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Weak and Strong Time Consistency

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ABSTRACT

We motivate and provide proofs of Başar and Olsder's (1995) theorems on the subject. The context is the increasing appreciation that the neoclassical framework is not the only model of the economy.

Keywords: Information Sets; Discretion

1. Introduction

The longstanding critique of the Dynamic Stochastic General Equilibrium (DSGE) model has gained widespread publicity as a consequence of the recent financial-real meltdown [1]. Of interest to us are an assumption and a theorem. The assumption of rational expectations is that agents access all available information. DSGE theorems leave no room for inefficient outcomes. Some of the profession's most respected scholars at the Hearings referred to drew attention to the weight of evidence suggesting that the choice of information sets by people is a nontrivial matter. Secondly, an outcome might be a Keynes equilibrium like involuntary unemployment. A suite of orientations is available to incorporate these insights [2,3].

A corollary of DSGE results is the time inconsistency of optimal policy. A game is played by the monetary authorities (MA) and the private sector. A Nash equilibrium in the inflation rate will exist in the present. However, in any following period, the MA has an incentive to generate a higher rate of inflation in order to stimulate activity. The gain is short term. In the long run, the economy will tend to the natural rate of unemployment with the higher rate of inflation. Consequently, the task is to remove discretion from the MA and subject monetary policy committees to rules.

Although the language of differential games is used, the nuance of the title of the paper will not be found in the economics literature. We exploit the distinction to show, under specified conditions, that discretion is not inferior to rules.

2. The Result

The following account is drawn from [4]. The loop model of dynamic games allows for two possible equilibrium solutions. In the *prior commitment* mode of play, decisions are made at the outset. *Feedback games*, on the other hand, are of the *delayed commitment type*. Each player waits to find out the current value of the state vector and then announces her action. Time inconsistency might be expected in prior commitment decisions. In the absence of devices that tie the players' hands in advance, there is an incentive for any one to recompute her strategy in each period based on the information that is forthcoming. In the macroeconomic illustrations, the payoff to the government (leader) by "cheating" thereby increases and the payoffs to members of the private sector (follower) fall. In what follows, therefore, we confine ourselves to the Feedback Nash Equilibrium (FNE) Solution.

Definition 1. A *two-person discrete-time deterministic infinite dynamic game* of fixed duration involves:

1) An index set $\mathbf{K} = \{1, \cdots, K\}$ denoting the *stages* of the game.

2) An infinite set *X* with some topological structure called the *state space* of the game to which the state of the game x_k belongs for all $k \in \mathbf{K} \cup K+1$.

3) An infinite set U_k^i with some topological structure defined for each $k \in \mathbf{K}$ and player *i* called the action set of **P***i* at stage *k*. Its elements are permissible actions u_k^i of **P***i* at stage *k*.

4) A function $f_k : X \times U_k^1 \times U_k^2 \to X$, defined for each $k \in \mathbf{K}$, so that $x_{k+1} = f_k\left(x_k, u_k^1, u_k^2\right)$, $k \in K$, for some $x_1 \in X$ called the *initial state* of the game. The difference equation above is called the *state equation* of the dynamic game.

5) A finite set η_k^i defined for each $k \in \mathbf{K}$ and player *i* as a subcollection of

$$\left\{x_1, \cdots, x_k; u_1^1, \cdots, u_{k-1}^1; u_1^2, \cdots, u_{k-1}^2\right\},$$

which determines the information gained and recalled by **P***i* at stage *k*. Specification of η_k^i for all stages *k* characterizes the *information structure* of **P***i* and the collection over both players of their information structures is the *information structure* of the game.

6) A set N_k^i defined for each $k \in \mathbf{K}$ and player *i* as an appropriate subset of

$$\left\{ \left(X_1 \times \cdots \times X_k \right) \times \left(U_1^1 \times \cdots U_{k-1}^1 \right) \times \left(U_1^2 \times \cdots U_{k-1}^2 \right) \right\},$$

compatible with $\eta_k^i \cdot N_k^i$ is called the *information space* of **P***i* at stage *k*.

7) A prespecified class Γ_k^i of mappings $\gamma_k^i : N_k^i \to U_k^i$ which are the permissible *strategies* of **P***i* at stage *k*. The aggregate mapping $\gamma^i = \left\{ \gamma_1^i, \gamma_2^i, \cdots, \gamma_K^i \right\}$ is a *strategy* for **P***i* in the game and the class Γ^i of all such mappings γ^i so that $\gamma_k^i \in \Gamma_k^i$, $k \in \mathbf{K}$, is the strategy space of **P***i*.

8) A functional

$$J^i : \left(X \times U_1^1 \times U_1^2 \right) \times \left(X \times U_2^1 \times U_2^2 \right) \times \cdots \times \left(X \times U_K^1 \times U_K^1 \right) \to \mathbf{R}$$

defined for each player *i* called the *cost functional* of **P***i*. The cost functional is said to be *stage-additive* if there exist $g_k^i : X \times X \times U_k^1 \times U_k^2 \to \mathbf{R}$, ($k \in \mathbf{K}$), so that

$$J^i \left(x, u^1, u^2 \right) = \sum_{k=1}^{K} g_k^i \left(x_{k+1}, u_k^1, u_k^2, x_k \right)$$

where $u^j = \left(u_1^j, \cdots, u_K^j \right).$

The information structures of relevance are as follows:

Definition 2. **P***i*'s information structure is said to be

open-loop (OL) if $\eta_k^i = \left\{ x_1 \right\}, k \in K,$

closed-loop perfect state (CLPS) if

$$\eta_k^i = \left\{ x_1, \cdots, x_k \right\},\ k \in K,$$

memoryless perfect state (MPS) if

$$\eta_k^i = \left\{ x_1, x_k \right\},\ k \in K,$$

feedback perfect state (FB) if $\eta_k^i = \left\{ x_k \right\}, k \in K.$

An important distinction, for our purposes, rests between the CLPS and MPS structures, on the one hand, and the FB on the other. Rational expectations is consistent with the former. History matters. If, on the other hand, agents suffer from memory loss or choose not to access past data because of cognitive or out-of-pocket costs, the latter prevails. In the buildup to the present recession, for instance, people seem to have forgotten previous recessions and history given by the complete cycle.

The Noncooperative (Nash) Equilibrium Solution is given by

Definition 3. A pair of strategies $\left\{ \gamma^{1*}, \gamma^{2*} \right\}$ with $\gamma^{i*} \in \Gamma^i, i = 1, 2$ is said to constitute a Nash Equilibrium Solution if, and only if, the following inequalities are satisfied for all $\left\{ \gamma^i \in \Gamma^i; i = 1, 2 \right\}$:

$$J^{1*} \cong J^1 \left(\gamma^{1*}, \gamma^{2*} \right) \le J^1 \left(\gamma^1, \gamma^{2*} \right)$$

$$J^{2*} \cong J^2 \left(\gamma^{1*}, \gamma^{2*} \right) \le J^2 \left(\gamma^{1*}, \gamma^2 \right)$$

In the next section we restrict our attention to feedback games where at the time of her act each player has perfect information concerning the current level of play. In that case, the set of inequalities above is rewritten as

Definition 4. A pair of strategies $\left\{ \gamma^{i*}, \gamma^{2*} \right\}$ constitutes a *Feedback Nash Equilibrium Solution* if it satisfies the following inequalities for all $\gamma_K^I \in \Gamma_K^I, i = 1, 2,\ k \in K.$

$$J^{1*} \le J^1 \left(\gamma_1^1, \cdots, \gamma_K^1; \gamma^{2*} \right)$$

$$J^{2*} \le J^2 \left(\gamma^{1*}; \gamma_1^2, \cdots, \gamma_K^2 \right)$$

On any pair of Nash equilibrium strategies that satisfies the above inequalities, impose the further restriction that it satisfies the following *K* inequalities:

At stage *K*

$$J^1 \left(\gamma_1^1, \cdots, \gamma_{K-1}^1, \gamma_K^{1*}; \gamma_1^2, \cdots, \gamma_{K-1}^2, \gamma_K^{2*} \right) \le J^1 \left(\gamma_1^1, \cdots, \gamma_{K-1}^1, \gamma_K^1; \gamma_1^2, \cdots, \gamma_{K-1}^2, \gamma_K^{2*} \right)$$

$$J^2 \left(\gamma_1^1, \cdots, \gamma_{K-1}^1, \gamma_K^{1*}; \gamma_1^2, \cdots, \gamma_{K-1}^2, \gamma_K^{2*} \right) \le J^2 \left(\gamma_1^1, \cdots, \gamma_{K-1}^1, \gamma_K^{1*}; \gamma_1^2, \cdots, \gamma_{K-1}^2, \gamma_K^2 \right)$$

At stage $K - 1$

$$J^1 \left(\gamma_1^1, \cdots, \gamma_{K-2}^1, \gamma_{K-1}^{1*}, \gamma_K^{1*}; \gamma_1^2, \cdots, \gamma_{K-2}^2, \gamma_{K-1}^{2*}, \gamma_K^{2*} \right) \le J^1 \left(\gamma_1^1, \cdots, \gamma_{K-2}^1, \gamma_{K-1}^1, \gamma_K^{1*}; \gamma_1^2, \cdots, \gamma_{K-2}^2, \gamma_{K-1}^{2*}, \gamma_K^{2*} \right)$$

$$J^2 \left(\gamma_1^1, \cdots, \gamma_{K-2}^1, \gamma_{K-1}^{1*}, \gamma_K^{1*}; \gamma_1^2, \cdots, \gamma_{K-2}^2, \gamma_{K-1}^{2*}, \gamma_K^{2*} \right) \le J^2 \left(\gamma_1^1, \cdots, \gamma_{K-2}^1, \gamma_{K-1}^{1*}, \gamma_K^{1*}; \gamma_1^2, \cdots, \gamma_{K-2}^2, \gamma_{K-1}^2, \gamma_K^{2*} \right)$$

At stage 1

$$J^1 \left(\gamma^{1*}, \gamma^{2*} \right) \le J^1 \left(\gamma_1^1, \gamma_2^{1*}, \cdots \gamma_K^{1*}; \gamma^{2*} \right)$$

$$J^2 \left(\gamma^{1*}, \gamma^{2*} \right) \le J^2 \left(\gamma^{1*}; \gamma_1^2, \gamma_2^{2*}, \cdots \gamma_K^{2*} \right)$$

For appreciating the notion of time consistency, the following notation and definitions are required. Following Başar and Jan Olsder, we abuse notation by using the denoting the discrete time period by the continuous time period. Thus, $\left[0, T \right] = \left\{ 1, \cdots, K \right\}$. Let

$$D\left(\Gamma; \left[0, T \right] N \right)$$

denote a two-person dynamic game where Γ is the product strategy space, $\left[0, T \right]$ is the decision interval and *N* stands for the Nash solution concept. Also, let

$$\gamma_{[s,t]} \in \Gamma_{[s,t]}, \gamma_{[s,t]}^i \in \Gamma_{[s,t]}^i$$

denote the truncations of $\gamma \in \Gamma$ and $\gamma^i \in \Gamma^i$, respectively, to the time period $\left[s, t \right] \subset \left[0, T \right]$, and let

$$D^{\beta}_{[s,t]} \equiv D\left(\left\{\gamma \in \Gamma : \gamma_{[0,s)} = \beta_{[0,s)}, \gamma_{(s,T]} = \beta_{(s,T]}, \gamma_{[s,t]} \in \Gamma_{[s,t]}\right\}; [0,T]; N\right)$$

denote a version of $D(\Gamma;[0,T]N)$ where the strategies of both players i in the intervals $[0,s)$ and $(t,T]$ are fixed as $\beta^i_{[0,s)}, \beta^i_{(s,T]}$. In that case, we have

Definition 5. A pair of strategies $\{\gamma^{1*},\gamma^{2*}\}$ solving the dynamic game $D(\Gamma;[0,T]N)$ is said to be *weakly time consistent* (WTC) if its truncation to the interval $[s,T], \gamma^*_{[s,T]}$, is the solution of the truncated game $D^{\gamma^*}_{[s,T]}$, for all $s \in (0,T]$. If a solution $\gamma^* \in \Gamma$ is not WTC, it is time inconsistent.

Definition 6. A pair of strategies $\{\gamma^{1*},\gamma^{2*}\}$ solving the dynamic game $D(\Gamma;[0,T]N)$ is said to be *strongly time consistent* (STC) if its truncation to the $[s,T], \gamma^*_{[s,T]}$, is the solution of the truncated game $D^{\beta}_{[s,T]}$, for every $\beta_{[0,s)} \in \Gamma_{[0,s)}$, this being so for every $s \in (0,T]$.

In the case of Definition 5 there is no basis at k for cheating at future stages only if the past actions are consistent with the original solution, whereas in the instance of Definition 6, this is true even if there have been deviations in the past from the actions dictated by the optimal strategy.

Definition 7. A strategy $\gamma^* \in \Gamma$ is said to be a representation of another strategy $\tilde{\gamma} \in \Gamma$ if

1) they both generate the same unique state trajectory and

2) they both have the same open-loop value on this trajectory.

The importance of the notion of representations is that it enables the construction of equivalence classes of equal open-loop value strategies in the general class of closed-loop strategies. The procedure is as follows. For the system defined in Definition 1, first determine the set of all elements of Γ which are strategies that depend only on the initial state x_1. The set thus constructed is the class of permissible open-loop controls in Γ. Now let

$$\gamma^* = \left\{\gamma^*_k(x_1), k \in K\right\}$$

be an element of the set of open-loop controls which generates by substitution into the state equation of Definition 1, a unique trajectory $\{x^*_k, k \in K\}$. Then, consider all elements $\gamma = \{\gamma_k(.), k \in K\}$ of Γ with the properties

1) γ generates the same trajectory as γ^* and

2) $\gamma_k\left(x^*_k, x^*_{k-1}, \cdots, x^*_2, x_1\right) \equiv \gamma^*_k(x_1), k \in K$.

If this procedure is repeated for every element of the set of open-loop controls, the construction of all equivalence classes of Γ becomes complete. In our terms, outcome space is dense with macroeconomic configurations. The DSGE class is one. The analyst is free to choose from a multiplicity of models.

The basic proposition is as follows:

Theorem 1) *The FNE Solution under the CLPS or MPS information pattern is strongly time consistent.*

2) *The FNE Solution under the FB information pattern is weakly time inconsistent.*

Proof. 1) Consider a FNE solution $\{\gamma^{1*},\gamma^{2*}\}$ till stage k. Note that the star is only used for notational brevity and does not rule out the possibility of different equilibria for subgames preceding stage k. At that stage, **P**1, say, decides to renege on this solution. It must be the case that for some strategy $\tilde{\gamma}^1_k$,

$$J^1\left(\gamma^{1*}_1,\cdots,\gamma^{1*}_{k-1},\gamma^{1*}_k,\gamma^{1*}_{k+1},\cdots,\gamma^{1*}_K;\gamma^{2*}_1,\cdots,\gamma^{2*}_{k-1},\gamma^{2*}_k,\gamma^{2*}_{k+1},\cdots,\gamma^{2*}_K\right) \geq J^1\left(\gamma^{1*}_1,\cdots,\gamma^{1*}_{k-1},\tilde{\gamma}^1_k,\gamma^{1*}_{k+1},\cdots,\gamma^{1*}_K;\gamma^{2*}_1,\cdots,\gamma^{2*}_{k-1},\gamma^{2*}_k,\gamma^{2*}_{k+1},\cdots,\gamma^{2*}_K\right)$$

The defection under CLPS or MPS implies entry into an information equivalence class where

$$\gamma_k\left(\tilde{x}_k,\tilde{x}_{k-1},\cdots,\tilde{x}_2,x_1\right) \equiv \tilde{\gamma}_k(x_1), k \in K.$$

Accordingly, the evolution of the state till stage k is given by $\tilde{x}_{k+1} = f_k\left(\tilde{x}_k,\tilde{u}^1_k,\tilde{u}^2_k\right), \tilde{x}_1 = x_1$. It turns out, therefore, that for agent **P**1, either

$$J^1\left(\gamma^{1*}_1,\cdots,\gamma^{1*}_k,\gamma^{1*}_{k+1},\cdots,\gamma^{1*}_K;\gamma^{2*}_1,\cdots,\gamma^{2*}_k,\gamma^{2*}_{k+1},\cdots,\gamma^{2*}_K\right) \geq J^1\left(\tilde{\gamma}^1_!,\cdots,\tilde{\gamma}^1_k,\tilde{\gamma}^1_{k+1},\cdots,\tilde{\gamma}^1_K;\tilde{\gamma}^2_1,\cdots,\tilde{\gamma}^2_k,\tilde{\gamma}^2_{k+1},\cdots\tilde{\gamma}^2_K\right)$$

or

$$J^1\left(\gamma^{1*}_1,\cdots,\gamma^{1*}_k,\gamma^{1*}_{k+1},\cdots,\gamma^{1*}_K;\gamma^{2*}_1,\cdots,\gamma^{2*}_k,\gamma^{2*}_{k+1},\cdots,\gamma^{2*}_K\right) \leq J^!\left(\tilde{\gamma}^1_!,\cdots,\tilde{\gamma}^1_k,\tilde{\gamma}^1_{k+1},\cdots,\tilde{\gamma}^1_K;\tilde{\gamma}^2_1,\cdots,\tilde{\gamma}^2_k,\tilde{\gamma}^2_{k+1},\cdots,\gamma^2_K\right)$$

If the first inequality holds, the assumption that the agents were parties to the FNE solution under the CLPS or MPS information pattern is violated. If the second inequality is valid, the assumption of the deviation from the FNE solution does not hold.

2) On the other hand, suppose the players were operating under the FB information pattern. At stage k, the players decide to recommit. Their inducement to do so must be

$$J^1\left(\gamma^{1*}_1,\cdots,\gamma^{1*}_{k-1},\gamma^{1*}_k,\gamma^{1*}_{k+1},\cdots,\gamma^{1*}_K;\gamma^{2*}_1,\cdots,\gamma^{2*}_{k-1},\gamma^{2*}_k,\gamma^{2*}_{k+1},\cdots,\gamma^{2*}_K\right) \geq J^1\left(\gamma^{1*}_1,\cdots,\gamma^{1*}_{k-1},\tilde{\gamma}^1_k,\gamma^{1*}_{k+1},\cdots,\gamma^{1*}_K;\gamma^{2*}_1,\cdots,\gamma^{2*}_{k-1},\gamma^{2*}_k,\gamma^{2*}_{k+1},\cdots,\gamma^{2*}_K\right)$$

and

$$J^2\left(\gamma^{1*}_1,\cdots,\gamma^{1*}_{k-1},\gamma^{1*}_k,\gamma^{1*}_{k+1},\cdots,\gamma^{1*}_K;\gamma^{2*}_1,\cdots,\gamma^{2*}_{k-1},\gamma^{2*}_k,\gamma^{2*}_{k+1},\cdots\gamma^{2*}_K\right) \geq J^2\left(\gamma^{1*}_1,\cdots,\gamma^{1*}_{k-1},\gamma^{1*}_k,\gamma^{1*}_{k+1},\cdots,\gamma^{1*}_K;\gamma^{2*}_1,\cdots,\gamma^{2*}_{k-1},\tilde{\gamma}^2_k,\gamma^{2*}_{k+2},\cdots,\gamma^{2*}_K\right)$$

where $\gamma_k^i = \psi(x_k), i = 1,2,$ is the feedback strategy under the FB information pattern. Their actions determine a new initial condition for the game $\tilde{x}_{k+1}$. We can design the set of strategies which only depend on this new initial state and the time periods that follow. This would be the class of open-loop controls. An element would be

$\{\tilde{\gamma} = \psi(\tilde{x}_{k+1})\}$ which generates by substitution into the state evolution equation a unique trajectory $\tilde{x}_{k+2}$. We can proceed to construct an equivalence class of representations all of which have the same value $\tilde{\gamma}$.

3. Conclusions

If agents access all the information they can command and the economy is described by a dynamic general equilibrium model, there is no incentive for any player to 'cheat' on the equilibrium. The DSGE model applies. On the other hand, if, under the same information conditions, the economy is characterized by a Keynesian model, agents will be locked into a "bad" equilibrium. Only a regime change will transfer the system into a superior equivalence class of solutions.

If, for reasons like myopia, agents are bound by the information provided in the present and the solution of the state equation of the game is an unemployment equilibrium, all of them are better off reneging on the Nash outcome into a dominating equivalence class of solutions. The mainstream response would that it is precisely when agents do not look to the past and, a fortiori, the future, that they are tempted out of the noncooperative equilibrium. However, no player is worse off in these circumstances. Credibility and a good reputation, in this instance, are at a discount.

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