455\right)^{59} \tag{76}$$

$$= \left(0.427797\right)\left(1.155655\right) \tag{77}$$

$$= 0.494386 \tag{78}$$

Proceeding similarly with the variables in the second factor yields:

$$\left(1+\kappa_{1900\text{-}1949}\right)^{49}\left(1+\kappa_{1950\text{-}2009}\right)^{59} = \left(1-0.006774\right)^{49}\left(1+0.013213\right)^{59} \tag{79}$$

$$= \left(0.716695\right)\left(2.169487\right) \tag{80}$$

$$= 1.554860 \tag{81}$$

The last two results show that the main cause of the decline of the $\frac{r}{g}$ ratio during the period 1900-2009 was the reduction of the profit share that took place during the period 1900-1949. In this respect, it is important to mention that during that half century the profit share attained its lowest level of the period 1820-2010 (Piketty [1], Figure 6.2, p. 201). The $\kappa_t$ residue also contributed to the decline of the $\frac{r}{g}$ ratio with a reduction during the same period that was half of the one affecting the profit share. However, the reduction of the $\kappa_t$ residue was compensated by its increase during the period 1950-2009 while, although the profit share also had an increase in this period, it was not enough to compensate its previous reduction.

Finally, to understand the behavior of the profit share, it is useful to consider that it is the complement to unity of the labor share. Indeed, the labor share is equal to the (real labor income)/(productivity of labor) ratio (see [14], p. 8). For this reason, the changes of the profit share in the French Economy during the periods 1900-1949 and 1950-2009 can be viewed, in a first approximation, as resulting from the increase of the real labor income relative to the productivity of labor in the first period and by its decrease in the second one. I intend to study these relative changes in a separate paper that will follow this one. The next two sections present some propositions concerning the residue $\kappa$.

## 7. $\gamma$ and the Residue $\kappa$ When the Two $\kappa$ Variables Are Constant

Let us assume that in a succession of three production cycles the savings rate ($s$) and the income/capital ratio growth rate ($f$) are constant. Then, for given values of the income/capital ratio ($E$) and of the growth rate of the profit share ($v$), Equation (35) can be written as follows:

$$1+\gamma_t=(1+v)\left(\frac{sE+f+f^2}{sE+f}\right) \tag{82}$$

$$=(1+v)\left(1+\frac{f^2}{sE+f}\right). \tag{83}$$

Inequality (24) implies that the second factor on the right-hand side of Equation (83) is greater than one either if $f>0$ or $f<0$ while it is equal to one if $f=0$, proving the following conclusion.

**Proposition 2**. If the savings rate in cycles $t-2$ and $t-1$ as well as the growth rate of the income/capital ratio in cycles $t-1$ and $t$ are constant, then:

$$\gamma_t \ge v_t. \tag{84}$$

In this relation, there is equality if the income/capital ratio keeps constant. Otherwise there is inequality whether this ratio increases or decreases.

Under the conditions of Proposition 2, the $\frac{r}{g}$ ratio increases whenever the income/capital ratio changes unless there is a sufficient decrease in the profit share.

**Corollary to Proposition 2**. If the profit share is constant, the value of $\gamma_t$ is equal to zero if the income/capital ratio keeps constant and it is greater than zero if this ratio changes whether it increases or decreases.

A similar result was established in Benítez [10], the difference being that, in that case, it refers to the growth rate of production as a whole (national income plus capital depreciation) while here it refers to the growth rate of the national income.

It is also important to present the following four comments.

*Remark* 1. Equations (36) and (83), taken together, imply that:

$$\kappa_t=\frac{f^2}{sE+f}. \tag{85}$$

This formula is useful for the study of a period $t_0 - t_1$ of more than three production cycles. Indeed, under the assumption that the conditions indicated in Proposition 2 are satisfied in an average year, substituting each variable on the right-hand side of Equation (85) by its corresponding average value permits to calculate the residue $\kappa_{t_0 - t_1}$. Then, comparing the result with the average value of the residue calculated using Equation (59), it becomes possible to have an idea of the part of the last value that is attributable to the fact that the conditions just mentioned have not been satisfied.

**Example 2**. Column [1] of Table FR.6e by Piketty and Zucman [13] presents the decennial averages of the (national capital)/(national income) ratio in France during the period 1870-2010. It follows from these data and from Equation (1) that $E_{1900\text{-}1909} = \frac{1}{7.1}$, $E_{1940\text{-}1949} = \frac{1}{3.2}$, $E_{1950\text{-}1959} = \frac{1}{2.98}$ and $E_{2000\text{-}2009} = \frac{1}{5.03}$. Substituting each variable on Equation (49) by its value in the period 1900-1949 yields $f_{1900\text{-}1949} = \left(\frac{7.1}{3.2}\right)^{\frac{1}{40}} - 1 = 0.020123$. Proceeding in the same way with the data from the period 1950-2009 we get $f_{1950\text{-}2009} = \left(\frac{2.98}{5.03}\right)^{\frac{1}{50}} - 1 = -0.010415$. Concerning the income/capital ratio in the period 1900-1949, I equate $E_{1900\text{-}1949}$ to the average value of the decennial averages of the income/capital ratio at the beginning and the end of the period. Thus, $E_{1900\text{-}1949} = \frac{\frac{1}{7.1} + \frac{1}{3.2}}{2} = 0.226672$. Proceeding in a similar manner for the following period gives $E_{1950\text{-}2009} = \frac{\frac{1}{2.98} + \frac{1}{5.03}}{2} = 0.267188$. Furthermore, it follows from Column [13] of Table FR.3c from Piketty and Zucman [13] that the average rate of savings for both periods 1900-1950 and 1950-2010 was 12%.

Substituting each variable in Equation (85) by its value in the period 1900-1950 results in:

$$\kappa_{1900\text{-}1949} = \frac{(0.020123)^2}{(0.12)(0.226672) + 0.020123} \tag{86}$$

$$= 0.008556 \tag{87}$$

This would have been the value of $\kappa_{1900\text{-}1950}$ if the growth rate of the income/capital ratio and the savings rate had been constant during the period 1900-1949. Therefore, the value obtained in Example 1 (see Equation (67)) is attributable to the instability of the variables just mentioned during that 50 years period.

Proceeding in a similar manner with the data corresponding to the period 1950-2009 yields:

$$\kappa_{1950\text{-}2009} = \frac{(-0.010415)^2}{(0.12)(0.267188) - 0.010415} \tag{88}$$

$$= 0.005011 \tag{89}$$

This would have been the value of $\kappa_{1950\text{-}2009}$ if the growth rate of the income/capital ratio and the savings rate had been constant during the period 1950-2009. The difference between this value and the one obtained in Example 1 (see Equation (69)) is attributable to the instability of the variables just mentioned. Thus, the fraction

$$\frac{0.013213 - 0.005011}{0.013213} = 0.620667,$$

that is to say 62.0667% of the value of $\kappa_{1950\text{-}2009}$ calculated in Example 1, can be attributed to that instability and 37.9247% to the average values of the variables involved. The next section presents further analysis of the general effects on the residue $\kappa_t$ of changes in the two $\kappa$ variables and, in particular, of their effects in the two cases studied in this example.

*Remark* 2. For any $\delta \neq 0$ and such that $sE + \delta > 0$, the value of $\gamma_t$ obtained in Equation (79) with $f = -|\delta|$ is greater than the one obtained with $f = |\delta|$. This means that, for any given absolute value of the income/capital ratio growth rate satisfying condition (24), the rate $\gamma_t$ is greater when the income/capital ratio decreases than when it increases.

*Remark* 3. The rate $\gamma_t$ is an increasing function of the absolute value of the income/capital ratio growth rate.

The derivative of the second factor on the right-hand side of Equation (85) with respect to *f* is:

$$\frac{2f(sE+f) - f^2}{(sE+f)^2} = \frac{2fsE + 2f^2 - f^2}{(sE+f)^2} \tag{90}$$

$$= \frac{f(2sE+f)}{(sE+f)^2}. \tag{91}$$

This result and inequality (24), taken together, imply that the sign of the derivative is equal to that of *f*. For this reason, $\gamma_t$ is an increasing function of *f* if $f > 0$ and a decreasing function of *f* if $f < 0$ proving the remark.

## 8. Effects on the Residue *κ* of Changes in the Two *κ* Variables

In order to study the effects on the residue $\kappa_t$ of changes in the two $\kappa$ variables, it is useful to introduce the following two variables defined for $t > 2$:

$$\Delta s_{t-1} = s_{t-1} - s_{t-2}, \tag{92}$$

and

$$\Delta f_t = f_t - f_{t-1}. \tag{93}$$

Substituting $s_{t-1}$ by $s_{t-2} + \Delta s_{t-1}$ and $f_t$ by $f_{t-1} + \Delta f_t$ in Equation (36) yields:

$$\kappa_t = \frac{s_{t-2}E_t + f_{t-1} + f_{t-1}(f_{t-1} + \Delta f_t)}{(s_{t-2} + \Delta s_{t-1})E_t + f_{t-1} + \Delta f_t} - 1. \tag{94}$$

In this formulation, it can be readily appreciated that the residue $\kappa_t$ increases monotonously if $\Delta s_{t-1}$ decreases and vice versa. Regarding $\Delta f_t$, it is useful to multiply and divide the second term on the right-hand side of this equation by the denominator of the first term. After simplifying, results in:

$$\kappa_t = \frac{s_{t-2}E_t + f_{t-1} + f_{t-1}\left(f_{t-1} + \Delta f_t\right) - \left(s_{t-2} + \Delta s_{t-1}\right)E_t - f_{t-1} - \Delta f_t}{\left(s_{t-2} + \Delta s_{t-1}\right)E_t + f_{t-1} + \Delta f_t} \tag{95}$$

$$= \frac{f_{t-1}\left(f_{t-1} + \Delta f_t\right) - \Delta s_{t-1}E_t - \Delta f_t}{\left(s_{t-2} + \Delta s_{t-1}\right)E_t + f_{t-1} + \Delta f_t} \tag{96}$$

$$= \frac{f_{t-1}^2 + f_{t-1}\Delta f_t - \Delta s_{t-1}E_t - \Delta f_t}{\left(s_{t-2} + \Delta s_{t-1}\right)E_t + f_{t-1} + \Delta f_t} \tag{97}$$

$$= \frac{f_{t-1}^2 + \Delta f_t\left(f_{t-1} - 1\right) - \Delta s_{t-1}E_t}{\left(s_{t-2} + \Delta s_{t-1}\right)E_t + f_{t-1} + \Delta f_t}. \tag{98}$$

For each $t > 1$, I assume that:

$$f_t - 1 < 0, \tag{99}$$

which I find justified because, as far as I know, the income/capital ratio changes from one year to the next at rates much lesser than 100% (see Examples 2 and 3). Under this condition, the numerator and denominator on the right-hand side of Equation (98) respectively increase and decrease monotonously if $\Delta f_t$ decreases and, inversely, they decrease and increase monotonously in the opposite case. Therefore, the residue $\kappa_t$ increases monotonously if $\Delta f_t$ decreases and vice versa. This result, taken together with the remark presented just below Equation (94), permit us to formulate the following conclusion.

**Proposition 3.** The residue $\kappa_t$ is a monotonous decreasing function of the increase, from one production cycle to the next, of both the savings rate and the growth rate of the income/capital ratio.

**Example 3**. Column [4] of Table FR.5b by Piketty and Zucman [13], presents the yearly values of the (national capital)/(national income) ratio in France during the period 1870-2010. It follows from these data and from Equation (1) that: $E_{1900} = \frac{1}{6.71}$, $E_{1909} = \frac{1}{6.97}$, $E_{1940} = \frac{1}{3.58}$, $E_{1949} = \frac{1}{2.56}$, $E_{1950} = \frac{1}{2.61}$, $E_{1959} = \frac{1}{3.41}$, $E_{2000} = \frac{1}{3.95}$ and $E_{2009} = \frac{1}{5.98}$. Substituting each variable on Equation (49) by its value in period 1900-1909 gives:

$$f_{1900\text{-}1909} = \left(\frac{6.71}{6.97}\right)^{\frac{1}{9}} - 1 = -0.004215.$$

Proceeding in the same way with the data from the periods 1940-1949, 1950-1959 and 2000-2009 yields respectively:

$$f_{1940\text{-}1949} = \left(\frac{3.58}{2.56}\right)^{\frac{1}{9}} - 1 = 0.037964, \quad f_{1950\text{-}1959} = \left(\frac{2.61}{3.41}\right)^{\frac{1}{9}} - 1 = -0.029269$$

$$\text{and} \quad f_{2000\text{-}2009} = \left(\frac{3.95}{5.98}\right)^{\frac{1}{9}} - 1 = -0.045032.$$

It follows from these results that:

$$\Delta f_{1900\text{-}1949} = \frac{0.037964 - (-0.0042151)}{40} = 0.001054$$

$$\text{and} \quad \Delta f_{1950\text{-}2009} = \frac{-0.045032 - (-0.029269)}{50} = -0.000315.$$

Moreover, Column [13] of Table FR.3c by Piketty and Zucman [13] presents the following decennial averages: $s_{1900\text{-}1909} = 0.12$, $s_{1940\text{-}1949} = 0.14$; $s_{1950\text{-}1959} = 0.14$ and $s_{2000\text{-}2010} = 0.11$. Therefore,

$$\Delta s_{1900\text{-}1949} = \frac{0.14 - 0.12}{40} = 0.0005 \quad \text{and} \quad \Delta s_{1950\text{-}2009} = \frac{0.11 - 0.14}{50} = -0.0006.$$

Substituting in Equation (98) each variable by its corresponding average value during the period 1900-1949 yields:

$$\kappa_{1900\text{-}1949} = \frac{(0.020123)^2 + (0.001054)(0.020123 - 1) - (0.0005)(0.226672)}{(0.12 + 0.0005)(0.226672) + 0.020123 + 0.001054} \tag{100}$$

$$= \frac{0.000404 - 0.001033 - 0.000113}{0.027313 + 0.021177} \tag{101}$$

$$= \frac{-0.000741}{0.048491} \tag{102}$$

$$= -0.015292 \tag{103}$$

This result permits to conclude that the difference between the values of $\kappa_{1900\text{-}1949}$ obtained in the first and second Examples is due to the fact that the second one does not take into consideration the increase in the two $\kappa$ variables that took place during that period. In turn, the difference between the values of $\kappa_{1900\text{-}1949}$ obtained in the first Example and in this one is probably due mainly to the fact that here it is assumed that the increases indicated where constant along the period, which was not the case.

Proceeding in a similar manner with the data corresponding to the period 1950-2009 yields:

$$\kappa_{1950\text{-}2009} = \frac{(-0.010415)^2 + (-0.000315)(-0.010415 - 1) - (-0.0006)(0.267188)}{(0.12 - 0.0006)(0.267188) - 0.010415 - 0.000315} \tag{104}$$

$$= \frac{0.000108 + 0.000318 + 0.000160}{0.032222 - 0.010730} \tag{105}$$

$$= \frac{0.000679}{0.021492} \tag{106}$$

$$= 0.031613 \tag{107}$$

This result permits to conclude that the difference between the values of $\kappa_{1950\text{-}2009}$ obtained in the first and second Examples is due to the fact that the second one does not take into consideration the decrease in the two $\kappa$ variables that took place during that period. In turn, the difference between the values of $\kappa_{1950\text{-}2009}$ obtained in the first Example and in this one is probably due mainly to the fact that here it is assumed that the decreases indicated where constant along the period, which was not the case.

The analysis developed in the three Examples permits to explain the evolution of the $\gamma_{t_0-t_1}$ rate in the French Economy during the periods 1900-1949 and 1950-2009. Indeed, during the first period, the profit share decreased while the two $\kappa$ variables increased. The combined effect of these two negative factors over the $\gamma_{1900\text{-}1949}$ rate nullified the positive effect on the rate due to the steady increase of the income/capital ratio. In the second period, the profit share increased and the two $\kappa$ variables decreased. The combined effect of these two positive factors over the $\gamma_{1950\text{-}2009}$ rate added up to the positive effect on the rate due to a steady decrease of the income/capital ratio.

However, it is important to add that the examples are intended mainly to illustrate the theoretical conclusions by means of some rough calculations. For this reason, the choice made of certain empirical measurements among several—for instance that of the profit share before taxes instead of the profit share after taxes—, does not imply a special need to use that particular measurement. The discussion of the advantages and disadvantages of using the different measurements disposable for each variable is a task beyond the scope of the present paper.

## 9. The $\frac{r}{g}$ Ratio and the Inequality $r > g$

A given succession of production cycles $t_0 - t_1$ may or may not satisfy the condition:

$$\gamma_{t_0-t_1} > 0. \tag{108}$$

If it does, Piketty's inequality $r > g$ results as a consequence of the gradual increase of the $\frac{r}{g}$ ratio when the succession of production cycles is sufficiently long although, in certain cases, reaching that result might require too many cycles (see Benítez [10], p. 1049).

Furthermore, Condition (108) is satisfied when both $v_{t_0-t_1}$ and $\kappa_{t_0-t_1}$ are greater than zero. Also, when one of the two variables is smaller than zero, if the other one is big enough to compensate this fact. Next, some cases corresponding to the second possibility are further discussed.

**Proposition 4**. When the income/capital ratio changes steadily, whether it increases or decreases, the inequality in relation (84) is true excepts if either the savings rate in cycle $t-1$, the growth rate of the income/capital ratio in cycle $t$ or both increase above their respective previous values surpassing certain limits.

Indeed, Propositions 3 implies that the inequality in relation (84) is true either if the last two variables indicated are constant or if they decrease. Due to the continuity of the functions involved, the inequality is also true if the variables do not increase surpassing certain limits, which depend on the numerical values of the other variables present in the second factor on the right-hand side of Equation (35). Hence, it is possible to formulate the following conclusion.

**Corollary to Proposition 4**. If the conditions of Proposition 4 are satisfied, inequality (108) is true unless the profit share decreases enough to compensate the increase of the second factor on the right-hand side of Equation (35).

Finally, considering a succession of production cycles where each one of the two $\kappa$ variables fluctuates around a particular value, it follows from Proposition 3 that the increasing and decreasing effects of these changes over $\gamma_{t_0-t_1}$ will tend to cancel reciprocally. Therefore, also in these cases, if the income/capital ratio changes steadily either increasing or decreasing, as a general rule, $\gamma_{t_0-t_1}$ will satisfy inequality (108) unless there is a compensatory decrease in $v_{t_0-t_1}$.

## 10. Conclusion

The preceding analysis presents the two factors that determine the evolution of the $\frac{r}{g}$ ratio, which depend respectively on the variables $v$ and $\kappa$. Also, it identifies another three variables which, in turn, determine $\kappa$, that is, the income/capital ratio and the two $\kappa$ variables. As a result of the difference in the possible configurations of the variables involved, the product of the two factors can mean a decrease, stagnation, or an increase in the $\frac{r}{g}$ ratio and no reason was identified for the preponderance of any of the three possibilities. Therefore, the analysis allows us to explain the origin of the inequality $r > g$ as a long term product of the growth of the $\frac{r}{g}$ ratio, which in turn results in each empirical case from one particular configuration of the variables involved, but it does not show that there is a general trend towards this inequality within the studied model. From a different angle, Example 1 shows that the decrease in the profit share taking place in France during the period 1900-1949 was the main cause of the decrease of the $\frac{r}{g}$ ratio during that period and, as a consequence, also in the period 1900-2009. Since the profit share in France reached exceptionally low levels during the first half of the last century, the fact could be that the decrease of the $\frac{r}{g}$ ratio just mentioned was also an exception on the long term evolution of this variable. For this reason, we can expect that identifying the periods of growth and decline of the $\frac{r}{g}$ ratio in different countries as well as the configurations of the relevant variables in each case, will permit us to advance in the explanation of the inequality $r > g$ as a general trend of modern economies.

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## References

[1] Piketty, T. (2014) Capital in the Twenty-First Century. Harvard University Press,

Cambridge, MA. https://doi.org/10.4159/9780674369542

[2] Boushey, H., DeLong, J.B. and Steinbaum, M. (2017) After Piketty: The Agenda for Economics and Inequality. Harvard University Press, Cambridge, MA.

[3] Galbraith, J.K. (2014) Unpacking the First Fundamental Law. *Real-World Economics Review*, **69**, 145-149. http://www.paecon.net/PAEReview/issue69/Galbraith69.pdf

[4] Grantham G. (2015) Capitalism in the Twenty-First Century: An Overview. *Basic Income Studies*, 10(1), 7-28. https://doi.org/10.1515/bis-2015-0019

[5] Lindert, P.H. (2014) Making the Most of Capital in the 21st Century. *National Bureau of Economic Research*, Working Paper Number 20232. http://www.nber.org/papers/w20232 https://doi.org/10.3386/w20232

[6] Solow, R. (2014) Thomas Piketty Is Right. *The New Republic*, April 22. https://newrepublic.com/article/117429/capital-twenty-first-century-thomas-piketty reviewed

[7] Delsol, J.P., Lecaussin, N. and Martin, E. (2017) Anti-Piketty: Capital for the 21st Century. Cato Institute, Washington DC.

[8] Mankiew, G. (2015) Yes, $r > g$. So What? *American Economic Review*, **105**, 43-47. https://doi.org/10.1257/aer.p20151059

[9] Rognlie, M. (2015) Deciphering the Fall and Rise in the Net Capital Share. Brooking Economic Papers on Economic Activity, Conference Draft, March, 19-20. http://www.brookings.edu/~/media/Projects/BPEA/Spring-2015/2015a_rognlie.pdf?la=en

[10] Benítez, A. (2016) Piketty's $r > g$ Explained by Changes in the Average Productivity of Capital. *Theoretical Economic Letters*, **6**, 1034-1059. https://doi.org/10.4236/tel.2016.65102

[11] Benítez, A. (2016) Piketty's Inequality between the Profit and Growth Rates and its Implications for the Reproduction of Economic Elites. *Theoretical Economic Letters*, **6**, 1363-1392. https://doi.org/10.4236/tel.2016.66125

[12] Benítez A. (2017) Erratum to "Piketty's Inequality between the Profit and Growth Rates and its Implications for the Reproduction of Economic Elites" [*Theoretical Economic Letters*, **6**, 1363-1392. https://doi.org/10.4236/tel.2016.66125]. *Theoretical Economic Letters*, **7**, 1794-1796. https://doi.org/10.4236/tel.2017.76121

[13] Piketty, T. and Zucman, G. (2014) Capital Is Back: Wealth-Income Ratios in Rich Countries 1700-2010. *The Quarterly Journal of Economics*, Online Appendix. http://piketty.pse.ens.fr/en/capitalisback

[14] Giovannoni O. (2014), What Do We Know About the Labor Share and the Profit Share? Part III: Measures and Structural Factors, Working Paper No. 8*05.* Levy Economics Institute. http://www.levyinstitute.org/publications/what-do-we-know-about-the-labor-share-and-the-profit-share-part-3-measures-and-structural-factors