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The Model-Free Equivalence Condition for American Spread Options

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Abstract

A spread option involves the right to obtain the spread between two asset prices at a predefined strike price. This type of derivative security is frequently used in financial markets and academic finance. Furthermore, analysts use the spread option technique for real option modeling purposes. Some spread options are American-type in the sense that an option holder may exercise her option prior to the expiration. In this paper, we propose an equivalence condition for American spread options under which they are not exercised early, and are therefore equivalent to European options. Our theoretical results, developed within a model-free economic setting, suggest that the equivalence conditions documented by previous papers do not hold in a distribution-free environment. Traders, quantitative modelers, and financial programmers in various derivatives markets and the real option modeling area may use our results.

Keywords

Spread Option, American Option, Early Exercise, Real Option

1. Introduction

In financial markets, a derivative security refers to a contract whose value depends on the values of other assets (underlying assets). One of the simplest examples is a call option that gives the right but not the obligation to buy an underlying asset at a strike price. A call option value depends on only one underlying asset price, but other derivative securities, e.g., exchange and spread options, have two underlying assets. A spread option involves the right to obtain the spread between two asset prices at a predefined strike price ([1]). An exchange option is a special case of a spread option where the strike price is set to zero.

Spread options are used in various markets: the Eurodollar-Treasury rates spread in fixed income markets, locational spreads in commodity markets, heating oil/crude oil spread in energy markets, and index spreads in equity markets, to name a few. Some of these spread options are American-style in the sense that the option holder can exercise her option on or before the pre-defined expiration date. Both CBOE and CME list American spread options, but the majority of the volume comes from over-the-counter (hereafter, "OTC") markets ([2]). American spread options are also important in real options modeling, because the production decisions of manufacturers, refineries, and power plants depend on both the raw material and product markets.

Prior to the expiration, an American option holder needs to make an optimal exercise decision. If the underlying prices are sufficiently high or low, it is optimal for the option holder to exercise her option. It is also interesting for the option holder to know under what conditions the probability of exercising early is trivially zero. The equivalence condition is defined as a condition under which an early exercise is not optimal, regardless of the underlying asset prices. If the equivalence condition is satisfied, an American option is never exercised early, and is therefore equivalent to a European option, which is computationally less expensive to value.

Research in equivalence conditions dates back to Merton ([3]), who showed that an early exercise of American call options is not optimal if the dividend payout rate is zero. Carr ([1]) argued that an American exchange option is not optimally exercised early and is equivalent to a European exchange option under certain equivalence conditions. However, Carr's results for exchange options do not apply to spread options. Villeneuve ([4]) proposed an equivalence condition for an American spread option. However, both Carr's and Villeneuve's equivalence conditions assume bivariate diffusive models. In other words, there is no guarantee that their results hold for unspecified statistical distributions.

In this paper, we study an equivalent condition of American spread options in a model-free economic setting. Our innovation is as follows: We propose a new equivalence condition for unspecified statistical distributions and show that Carr's and Villeneuve's equivalence conditions in a bivariate diffusive setting are not true in a model-free setting. Our second innovation is as follows: We propose a model-free non-early exercise bound that may be useful for traders, quantitative modelers, and financial programmers. With these model-free results, we contribute to both the spread options literature and the literature on the early exercise of American options.

The remainder of this article is structured as follows. Section 2 discusses our model-free economic setting. Section 3 discusses our main theoretical results, and Section 4 compares our results with [1] and [4] results. Section 5 discusses the implications on financial modeling and Section 6 concludes.

2. Definitions and Assumptions

Consider an American-type spread option with expiration T and a payoff

$$\max\left[X(t_i)-Y(t_i)-K,0\right] \tag{1}$$

where $X(t_i)>0$ and $Y(t_i)>0$ are two underlying asset prices at time t_i, $0=t_0<t_1<\cdots<t_N=T$, and K is a strike price. The conditional probabilities of $X(t_{i+1})$ and $Y(t_{i+1})$ at t_i follow an unspecified joint evolution with information available at t_i. This American option has a total of $N+1$ exercise possibilities.

Assume that given $X(t_i)$ and $Y(t_i)$ at time t_i, the conditional probability of the spread option being in-the-money in the next period $\Pr_{t_i}\left[X(t_{i+1})-Y(t_{i+1})>K\right]$ is not extremely as low as 0 and is not extremely as high as 1; this technical condition is always true if $X(t_{i+1})$ and $Y(t_{i+1})$ conditional on $X(t_i)$ and $Y(t_i)$ are continuous random variables with the support of $(0,\infty)$ and $X(t_{i+1})$ and $Y(t_{i+1})$ are not perfectly correlated.

A risk-free interest rate r is predictable. Let $(r-q_X)$ and $(r-q_Y)$ be the net holding costs: In the case of commodities, q_X and q_Y are the rates of the net convenience yield (*i.e.*, the convenience yield *minus* storage costs); in the case of equities, q_X and q_Y are the dividend payout rate; and in the case of futures contracts, $q_X=q_Y=r$. It is well known that $(r-q_X)$ and $(r-q_Y)$ are the expected rates of growth of the two underlying asset prices under a risk-neutral measure. Let $F_X(t_i,t_j)$ and $F_Y(t_i,t_j)$ be time-t_i futures prices maturing at time t_j where $i\le j$; the spot prices corresponding to futures prices $F_X(t_i,t_j)$ and $F_Y(t_i,t_j)$ are $X(t_i)$ and $Y(t_i)$.[1] A no-arbitrage condition implies that $F_X(t_i,t_j)=X(t_i)\exp\left[+(r-q_X)(t_j-t_i)\right]$ and $F_Y(t_i,t_j)=Y(t_i)\exp\left[+(r-q_Y)(t_j-t_i)\right]$. If an analyst builds his/her modeling on the risk premium theory (e.g., [5], among many others), as opposed to non-arbitrage arguments, he or she may consider q_X and q_Y as the rates of the futures risk premium *minus* the spot drift rates in excess of the risk-free interest rate.[2]

3. The Main Results

Lemmas 1 and 2 hold within this model-free economic setting and will be used to derive the main results.

Lemma 1. *The time-*t_i *continuation value of an American spread option with payoff* $\max\left[X(t_i)-Y(t_i)-K,0\right]$, *where* $t_i<T$, *is strictly greater than* $\max\left[X(t_i)\mathrm{e}^{-q_X(t_{i+1}-t_i)}-Y(t_i)\mathrm{e}^{-q_Y(t_{i+1}-t_i)}-K\mathrm{e}^{-r(t_{i+1}-t_i)},0\right]$ *for* $i=0,\cdots,N-1$.

Proof. Let $E_{t_i}^{Q}[\cdot]$ be the conditional expectation on time t_i under a risk-neutral measure. Let $Z\equiv X(t_{i+1})-Y(t_{i+1})$ and $h(z)\equiv\max[z-K,0]$. Because r between t_i and t_{i+1} is known at time t_i,

$$E_{t_i}^{Q}\left[\mathrm{e}^{-r(t_{i+1}-t_i)}\max\left[X(t_{i+1})-Y(t_{i+1})-K,0\right]\right]=\mathrm{e}^{-r(t_{i+1}-t_i)}E_{t_i}^{Q}\left[h(Z)\right]. \tag{2}$$

Because $h(z)$ is weakly convex, Jensen's inequality gives $\mathrm{e}^{-r(t_{i+1}-t_i)}E_{t_i}^{Q}\left[h(Z)\right]\ge\mathrm{e}^{-r(t_{i+1}-t_i)}h\left(E_{t_i}^{Q}[Z]\right)$. Furthermore, the strict inequality holds because $\Pr[Z>K]>0$, $\Pr[Z>K]<1$, and $h(z)$ is non-linear and weakly

[1]In this paper, we use a futures price and a forward price interchangeably.

[2]The seasonality of a commodity (e.g., natural gas) shows up in the spot drift rate.

convex on the support of Z:

$$\mathrm{e}^{-r(t_{i+1}-t_i)}E_{t_i}^{Q}\left[h(Z)\right]>\mathrm{e}^{-r(t_{i+1}-t_i)}h\left(E_{t_i}^{Q}[Z]\right). \tag{3}$$

Combining (2), (3), and the definitions of Z and $h(\cdot)$, we have:

$$E_{t_i}^{Q}\left[\mathrm{e}^{-r(t_{i+1}-t_i)}\max\left[X(t_{i+1})-Y(t_{i+1})-K,0\right]\right]>\max\left[X(t_i)\mathrm{e}^{-q_X(t_{i+1}-t_i)}-Y(t_i)\mathrm{e}^{-q_Y(t_{i+1}-t_i)}-K\mathrm{e}^{-r(t_{i+1}-t_i)},0\right]$$

Because the continuation value of an American spread option at time t_i is greater than or equal to $E_{t_i}^{Q}\left[\mathrm{e}^{-r(t_{i+1}-t_i)}\max\left[X(t_{i+1})-Y(t_{i+1})-K,0\right]\right]$, we obtain the required result.

Lemma 2. *If*

$$\max\left[X(t_i)\mathrm{e}^{-q_X(t_{i+1}-t_i)}-Y(t_i)\mathrm{e}^{-q_Y(t_{i+1}-t_i)}-K\mathrm{e}^{-r(t_{i+1}-t_i)},0\right]\geq\max\left[X(t_i)-Y(t_i)-K,0\right]$$

an American spread option is not exercised early in an optimal manner.

Proof. From Lemma 1,

$$\begin{aligned}\text{The continuation value}&>\max\left[X(t_i)\mathrm{e}^{-q_X(t_{i+1}-t_i)}-Y(t_i)\mathrm{e}^{-q_Y(t_{i+1}-t_i)}-K\mathrm{e}^{-r(t_{i+1}-t_i)},0\right]\\&\geq\max\left[X(t_i)-Y(t_i)-K,0\right]=\text{The immediate exercise value.}\end{aligned}$$

Invoking Lemma 2, Proposition 1 gives the model-free equivalence condition of American spread options.

Proposition 1. *If* $q_X\leq 0\leq q_Y$, *and* $r\geq 0$, *an American spread option is not exercised early in an optimal manner.*

Proof. If $q_X\leq 0$, $q_Y\geq 0$, and $r\geq 0$, it holds that

$$\max\left[X(t_i)\mathrm{e}^{-q_X(t_{i+1}-t_i)}-Y(t_i)\mathrm{e}^{-q_Y(t_{i+1}-t_i)}-K\mathrm{e}^{-r(t_{i+1}-t_i)},0\right]\geq\max\left[X(t_i)-Y(t_i)-K,0\right]$$

Invoking Lemma 2, we obtain the required result.

Lemma 2 is more general than that of Proposition 1. In addition, the parameters for contango/backwardation in Lemma 2 (*i.e.*, q_X and q_Y) are not observable. Hence, we present Proposition 2, which is as general as Lemma 2, but is written in terms of observable futures prices.

Proposition 2. *If*

$$\mathrm{e}^{-r(t_{i+1}-t_i)}\max\left[F_X(t_i,t_{i+1})-F_Y(t_i,t_{i+1})-K,0\right]\geq\max\left[X(t_i)-Y(t_i)-K,0\right]$$

an American spread option is not exercised early in an optimal manner.

Proof. Because

$$F_X(t_i,t_{i+1})=X(t_i)\exp\left[+(r-q_X)(t_{i+1}-t_i)\right]$$

and

$$F_Y(t_i,t_{i+1})=Y(t_i)\exp\left[+(r-q_Y)(t_{i+1}-t_i)\right]$$

it holds that,

$$\mathrm{e}^{-r(t_{i+1}-t_i)}\max\left[F_X(t_i,t_{i+1})-F_Y(t_i,t_{i+1})-K,0\right]\geq\max\left[X(t_i)-Y(t_i)-K,0\right]$$

is equivalent to

$$\max\left[X(t_i)\mathrm{e}^{-q_X(t_{i+1}-t_i)}-Y(t_i)\mathrm{e}^{-q_Y(t_{i+1}-t_i)}-K\mathrm{e}^{-r(t_{i+1}-t_i)},0\right]\geq\max\left[X(t_i)-Y(t_i)-K,0\right]$$

Invoking Lemma 2, we obtain the required result.

Proposition 2 is not an equivalence condition, because the sufficiency part depends on $X(t_i)$ and $Y(t_i)$. However, Proposition 2 provides a model-free non-early exercise bound. Proposition 2 can be interpreted in two ways. First, if the spread of the two futures prices is sufficiently large in comparison to the spread of the two spot prices, the option holder should wait. Second, if the first asset (optioned asset) is sufficiently in contango and the second asset (delivery asset) is sufficiently backwardated, the option holder should defer the exercise of the American spread option.

4. Comparison with Carr's (1988) and Villeneuve's (1999) Results

Assuming bivariate diffusive processes and a dynamically complete market, [1] proposed a equivalence condition of exchange options with a payoff of

$$\max\left[X(t_i)-Y(t_i),0\right]. \tag{4}$$

Observe that an exchange option (4) is a special case of a spread options (1) where $K=0$. Carr found that $q_X=q_Y=0$ or $0\le q_X\le q_Y$ is an equivalent condition. Carr's equivalence condition $q_X=q_Y=0$ is trivially nested by our Proposition 1 if $r\ge 0$.[3] In Proposition 3, we show that Carr's second equivalence condition $0\le q_X\le q_Y$ is *not* true in our model-free setting. We obtain a different result from Carr's because we relax his bivariate diffusive assumption.

Proposition 3. $0\le q_X\le q_Y$, *and* $r\ge 0$ *is not a sufficient condition under which an American spread option is not exercised early in an optimal manner.*

Proof. First of all, I will give an example satisfying $0\le q_X\le q_Y$, and $r\ge 0$ in which the immediate exercise value $\max\left[X(t_i)-Y(t_i)-K,0\right]$ is strictly greater than the greatest lower bound of the continuation value $\max\left[X(t_i)\mathrm{e}^{-q_X(t_{i+1}-t_i)}-Y(t_i)\mathrm{e}^{-q_Y(t_{i+1}-t_i)}-K\mathrm{e}^{-r(t_{i+1}-t_i)},0\right]$ for some $X(t_i)$ and $Y(t_i)$. Let $X(t_i)=10$, $Y(t_i)=5$, $K=0$, $q_X=0.05$, $q_Y=0.06$, and $(t_{i+1}-t_i)=1$. Then,

$$\begin{aligned}\max\left[X(t_i)-Y(t_i)-K,0\right]&=5>4.805=X(t_i)\mathrm{e}^{-q_X(t_{i+1}-t_i)}-Y(t_i)\mathrm{e}^{-q_Y(t_{i+1}-t_i)}\\&=\max\left[X(t_i)\mathrm{e}^{-q_X(t_{i+1}-t_i)}-Y(t_i)\mathrm{e}^{-q_Y(t_{i+1}-t_i)}-K\mathrm{e}^{-r(t_{i+1}-t_i)},0\right].\end{aligned}$$

Because the continuation value can be anywhere between $\max\left[X(t_i)\mathrm{e}^{-q_X(t_{i+1}-t_i)}-Y(t_i)\mathrm{e}^{-q_Y(t_{i+1}-t_i)}-K\mathrm{e}^{-r(t_{i+1}-t_i)},0\right]$ and $+\infty$, depending on the time-t_i conditional joint distribution of $X(t_{i+j})$ and $Y(t_{i+j})$ for $i+j=i+1,\cdots,N$, the immediate exercise value can be greater than the continuation value.

[4] studied the early exercise bounds of derivative securities with multiple underlying assets. Differently from [1], he proposed a non-early exercise condition of spread options. However, our results are different from Villeneuve's in several ways. First, Villeneuve assumed a multivariate diffusive economic setting, but our results are distribution-free. Second, we propose not only an

[3] Carr's equivalence theorem holds regardless of the sign of *r*.

equivalence condition (Proposition 1) but also model-free non-exercise bounds (Proposition 2). Third, our conclusion of the equivalence condition ($q_X \le 0 \le q_Y$) is different from Villeneuve's ($q_X \le 0$). Our Proposition 1 shows that Villeneuve's equivalence condition $q_X \le 0$ is *not* true in our model-free setup.

Proposition 4. $q_X \le 0$ *and* $r \ge 0$ *is not a sufficient condition under which an American spread option is not exercised early in an optimal manner.*

Proof. First of all, I will provide an example satisfying $q_X \le 0$, and $r \ge 0$ in which the immediate exercise value $\max\left[X(t_i) - Y(t_i) - K, 0\right]$ is strictly greater than the greatest lower bound of the continuation value $\max\left[X(t_i)\mathrm{e}^{-q_X(t_{i+1}-t_i)} - Y(t_i)\mathrm{e}^{-q_Y(t_{i+1}-t_i)} - K\mathrm{e}^{-r(t_{i+1}-t_i)}, 0\right]$ for some $X(t_i)$ and $Y(t_i)$. Let $X(t_i) = 10$, $Y(t_i) = 9$, $K = 0$, $q_X = 0$, $q_Y = -0.03$, and $(t_{i+1} - t_i) = 1$. Then,

$$\begin{aligned} &\max\left[X(t_i) - Y(t_i) - K, 0\right] = 1 > 0.7259 \\ &= \max\left[X(t_i)\mathrm{e}^{-q_X(t_{i+1}-t_i)} - Y(t_i)\mathrm{e}^{-q_Y(t_{i+1}-t_i)} - K\mathrm{e}^{-r(t_{i+1}-t_i)}, 0\right]. \end{aligned}$$

The remainder of the proof is similar to that of Proposition 3's.

5. Discussion

We derive the model-free equivalence condition for American spread options (Proposition 1). In addition, we also propose a model-free non-early exercise condition (Proposition 2). These novel results are useful to traders, quantitative modelers, and programmers in various financial markets. First, one may use a spread option for real option modeling purposes. Consider a real option decision where a firm shuts down asset $Y(t_i)$ and invests in asset $X(t_i)$. This asset replacement decision is irreversible, but a firm can delay this investment decision to the next period. Let K be the costs of acquiring asset $X(t_i)$ and decommissioning asset $Y(t_i)$. For example, $X(t_i)$ is a natural gas-fired electricity generation plant, $Y(t_i)$ is a coal-fired plant, and K is the costs of building a natural gas plant and decommissioning a coal plant (See, e.g., [6] [7]). Because the two asset prices can be far more complicated than diffusive processes, our model-free results are relevant.

Second, our results have implications for exchange-traded American spread options on futures contracts. Since 1992, NYMEX has listed American-style crack spread options, defined as an option on the spread between an oil product and crude oil futures prices. End-users of crack spread options include refineries and inventory holders. If the two underlying assets are futures contracts, then $q_X = q_Y = r > 0$, and the sufficient conditions in Proposition 1 (*i.e.*, $q_X \le 0 \le q_Y$, and $r \ge 0$) are not satisfied. Furthermore, it is easy to see that the sufficient condition in Proposition 2 is satisfied only when the spread futures option is out-of-the-money. Therefore, there are no grounds to say that American spread options on futures contracts are equivalent to their European counterparts.

Third, a trader who is interested in an optimal exercise decision of American spread options may use Proposition 2 because spot and futures (or forward) prices are available in the market. If the optioned asset's forward curve $\left[X(t_i), F_X(t_i, t_{i+1})\right]$ is sufficiently contango or the delivery asset

$\left[Y(t_i), F_Y(t_i, t_{i+1})\right]$ is sufficiently backwardated, the trader should not exercise her option. For the model-free non-early exercise condition, only the front parts of the forward curves from t_i to t_{i+1} matter.

Finally, our propositions have implications for quant modelers and financial programmers in various OTC markets. If a closed-form solution of a spread option does not exist, a modeler should rely on a numerical method. He/she may be interested in an underlying process beyond diffusive processes such as a jump-diffusion model or a regime-switching model. If his/her program exercises an American option early, and the equivalence condition (Proposition 1) is satisfied, it is very likely that his/her program has a bug. A quant modeler may simulate a forward curve. If an early exercise is made even when the sufficiency part in Proposition 2 is satisfied, he/she needs to double-check his/her code.

6. Conclusions

We have derived a model-free equivalence condition for American spread options under which an American spread option is equivalent to a European spread option because an early exercise is not allowed. In addition, we also propose a model-free non-early exercise condition. These results are new, because different from previous papers, we use a model-free economic set-up.

This paper focuses on an American spread option's equivalence condition. We do not provide an insight into other complex American option's (e.g. American rainbow option) model-free equivalence condition. This is a limitation of our paper, and we leave this interesting and important topic as a future research agenda.

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