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A Biased Expectation Equilibrium in Indeterminate DSGE Models*

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ABSTRACT

The aim of this article is to introduce a solution method for an indeterminate dynamic stochastic general equilibrium (DSGE) model. The method uses the concept of a biased expectation equilibrium, which is defined in this paper and means that expectations of certain variable are mechanically biased against those that would be rational. Our method should be particularly useful in terms of empirical estimation using DSGE models, because it will allow researchers to estimate how much agents' expectations are biased in the case where a model has indeterminacy.

Keywords: DSGE Modeling; Rational Expectations Model; Indeterminacy

1. Introduction

There are many methods available for computing the saddle path of a linear rational expectations model. However, a model of this sort can nevertheless be in a condition of indeterminacy. For example, it is well known that in a simple New Keynesian model, comprised of the consumption Euler equation, the New Keynesian Phillips curve, and a monetary policy rule, equilibrium is indeterminate if the monetary policy is accommodative of inflation; for example, see Leeper [1].

As surveyed in Fernández-Villaverde [2], the methods for estimating DSGE models have been developed and many empirical studies, such as that of Smets and Wouters [3] have been conducted. Notably, most of these studies assume that an economy is always on a saddle path, although an economy can have a possibility of indeterminacy (The exception is Lubik and Schorfheide [4]. They evaluate the US economy using a simple New Keynesian model that allows for indeterminacy). However, this assumption may be restrictive It is hence useful to develop a method that can handle indeterminacy, especially for empirical research programs. Furthermore, with such a method, the forecasting performance of the DSGE-VAR model, which is introduced by Del Negro and Schorfheide [5], would be improved.

To handle indeterminacy, this article introduces 1) a biased expectations equilibrium in which certain variables' expectations are biased against rational expectations and 2) a method of computing that equilibrium. Lubik and Schorfheide [6] have demonstrated how to handle indeterminacy through exogenous sunspot shocks, based on Sims [7]. Our approach is different from theirs in that the path of economic variables is here determined through subjective expectations that may be independent of the underlying structural model. This paper simply expresses "subjective expectations" as a situation under which expectations are mechanically biased with respect to rational expectations.

Our paper is organized as follows. Section 2 sets a linear rational expectations model. Section 3 introduces the definition of a biased expectation equilibrium and shows how to compute a biased equilibrium by employing subjective expectations. Section 4 constructs a simple New Keynesian model, and we apply our method to it.

2. Settings of the Model

Consider the following linear rational expectations model:

$$AE_t\left[x_{t+1}\right] + Bx_t + Cx_{t-1} + Dz_t = 0\,, \tag{1}$$

$$z_t = Nz_{t-1} + \varepsilon_t\,,\quad E_t\left[\varepsilon_{t+j}\right] \text{ for all } j > 0\,,$$

where x_t denotes a vector of endogenous variables that comprise k state variables and n jump variables, and z_t represents a vector of the exogenous variables. Then, what we are looking for is the following expression:

$$x_t = Px_{t-1} + Qz_t\,, \tag{2}$$

Note that in these settings, the lagged jump variables

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are taken as state variables.[1] Substituting Equation (2) into Equation (1), the following conditions are obtained:

$$APP + BP + C = 0, \quad (3)$$

$$(AP + B)Q + AQN + D = 0, \quad (4)$$

The matrix quadratic Equation (3) can be used to check whether the solution is none, indeterminate, or unique by using the following matrices:

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$$

and

$$\tilde{B} = \begin{bmatrix} -B & -C \\ I & 0 \end{bmatrix},$$

where I represents the identity matrix. To solve Equation (3), we apply QZ decomposition to $\tilde{A}$ and $\tilde{B}$. This decomposition yields upper triangular matrices S and T and orthogonal matrices Σ and Z, such that

$$\tilde{A} = \Sigma S Z$$

and

$$\tilde{B} = \Sigma T Z .$$

This decomposition can be arranged such that the absolute ratio of diagonal entries of S and T (that is $|S_{ii}/T_{ii}|$), is in ascending order. If the number of the ratio exceeding one denoted by r is equal to n, the model has a saddle path. On the other hand, if $r < n$, the solution is indeterminate.[2]

By QZ decomposition and counting r, the following restrictions for P are obtained:

$$Z_{21}P + Z_{22} = 0 \quad (5)$$

where Z_{21} and Z_{22} denote block matrices of size $r \times n$. If $n = r$, then P can be computed as:

$$P = -Z_{21}^{-1}Z_{22} \quad (6)$$

If $r < n$, then P cannot be determined uniquely and this case (indeterminacy) will be explained in the next section.

The matrix $\boldsymbol{Q}$ is calculated from (4) as follows

$$\text{vec}(\boldsymbol{Q}) = \left[N' \otimes A + I' \otimes (AP + B)\right]^{-1} \text{vec}(D),$$

where "vec" operator denotes column-wise vectorization.

3. A Biased Expectations Equilibrium

Assume the indeterminate economy, that is, $p \equiv n - r > 0$. In this case, P cannot be uniquely obtained. Therefore, additional restrictions for P are needed. To set these restrictions, we assume the existence of forecasters who form expectations of certain variables based on their subjective knowledge. Furthermore, this paper assumes that agents in the economy use subjective expectation in their decision-making. This paper simply assumes that the subjective expectation is expressed as a situation under which it there is mechanical bias against rational expectations.

3.1. Definition

This section defines a biased expectation equilibrium. To do this, we assume that subjective expectations are related to rational expectations as follows. Letting x_t^i be the ith element of x_i,

$$E_t^s\left[x_{t+1}^i\right] = \alpha_i E_t\left[x_{t+1}^i\right] \quad (7)$$

where the "E_t^s" operator is the operator of subjective expectations. Suppose that this forecaster always forms the biased expectations that are represented by the scalar α_i. Assume that the biased expectations of the ith element of x_i are formed in the following manner (for example, by traditional macro-econometric models):

$$E_t^s\left[x_{t+1}^i\right] = \beta_i x_t, \quad (8)$$

where β_i denotes an $n-$dimensional row vector. Note that Equation (8) may be independent of the underlying economic structure, and that if $\alpha_i = 1$, subjective expectation reduces to adaptive expectation.

Next note that p forecasts, expressed as Equation (8), are required to compute P. In order to choose the variables predicted by the forecasters from x_i, define the choice matrix $\boldsymbol{v}$ of size $p \times n$ such that for each row, the ith column takes one if the ith variable of x_i is to be forecasted and zero otherwise. For example, if $p = 2$ and the forecasters predict the first and the second variable of x_i, then

$$\boldsymbol{v} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix}.$$

From this choice matrix, the following equation is obtained

$$vE_t^s\left[x_{t+1}\right] = v_\alpha E_t\left[x_{t+1}\right] \quad (9)$$

where v_α is a $p \times n$ matrix such that one in v is replaced by α_i. Furthermore, v and Equation (8) yield

$$vE_t^s\left[x_{t+1}\right] = v_\beta x_t, \quad (10)$$

where v_β represents a $p \times n$ matrix such that each row is β_i. For expectations other than vx_t, it is assumed that agents can form rational expectations. Here, we define a biased expectations equilibrium path as follows.

[1]By applying these settings to the method of undetermined coefficients, the full column rankness of C in the framework based on Uhlig [8] is not needed.

[2]If r = 0, one can solve the model by using "sunspot shocks." This method was first introduced by Farmer and Guo [9]. The approach using sunspot shocks is applicable even if 0 < r < n. See Lubik and Schorfheide [6].

3.2. Definition: A Biased Expectations Equilibrium

Suppose $n-r>0$. We call the sequence $\{x_t\}$ a biased expectations equilibrium path if it satisfies Equations (1), (9), and (10).

3.3. Computation Method

A biased expectations equilibrium path can be easily computed as follows. If $n-r>0$, then the model represented by Equation (1) can be replaced by

$$\begin{aligned}&A\left\{(I-\Pi)E_t\left[x_{t+1}\right]+\Pi E_t^s\left[x_{t+1}\right]\right\}\\&+Bx_t+Cx_{t-1}+Dz_t=0,\end{aligned} \tag{11}$$

where Π represents a $n\times n$ matrix such that the *i*th diagonal element is one if the *i*th variable of x_t is to be forecasted and zero otherwise. To facilitate computation, define Ψ as an $n\times n$ diagonal matrix such that the *i*th diagonal element is α_i if the *i*th variable of x_t is to be forecasted and one otherwise. The off-diagonal entries of Π and Ψ are zero. In the example with Π and Ψ, if $n=3$, $p=2$, and the forecasted variables are the first and second elements of x_t,

$$\Pi=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$$

and

$$\Psi=\begin{bmatrix}\alpha_1&0&0\\0&\alpha_2&0\\0&0&1\end{bmatrix}$$

In this case, noting

$$(I-\Pi)E_t\left[x_{t+1}\right]+\Pi E_t^s\left[x_{t+1}\right]=(I-\Pi+v\Psi)E_t\left[x_{t+1}\right],$$

Equations (3) and (4) are replaced by

$$A(I-\Pi+v\Psi)PP+BP+C=0 \tag{12}$$

$$\begin{aligned}&\left\{A(I-\Pi+v\Psi)P+B\right\}Q\\&+A(I-\Pi+v\Psi)QN+D=0\end{aligned} \tag{13}$$

Equations (2), (9), and (10) yield $v_\alpha Px_t=v_\beta x_t$. Therefore, the following equations are obtained:

$$\left(v_\alpha P-v_\beta\right)=0. \tag{14}$$

Using the restrictions that are implied by Equation (11), we can compute the unique *P* as follows:

$$P=\begin{bmatrix}Z_{21}\\v_\alpha\end{bmatrix}^{-1}\begin{bmatrix}-Z_{22}\\v_\beta\end{bmatrix}. \tag{15}$$

where Z_{21} and Z_{22} are obtained by the QZ decomposition of $\tilde{A}$ in which *A* is replaced by $A(I-\Pi+v\Psi)$, and $\tilde{B}$. Note that to obtain a biased expectation equilibrium, it is needed that the underlying model has $n-r>0$ under this replaced $\tilde{A}$.

4. Example

This section provides examples of the application of our method to an indeterminate economy. Consider the following simple New Keynesian Model described by

$$E_t\left[y_{t+1}\right]+\sigma E_t\left[\pi_{t+1}\right]=y_t+\sigma R_t,$$

$$\pi_t=\beta E_t\left[\pi_{t+1}\right]+\kappa y_t,$$

$$R_t=\xi\pi_t+\varepsilon_t,$$

where y_t denotes the output, π_t the inflation, and R_t the nominal interest rate. All variables are deviations from the steady state; ε_t denotes a monetary policy shock, and $E_t\left[\varepsilon_{t+j}\right]$ for all $j>0$. Using the notation of the model (1), $x_t=\left[y_t \quad R_t \quad \pi_t\right]'$ and $z_t=\left[\varepsilon_t\right]$. For the coefficients matrices,

$$A=\begin{bmatrix}1&0&\sigma\\0&0&-\beta\\0&0&0\end{bmatrix},\ B=\begin{bmatrix}-1&-\sigma&0\\0&-\kappa&1\\0&1&-\xi\end{bmatrix},\ C=\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix},$$

and

$$D=\begin{bmatrix}0\\0\\-1\end{bmatrix}.$$

It is well known that if $\xi>1$ (the Taylor principle), the model has a unique path. Note that in this case *P* is the zero matrix, and if $\xi<1$, the solution is indeterminate.

Suppose $\left[\sigma \quad \kappa \quad \beta \quad \xi\right]=\left[1 \quad 0.5 \quad 0.99 \quad 0.95\right]$. In this case, we have $p=1$. Further, assume that

$$v=\left[0 \quad 0 \quad 1\right],$$

$$v=v_\alpha,$$

and

$$E_t^s\left[\pi_{t+1}\right]=0.5\pi_t.$$

Note that $v=v_\alpha$ implies that agents' expectations are adaptive. These imply that

$$v_\beta=\left[0 \quad 0 \quad 0.5\right],$$

$$\Pi=\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix},$$

and

$$\Psi=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

Applying our method, we obtain

$$\begin{bmatrix} y_t \\ R_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.055 \\ 0 & 0 & 0.475 \\ 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ R_{t-1} \\ \pi_{t-1} \end{bmatrix} + \begin{bmatrix} -0.719 \\ 0.324 \\ -0.712 \end{bmatrix} \varepsilon_t ,$$

Next, consider that the forecasters overestimate expected inflation in the sense of $\alpha = 2$, which implies

$$v_\alpha = [0 \quad 0 \quad 2]$$

in the above settings.[3] In this case,

$$\begin{bmatrix} y_t \\ R_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.015 \\ 0 & 0 & 0.238 \\ 0 & 0 & 0.25 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ R_{t-1} \\ \pi_{t-1} \end{bmatrix} + \begin{bmatrix} -0.699 \\ 0.343 \\ -0.692 \end{bmatrix} \varepsilon_t .$$

This result implies that if the forecasters' estimates are overshot, then the effect of the monetary policy will be mitigated.

5. Final Remarks

In this paper, we presented a way to handle indeterminacy in DSGE models by introducing a biased expectations equilibrium in which certain variables' expectations are biased against rational expectations. We then introduced a method for such an equilibrium. The method is particularly useful in terms of the empirical estimation of DSGE models, because in an indeterminate economy, researchers can estimate how much agents' expectations are biased.

Whether a biased equilibrium in an indeterminate economy as defined in this paper is actually valid is an empirical matter and a task for future research.

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[3]This says that for example, if $E_t[\pi_{t+1}] = 0.02$, then $E_t^s[\pi_{t+1}] = 0.04$.

