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# Exponential Ergodicity and *β*-Mixing Property for Generalized Ornstein-Uhlenbeck Processes

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## ABSTRACT

The generalized Ornstein-Uhlenbeck process is derived from a bivariate Lévy process and is suggested as a continuous time version of a stochastic recurrence equation [1]. In this paper we consider the generalized Ornstein-Uhlenbeck process and provide sufficient conditions under which the process is exponentially ergodic and hence holds the exponentially *β*-mixing property. Our results can cover a wide variety of areas by selecting suitable Lévy processes and be used as fundamental tools for statistical analysis concerning the processes. Well known stochastic volatility model in finance such as Lévy-driven Ornstein-Uhlenbeck process is examined as a special case.

**Keywords:** *β*-Mixing; Generalized Ornstein-Uhlenbeck Process; Exponential Ergodicity; Lévy-Driven Ornstein-Uhlenbeck Process

## 1. Introduction

Many continuous time processes are suggested and studied as a natural continuous time generalization of a random recurrence equation, for example, diffusion model of Nelson [2], continuous time GARCH (COGARCH) (1,1) process of Klüppelberg *et al.* [3] and Lévy-driven Ornstein-Uhlenbeck (OU) process of Barndorff-Nielsen and Shephard [4] etc. Continuous time processes are particularly appropriate models for irregularly spaced and high frequency data [5]. We consider the generalized Ornstein-Uhlenbeck (GOU) process $(V_t)_{t\geq 0}$ which is defined by

$$V_t = V_0 e^{-\xi_t} + e^{-\xi_t}\int_0^t e^{\xi_{s-}}\mathrm{d}\eta_s, \quad t \geq 0, \tag{1}$$

where $(\xi_t,\eta_t)_{t\geq 0}$ is a two-dimensional Lévy process and the starting random variable $V_0$ is independent of $(\xi_t,\eta_t)_{t\geq 0}$. Lévy processes are a class of continuous time processes with independent and stationary increments and continuous in probability. Since Lévy processes $\xi_t$ and $\eta_t$ are semimartingales, stochastic integral in Equation (1) is well defined.

The GOU process is a continuous time version of a stochastic recurrence equation derived from a bivariate Lévy process (de Haan and Karandikar [1]). The GOU process has recently attracted attention, especially in the financial modelling area such as option pricing, insurance and perpetuities, or risk theory. Stationarity, moment condition and autocovariance function of the GOU process are studied in Lindner and Maller [6]. Fasen [7] obtain the results for asymptotic behavior of extremes and sample autocovariance function of the GOU process. For related results, we may consult, e.g. Masuda [8], Klüppelberg *et al.* [3,9], Maller *et al.* [5] and Lindner [10] etc.

Mixing property of a stochastic process describes the temporal dependence in data and is used to prove consistency and asymptotic normality of estimators. For a stationary process $(X_t)_{t\geq 0}$, $F_t = \sigma(X_s : s \leq t)$ and $G_t = \sigma(X_s : s \geq t)$, let

$$\beta(t) = \sup \frac{1}{2}\sum_{i=1}^{I}\sum_{j=1}^{J}\left|P(A_i \cap B_j) - P(A_i)P(B_j)\right|,$$

where the supremum takes over $A_i \in F_u, B_i \in G_{u+t}$,

$$A_i \cap A_j = \varnothing, B_i \cap B_j = \varnothing,$$

if $i \neq j$ and $\cup_{i=1}^{I} A_i = \cup_{j=1}^{J} B_j = \Omega$. If $\beta(t) \to 0$ as $t \to \infty$, then $(X_t)_{t\geq 0}$ is called *β*-mixing. $(X_t)_{t\geq 0}$ is called exponentially *β*-mixing if $\beta(t) \leq Ke^{-at}$ for some $K, a > 0$ and all $t \geq 0$.

In this paper we prove the exponential ergodicity and exponentially *β*-mixing property of the GOU process $(V_t)_{t\geq 0}$ of Equation (1) and obtain the *β*-mixing property of the Lévy-driven OU process as a special case.

For more information on Markov chain theory, we refer to Meyn and Tweedie [11]. We refer to Bertoin [12] and Sato [13] for basic results and representations concerning Lévy processes.

 

## 2. Exponential Ergodicity of $(V_t)_{t\ge 0}$

### 2.1. The Model

A bivariate Lévy process $(\xi_t,\eta_t)_{t\ge 0}$ defined on a complete probability space $(\Omega, F, P)$ is a stochastic process in $R^2$, with càdlàg paths, $(\xi_0,\eta_0)=(0,0)$ and stationary independent increments, which is continuous in probability.

Consider the GOU process $V_t$ given by

$$V_t = e^{-\xi_t}\left(\int_0^t e^{\xi_{s-}}\mathrm{d}\eta_s + V_0\right), t\ge 0.$$

Assume that $V_0$ is independent of $(\xi_t,\eta_t)_{t\ge 0}$. Let

$$A_t^s = e^{-(\xi_t-\xi_s)},\ B_t^s = e^{-\xi_t}\int_s^t e^{\xi_{u-}}\mathrm{d}\eta_u. \quad (2)$$

Then we have that

$$V_{(n+1)h} = A_{(n+1)h}^{nh}V_{nh} + B_{(n+1)h}^{nh},\ \ h>0, n\ge 0. \quad (3)$$

Let $n$ denote an integer and $t$ a real number. We can easily show that $\left(A_{(n+1)h}^{nh}, B_{(n+1)h}^{nh}\right)_{n\ge 0}$ in Equation (2) is a sequence of independent and identically distributed random vectors and $(V_t)_{t\ge 0}$ in Equation (1) is a time homogeneous Markov process with $t$-step transition probability function

$$P^{(t)}(x,C) = P(V_t\in C|V_0 = x), x\in R, C\in B(R),$$

where $B(R)$ is a Borel $\sigma$-field of subsets of real numbers $R$.

We temporally assume that $h>0$ is fixed. $(V_{nh})_{n\ge 0}$ in Equation (3) can be considered as a discrete time Markov process with $n$-step transition probability function $P^{(nh)}(x,C) = P(V_{nh}\in C|V_0 = x), n\ge 1$. $(V_{nh})_{n\ge 0}$ is called the $h$-skeleton chain of $(V_t)_{t\ge 0}$. A Markov process $(V_{nh})_{n\ge 0}$ is $\phi$-irreducible if, for some $\sigma$-finite measure $\phi$, $\sum_n 2^{-n}P^{(nh)}(x,B)>0$ for all $x\in R$ whenever $\phi(B)>0$. $(V_t)_{t\ge 0}$ is said to be simultaneously $\phi$-irreducible if any $h$-skeleton chain is $\phi$-irreducible. It is known that if $(V_t)_{t\ge 0}$ is simultaneously $\phi$-irreducible, then any $h$-skeleton chain is aperiodic (Proposition 1.2 of Tuominen and Tweedie [14]).

For fixed $h>0$, we make the following assumptions:

(A1) $0<E(\xi_h)\le E|\xi_h|<\infty$ and $E\log^+|\eta_h|<\infty$.

(A2) $E\left|e^{-\xi_h}\right|^r<\infty, E\left|e^{-\xi_h}\int_0^h e^{\xi_{s-}}\mathrm{d}\eta_s\right|^r<\infty$ for some $r>0$.

**Theorem 2.1** *Under the assumption* (A1), $(V_{nh})_{n\ge 0}$ *defined by Equation* (3) *converges in distribution to a probability measure* π *which does not depend on* $V_0$. *Further*, π *is the unique invariant initial distribution for* $(V_{nh})_{n\ge 0}$.

*Proof.* The conclusion follows from Theorem 3.1 and Theorem 3.4 in de Haan and Karandikar [1]. Note that if the assumption (A1) holds, then it is obtained that

$E(\log A_h^0)<0$ and $E\left(\log^+|B_h^0|\right)<\infty$. □

**Remark 1** Assume that $0<E(\xi_h)\le E|\xi_h|<\infty$. Then $E\log^+|\eta_h|<\infty$ is also necessary for the existence of a strictly stationary solution. (See Theorem 2.1 in Lindner and Maller [6].)

**Remark 2** Suppose that there exist $\alpha>0$ and $p,q>1$ with $1/p+1/q=1$ such that

$$\Psi_\xi(\alpha)<0, E\left(e^{-\max\{1,\alpha\}p\xi_1}\right)<\infty, E|\eta_1|^{\max\{1,\alpha\}q}<\infty$$

where $\Psi_\xi(\alpha)$ denotes the Lévy exponent of the Lévy process $\xi$: $\Psi_\xi(\alpha) = \log Ee^{-\alpha\xi_1}$. If in addition, $E(\xi_1^+)<\infty$, then assumptions (A1) and (A2) hold (Proposition 4.1 in Lindner and Maller [6]).

### 2.2. Drift Condition for $(V_{nh})_{n\ge 0}$

A discrete time Markov process $(X_n)_{n\ge 0}$ is said to hold the drift condition if there exist a positive function $g$ on $R$, a compact set $K$, and constants $\nu>0$ and $0<\rho<1$ such that

$$E(g(X_{n+1})|X_n = x)\le\rho g(x)-\nu,\ x\in K^c$$

and $$\sup_{x\in K}E(g(X_{n+1})|X_n = x)<\infty.$$

**Theorem 2.2** *Under the assumptions* (A1) *and* (A2), $(V_{nh})_{n\ge 0}$ *given in Equation* (3) *satisfies the drift condition.*

*Proof.* For notational simplicity, let $A_1 = A_h^0, B_1 = B_h^0$. From assumptions, we have that $E(\log A_1)<0$ and $E|A_1|^r<\infty$ for some $r>0$. Then

$$E\left(|A_1|^r\right)^{1/r}\to e^{E(\log A_1)}$$

as $r\to 0$ ( Hardy *et al.* [15]). Here $E(\log A_1)<0$ implies the existence of $r^*<1$, $0<r^*<r$ such that $\rho^* := E|A_1|^{r^*}<1$. Now define a nonnegative test function $g$ on $R$ by $g(x) = |x|^{r^*}+1$. Then we have that

$$\begin{aligned}&E(g(A_1V_0+B_1)|V_0 = x)\\&= E|A_1x+B_1|^{r^*}+1\\&\le E|A_1|^{r^*}|x|^{r^*}+E|B_1|^{r^*}+1\\&=\rho^*g(x)+M,\end{aligned} \quad (4)$$

where $M = E|B_1|^{r^*}-\rho^*+1<\infty$, by assumption (A2). Since $g(x)$ increases to $\infty$ as $|x|$ increases to $\infty$, for any $\nu>0$, there exist $\rho, 0<\rho^*<\rho<1$ and $k>0$ with $K=\{x\,\|x|\le k\}$, such that

$$\rho^*g(x)+M\le\rho g(x)-\nu,\ x\in K^c. \quad (5)$$

Clearly,

$$\sup_{x\in K}E|A_1x+B_1|^{r^*}<\infty. \quad (6)$$

Combining Equations (4)-(6), the drift condition for $(V_{nh})_{n\ge 0}$ holds. □

 

## 2.3. Simultaneous $\phi$-Irreducibility of $(V_t)_{t\geq 0}$

For reader's convenience, we state the following theorems which play important roles to prove our main results.

**Theorem 2.3** (*Meyn and Tweedie* [11]) *Suppose that a Markov chain* $(X_n)_{n\geq 0}$ *has the Feller property. If* $(X_n)_{n\geq 0}$ *satisfies the drift condition for a compact set* $K$, *then there exists an invariant probability measure. In addition, if the process is* $\phi$*-irreducible and aperiodic, then the given process is geometrically ergodic.*

Theorem 2.3 shows that the crucial step to prove the geometric ergodicity of a Markov process is to show that the given process is $\phi$-irreducible and holds the drift condition. In many cases, however, proving irreducibility of a Markov process is an awkward task. Consulting the following Theorem 2.4, irreducibility of the process can be derived from connection between $\phi$-irreducibility and the uniform countable additivity condition. A Markov chain $(X_n)_{n\geq 0}$ is said to hold the uniform countable additivity condition (Liu and Susko [16]) if its one-step transition probability function satisfies that for any decreasing sequence $G_n \downarrow \varnothing$ inside compact sets,

$$\lim_{G_n\downarrow\varnothing}\sup_{x\in K} P(x, G_n) = 0 \text{ for every compact set } K.$$

**Theorem 2.4** (*Tweedie* [17]) *Suppose that the drift condition holds with a test set K and the uniform countable additivity condition holds for the same set K. Then there is a unique invariant measure for* $(X_n)_{n\geq 0}$ *if and only if* $(X_n)_{n\geq 0}$ *is* $\phi$*-irreducible.*

Let $K = \{x \| x| \leq k\}$ be the compact set defined in the proof of Theorem 2.2.

**Theorem 2.5** *Under the assumptions* (A1) *and* (A2), $(V_t)_{t\geq 0}$ *is simultaneously π-irreducible if for any* $h>0$, $P^{(h)}(x,\cdot)$ *has a probability density function* $p_h(x,y)$ (*with respect to the Lebesgue measure* $\mu$), *which is uniformly bounded on compacts for* $x\in K$.

*Proof.* Let $G_n$ be any decreasing sequence inside compact sets with $G_n \downarrow \varnothing$. Then

$$\begin{aligned}\sup_{x\in K} P^{(h)}(x,G_n) &= \sup_{x\in K}\int_{G_n} p_h(x,y)\,\mathrm{d}\mu(y)\\ &\leq \int_{G_n}\sup_{x\in K} p_h(x,y)\,\mathrm{d}\mu(y) = M_G\cdot\mu(G_n),\end{aligned} \tag{7}$$

where $M_G := \sup\{p_h(x,y) | x\in K, y\in G_1\} < \infty$.

The inequality in Equation (7) and the condition that $G_n$ is any sequence inside compact sets in $B(R)$ with $G_n \downarrow \varnothing$ imply that

$$\lim_{n\to\infty}\sup_{x\in K} P^{(h)}(x,G_n) \leq \lim_{n\to\infty} M_G\cdot\mu(G_n) = 0.$$

Therefore the uniform countable additivity condition holds for the compact set *K*. Theorem 2.4 and the existence of a unique invariant initial distribution for $(V_{nh})_{n\geq 0}$ yield the π-irreducibility of any *h*-skeleton chain $(V_{nh})_{n\geq 0}$.

To complete the proof, we need to show that the assumption (A1) and (A2) hold for all $h>0$. Since Lévy processes have stationary and independent increments, it is easy to show that the assumption (A1) and $E\left|e^{-\xi_h}\right|^r < \infty$ hold for all $h>0$. It remains to prove that

$$E\left|e^{-\xi_h}\int_0^h e^{\xi_{s-}}\mathrm{d}\eta_s\right|^r < \infty$$

for all $h>0$ with some $r>0$. We first define a finite Lévy process $(L_t)_{t\geq 0}$ as follows:

$$\begin{aligned}L_t &:= \eta_t + \sum_{0<s\leq t}\left(e^{-\Delta\xi_s}-1\right)\Delta\eta_s\\ &\quad - t\,Cov\left(B_{\xi,1},B_{\eta,1}\right),\quad t\geq 0.\end{aligned}$$

Then it is shown that $\forall t>0$,

$$\int_0^t e^{-\xi_{s-}}\mathrm{d}L_s \overset{D}{=} e^{-\xi_t}\int_0^t e^{\xi_{s-}}\mathrm{d}\eta_s.$$

(See Proposition 2.3 in Lindner and Maller [6]). Without loss of generality, we may assume that $0<r<1$. Choose any $l>0$. Then $l = nh+\alpha h$, where $n$ is a nonnegative integer, $0<\alpha<1$ and $h>0$ is in the assumptions (A1) and (A2), we have that

$$\begin{aligned}&E\left|e^{-\xi_l}\int_0^l e^{\xi_{s-}}\mathrm{d}\eta_s\right|^r\\ &= E\left|\int_0^l e^{-\xi_{s-}}\mathrm{d}L_s\right|^r = E\left|\sum_{j=1}^n\int_{(j-1)h}^{jh} e^{-\xi_{s-}}\mathrm{d}L_s + \int_{nh}^l e^{-\xi_{s-}}\mathrm{d}L_s\right|^r\\ &= E\left|\sum_{j=1}^n e^{-\xi_{(j-1)h}}\int_{(j-1)h}^{jh} e^{-\left(\xi_{s-}-\xi_{(j-1)h}\right)}\mathrm{d}(L_s - L_{(j-1)h}) + e^{-\xi_{nh}}\int_{nh}^l e^{-(\xi_{s-}-\xi_{nh})}\mathrm{d}\left(L_s - L_{nh}\right)\right|^r\\ &\leq \sum_{j=1}^n E\left|e^{-\xi_{(j-1)h}}\right|^r E\left|\int_0^h e^{-\xi_{s-}}\mathrm{d}L_s\right|^r + E\left|e^{-\xi_{nh}}\right|^r E\left|\int_0^{\alpha h} e^{-\xi_{s-}}\mathrm{d}L_s\right|^r\\ &= \sum_{j=1}^n E\left|e^{-\xi_{(j-1)h}}\right|^r E\left|e^{-\xi_h}\int_0^h e^{\xi_{s-}}\mathrm{d}\eta_s\right|^r + E\left|e^{-\xi_{nh}}\right|^r E\left|e^{-\xi_{\alpha h}}\int_0^{\alpha h} e^{\xi_{s-}}\mathrm{d}\eta_s\right|^r < \infty.\end{aligned} \tag{8}$$

The first inequality in Equation (8) follows from stationary and independent increments property of Lévy processes $(\xi_t)_{t\ge0}$ and $(L_t)_{t\ge0}$.

Therefore for any $h>0$, $h$-skeleton chain $(V_{nh})_{n\ge0}$ is $\pi$-irreducible and hence $(V_t)_{t\ge0}$ is simultaneously $\pi$-irreducible and $(V_{nh})_{n\ge0}$ is aperiodic.

### 2.4. Exponential Ergodicity of $(V_t)_{t\ge0}$

The next theorem is our main result.

**Theorem 2.6** *Suppose that the assumptions of Theorem* 2.5 *hold. Then the GOU process* $(V_t)_{t\ge0}$ *in Equation* (1) *is exponentially ergodic and holds the exponentially* $\beta$ *-mixing property.*

*Proof.* Theorem 2.5 shows that any $h$-skeleton chain $(V_{nh})_{n\ge0}$ is $\pi$-irreducible and aperiodic. Note that $(V_{nh})_{n\ge0}$ is a Feller chain, that is, $E\left(f\left(V_{(n+1)h}\right)\middle|V_{nh}=x\right)$ is a continuous function of $x$ whenever $f$ is continuous and bounded. Therefore any nontrivial compact set is a small set. Theorem 2.2 ensures that $(V_{nh})_{n\ge0}$ holds the drift condition and hence Theorem 2.5 and Theorem 2.3 imply that $(V_{nh})_{n\ge0}$ is geometrically ergodic, that is, there exists a constant $\rho\in(0,1)$ such that

$$\left\|P^{(nh)}(x,\cdot)-\pi(\cdot)\right\|=O\left(\rho^n\right), \tag{9}$$

$\pi$-a.a. $x$ as $n\to\infty$, where $\|\cdot\|$ denotes the total variation norm. Under simultaneous $\pi$-irreducibility condition of $(V_t)_{t\ge0}$, Equation (9) and Theorem 5 in Tuominen and Tweedie [14] guarantee the exponential ergodicity of $(V_t)_{t\ge0}$ in the following sense:

$$\left\|P^{(t)}(x,\cdot)-\pi(\cdot)\right\|=O\left(e^{-\alpha t}\right),$$

as $t\to\infty$, for some $\alpha>0$ and $\pi$-a.a. $x$. $\beta$-mixing property for the continuous time GOU process $(V_t)_{t\ge0}$ is also obtained.

### 2.5. Examples

In this example, we assume that $\xi_t=\rho t, \rho>0$. If $\eta_t$ is any Lévy process, then $V_t$ in Equation (1) is the Lévy-driven OU process which is studied by Barndorff-Nielsen and Shephard [4]. In particular, if $\eta_t$ is a subordinator, that is, $\eta_t$ has nondecreasing sample path, finite variation with nonnegative drift and Lévy measure concentrated on $(0,\infty)$, then $(V_t)_{t\ge0}$ is called the Lévy-driven stochastic volatility model. For the case that $\eta_t$ is a Brownian motion, $V_t$ is the classical OU process. Let $\Pi_\eta$ be the Lévy measure for the process $\eta_t$ and assume that $E|\eta_h|^r<\infty$ for some $h>0$ and $r>0$. Then $\int_{|z|>1}\log|z|\Pi_\eta(\mathrm{d}z)<\infty$. Here we can easily show that the assumptions (A1)and (A2) hold. Theorem 2.2 implies that $(V_{nh})_{n\ge0}$ holds the drift condition. Moreover, it is known that $P^{(t)}(x,\cdot)$ admits a $C_b^\infty$ density $p_t(x,y)$ for each $t>0$ (Sato and Yamazato [18]) and by Theorem 2.5, $(V_{nh})_{n\ge0}$ is $\phi$-irreducible. Above statements hold for any $h>0$ and hence $(V_t)_{t\ge0}$ is simultaneously $\phi$-irreducible. Therefore exponential ergodicity and exponential $\beta$-mixing property of $(V_t)_{t\ge0}$ follow from Theorem 2.6.

## 3. Conclusion

Recently, time series models in finance and econometrics are suggested as continuous time models which are particularly appropriate for irregularly spaced and high frequency data. The GOU process is a continuous time stochastic process driven by a bivariate Lévy process. The stationarity, moment conditions, autocovariance function and asymptotic behavior of extremes of the process are studied in [6,7], but exponential ergodicity does not seem to have been investigated as yet. In this paper, we give sufficient conditions under which the process is exponentially ergodic and $\beta$-mixing. The drift condition and the simultaneous $\phi$-irreducibility of the process that is induced from uniform countable additivity condition play a crucial role to prove the results. Our results are used to show, in particular, consistency and asymptotic normality of estimators.

## 4. Acknowledgements

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