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Solution for Rational Expectation Models Free of Complex Numbers

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Abstract

This paper approaches the problem of the potential for complex-valued solutions within linear macroeconomic models with rational expectations. It finds that these problems are associated with a specific solution method for the underlying model. The paper establishes that the danger of complex-valued solutions always can be eliminated by forcing those solutions to fulfill additional constraints. These constraints are essentially restrictions on the degrees of freedoms in indeterminate solutions.

Keywords: Rational Expectation Models, Indeterminacy, Potentially Complex Coefficients

1. Introduction

Within the solution of dynamic economic models with rational expectations regularly some kind of matrix decomposition technique is employed in order to separate the stable (generalized) eigenvalues from the unstable (generalized) eigenvalues. Examples are the eigen decomposition, the Schur decomposition or the generalized Schur decomposition. All these decompositions share the property that for real matrix polynomials it is sometimes less costly to compute complex decompositions than real decompositions. This raises the question whether we can nevertheless assert real-valued solution paths. [1] shows for the case of the eigen decomposition that this can be achieved by choosing appropriate constants associated with each eigenvalue. This paper discusses the same question for the case of the generalized Schur decomposition. It shows that while the structural properties of the decomposition methods support real-valued solutions, they do not always suffice to assert such solution paths. In particular, the structural properes of the generalized Schur decomposition can not assert real-valued solution paths for the case that the model's expectational error is explained as a function of the exenous shocks to the model as e.g. in [2]. Hence, in order to get a real-valued solution of the model for this case, it is necessary to introduce additional conditions to force the solution to the real domain.

The paper develops such conditions. Of course this requires some degrees of freedom within the solution of the model, since a unique solution does not allow for the development of additional conditions. Therefore the paper applies to solutions of rational expectations models which are characterized by indeterminacy. Within this solution it attempts to use the available degrees of freedom to solve for a constellation, in which the coefficients of endogenous and exogenous variables are forced into the appropriately dimensioned real spaces $\mathbb{R}^{n\times m}$. If such a constellation exists, the entire solution of the model is a real-valued one. To this purpose this paper uses the fact that the product of two complex matrices generates an imaginary part which consists in the sum of the products of the real part of one matrix and the imaginary one of the other matrix. Within this imaginary part eventually complex-valued exogenous components can be balanced by the imaginary part of the components containing the degrees of freedom. Thus the product's imaginary part is set to zero. Applying this to the coefficient mentioned it can be guaranteed that the coefficient is a real-valued matrix.

The conditions obtained are sufficient conditions for the existence of a real-valued solution. These conditions built on the model's transversality condition, which is integrated into the model's solution in a specific way that does not allow any complex values for the model's endogenous variables. Thus, if necessary, the conditions restrict the degrees of freedom available in the solution further beyond the scope of restriction already obtained by the mere integration of the transversality condition in [3]. For the case that the transversality condition does only imply real solutions, no further conditions beyond the fact that all degrees of freedom must be restricted to the real domain, are needed.

The paper is organized as follows. After this section's introduction section 2 presents the general method to be employed. In sections 3 and 4 separate algebraic conditions are developed which need to hold simultaneously in order to guarantee a pure real-valued solutions. Section 5 concludes.

2. General Method

According to the solution algorithms for linear dynamic models involving rational expectations, *i.e.*,

$$\mathbf{G}\begin{pmatrix} v_{t+1} \\ w_{t+1} \end{pmatrix} = \mathbf{H}\begin{pmatrix} v_t \\ w_t \end{pmatrix} + \mathbf{C}z_t + \tilde{\mathbf{G}}\eta_{t+1} \tag{1}$$

presented in [4,5,6-10], among others, any solution to this type of models can be written in form of a VAR(1)-process

$$\begin{pmatrix} v_{t+1} \\ w_{t+1} \end{pmatrix} = \Upsilon_1\begin{pmatrix} v_t \\ w_t \end{pmatrix} + \Upsilon_2 z_t + \sum_{q=1}^{\infty}\Upsilon_{2+q}E_t z_{t+q}. \tag{2}$$

Herein $\begin{pmatrix} v_t^T & w_t^T \end{pmatrix}^T$ denotes the vector of endogenous variables in period t, while z_t is an exogenous shock vector realized in period t. η_{t+1} is the vector of expectational errors in the endogenous variables denoting the difference between the unconditionally expected values based on information available in period t and the values actually realized in period $t+1$. The shock vector z_t might display some autocorrelation which explains the appearance of expected future shock terms in Equation (2). In addition, the model in Equation (1) might be required to fulfill the transversality condition

$$\lim_{q->\infty} E_t \mathbf{W}\begin{pmatrix} v_{t+q} \\ w_{t+q} \end{pmatrix} = 0. \tag{3}$$

Herein E_t denotes the operator for the unconditional expectations based on information available in t. As pointed out in [9] this requirement, if not guaranteed by appropriate initial conditions, essentially restricts the growth of the model's unstable exogenous variables.

Depending on the characteristics of the used solution method the coefficients $(\Upsilon_1, \cdots, \Upsilon_\infty)$ are defined as specific functions of the original coefficient matrices. The central idea of asserting a real-valued solution starts with the fact that often some of the coefficients are real by construction. This claim will be discussed in the next section for coefficients associated with the endogenous variables and the expectational error terms. Afterwards it will be analyzed with respect to the coefficient of the exogenous shock term in the subsequent section. For any coefficient for which this claim does not hold, we will use any available degrees of freedom within that coefficient in order to force it into the real domain. Thus the indeterminacy allows to guarantee a purely real-valued solution for the model. Therefore such a coefficient Υ will be decomposed into potentially complex, $\hat{\Upsilon}$, and real, *i.e.* $\tilde{\Upsilon}$, factors. The product of this factors has the form

$$\tilde{\Upsilon}\cdot(\Re(\hat{\Upsilon}) + i\Im(\hat{\Upsilon})) = \tilde{\Upsilon}\cdot\Re(\hat{\Upsilon}) + i\tilde{\Upsilon}\cdot\Im(\hat{\Upsilon}) \tag{4}$$

where i denotes the square root of -1, while $\Re$ ($\Im$) denotes the real (imaginary) part of its argument. Whenever the last summand contains enough degrees of freedom in order to be forced down to zero, the entire coefficient will take on values from the set of real numbers. Thus the entire solution of the macroeconomic model will not include any complex numbers.

As already indicated the paper restricts the analysis on the case of solution methods based on the generalized Schur decomposition. Nevertheless, similar arguments could be obtained for the case of solution methods based on eigenvalue decompositions, Jordan decompositions and ordinary Schur decompositions. This paper focuses on the generalized Schur decomposition, because models which can be solved by those methods nest all models solvable by the mentioned alternative decompositions.

In addition the paper distinguishes two approaches to balance the distorting influence of expectational errors appearing within any rational expectations model. The first approach has been presented in [8] and [10]. Herein the mentioned distortion is eliminated by explaining the expectational error's influence on the model's stable part as a function of it's influence on the model's unstable part, which itself is forced to be zero by the initial conditions. On the other hand [5] explains the expectational error directly as function of the exogenous shock term. As shown in [2] this does not exclude sunspot solutions because expectations might be driven by an additional component which do not contribute to the model's unstable part. For both approaches we also separate between the cases in which either a microfoundated transversality condition is explicitly integrated into the solution as presented in [3] or the transversality condition is only used in the traditional manner, e.g. [4], by forcing the non-state variables to take on appropriate initial values.

3. Real-Valued Coefficients for Endogenous Variables and Expectational Errors

In order to prove the claim that the coefficient of the endogenous variables as well as those of the expectational errors do not include complex numbers, some properties of the generalized Schur decomposition of the matrix pencil $(\mathbf{H},\mathbf{G})$ will be used. This decomposition takes the form

$$\mathbf{RSP}^H = \mathbf{G} \quad (5)$$

$$\mathbf{RTP}^H = \mathbf{H}, \quad (6)$$

where $\mathbf{R}$ and $\mathbf{P}$ are unitary matrices and $\mathbf{T}$ is an upper triangular matrix. Depending on the exact form of the decomposition $\mathbf{S}$ is either upper triangular or upper Hessenberg. The details of the computation of this decomposition are discussed in [11]. For the purpose of the present paper just one characteristic of this algorithm is needed. So long as $\mathbf{S}$ is only transformed up to upper Hessenberg form, *i.e.* the socalled real generalized Schur decomposition, the operations involved will secure that for any real-valued initial matrix-pencil both $\mathbf{S}$ and $\mathbf{T}$ are real-valued matrices, while the matrices $\mathbf{R}$ and $\mathbf{P}$ will be potentially complex-valued[1]. In order to exploit these facts, the decomposition in Equation (6) is premultiplied by the factor $\mathbf{R}(\mathbf{I}-\mathbf{\Phi})\mathbf{R}^H$ yielding the expression

$$\begin{pmatrix}\mathbf{T}_{11} & \mathbf{T}_{12}-\mathbf{\Phi}\mathbf{T}_{22}\end{pmatrix}\mathbf{P}^H = (\mathbf{R}_1^H - \mathbf{\Phi}\mathbf{R}_2^H)\mathbf{H}, \quad (7)$$

while the results $\mathbf{S}_{11}^{-1} = \mathbf{P}_1^H\mathbf{G}^+\mathbf{R}_1$ and $\mathbf{T}_{22} = \mathbf{R}_2^H\mathbf{HP}_2$ are derived from Equations (5) and (6), whereby $\mathbf{P}_i^H$ stands for the rows of matrix $\mathbf{P}^H$ associated with the model's stable (unstable) eigenvalues for the case of $i=1$ ($i=2$), while $\mathbf{S}_{ij}$ denotes essentially a block of $\mathbf{S}$ after partitioning the latter according to the criterion of stable and unstable eigenvalues. Symmetric notations hold for matrices $\mathbf{T}$ and $\mathbf{R}$. $\mathbf{X}^+$ denotes the pseudo-inverse of any matrix $\mathbf{X}$, while $\mathbf{X}^H$ symbolizes the conjugate transpose of that matrix.

For the solution methods based on generalized Schur decompositions [5] shows that the coefficients Υ_1 and Υ_{2+q} take on the following general forms

$$\Upsilon_1 = \mathbf{P}\begin{pmatrix}\mathbf{S}_{11}^{-1}\\ \mathbf{0}\end{pmatrix}\begin{pmatrix}\mathbf{T}_{11} & \mathbf{T}_{12}-\mathbf{\Phi}\mathbf{T}_{22}\end{pmatrix}\mathbf{P}^H \quad (8)$$

$$\Upsilon_{2+q} = \mathbf{P}\begin{pmatrix}\mathbf{S}_{11}^{-1}(\mathbf{S}_{12}-\mathbf{\Phi}\mathbf{S}_{22})\\ \mathbf{I}\end{pmatrix}. \sum_{q=1}^{\infty}(\mathbf{T}_{22}^{-1}\mathbf{S}_{22})^{q-1}\mathbf{T}_{22}^{-1}\mathbf{R}_2^H\mathbf{C}. \quad (9)$$

in which the matrix $\mathbf{\Phi}$ depends on the specific solution technique chosen and the question whether a transversality question is integrated or not[2]. Substituting Equation (7) and the expressions obtained for $\mathbf{S}_{11}^{-1}$ and $\mathbf{T}_{22}$ into these coefficients and simplifying yields:

$$\Upsilon_1 = \mathbf{P}_1\mathbf{P}_1^H\mathbf{G}^+\mathbf{R}_1\mathbf{R}_1^H\mathbf{H} - \mathbf{P}_1\mathbf{P}_1^H\mathbf{G}^+\mathbf{R}_1\mathbf{\Phi}\mathbf{R}_2^H\mathbf{H} \quad (10)$$

$$\Upsilon_{2+q} = (\mathbf{G}^+\mathbf{R}_1\mathbf{S}_{12}\mathbf{P}_2^H - (\mathbf{G}^+\mathbf{R}_1\mathbf{\Phi}\mathbf{R}_2^H\mathbf{G} + \mathbf{I})\mathbf{P}_2\mathbf{P}_2^H) \cdot \sum_{q=1}^{\infty}\mathbf{P}_2(\mathbf{P}_2^H\mathbf{H}^+\mathbf{G}\mathbf{P}_2)^{q-1}\mathbf{P}_2^H\mathbf{H}^+\mathbf{C}. \quad (11)$$

Because the product of any set of columns of a unitary factor of the generalized Schur decomposition with its pseudoinverse, *i.e.* with its conjugate transpose, is a real-valued matrix by construction[3] and the expression $\mathbf{R}_1\mathbf{S}_{12}\mathbf{P}_2^H$ can be proven to be real-valued as well, the two expressions in Equations (10) and (11) are real-valued matrices iff the expression $\mathbf{R}_1\mathbf{\Phi}\mathbf{R}_2^H$ is real-valued.

For the solution along the lines proposed in [10] the expression $\mathbf{R}_1\mathbf{\Phi}\mathbf{R}_2^H$ is a zero matrix,[4] if there exists an unique solution, while for any indeterminate solution $\mathbf{R}_1\mathbf{\Phi}\mathbf{R}_2^H$ depends on the question whether an explicit transversality condition is integrated into the solution or not. For the latter case there are no additional restrictions on the matrix $\mathbf{\Phi}$, which implies that for real-valued

$$\mathbf{R}_1\mathbf{\Phi}\mathbf{R}_2^H = \mathbf{R}_1\mathbf{R}_1^H\tilde{\mathbf{G}}\tilde{\mathbf{G}}^+\mathbf{R}_2\mathbf{R}_2^H + \mathbf{R}_1\mathbf{Z}\mathbf{R}_2^H(\mathbf{I}-\tilde{\mathbf{G}}\tilde{\mathbf{G}}^+\mathbf{R}_2\mathbf{R}_2^H)$$

the expression $\mathbf{R}_1\mathbf{Z}\mathbf{R}_2^H$ need to be real-valued as well. Without presenting details, it should be only mentioned that the only possible solution for this case would coincide with the case of a unique solution for the whole model, for which, as already stated, $\mathbf{\Phi}$ will be zero matrix of appropriate dimensions. For the case that the solution includes an explicit transversality condition

$$\mathbf{R}_1\mathbf{\Phi}\mathbf{R}_2^H = \mathbf{R}_1\mathbf{R}_1^H\tilde{\mathbf{G}}\tilde{\mathbf{G}}^+\mathbf{R}_2\mathbf{R}_2^H + \mathbf{R}_1\mathbf{T}_{12}\mathbf{P}_2^H\mathbf{H}^+\mathbf{R}_2\mathbf{R}_2^H \cdot \mathbf{B}^+\mathbf{B}\mathbf{R}_2\mathbf{R}_2^H - \mathbf{R}_1\mathbf{R}_1^H\mathbf{G}\mathbf{P}_1(\ker(\mathbf{WP}))_1 ((\ker(\mathbf{WP}))_1)^+\mathbf{P}_1^H\mathbf{G}^+\mathbf{R}_1\mathbf{T}_{12}\mathbf{R}_2^H\mathbf{B}^+\mathbf{B}\mathbf{R}_2\mathbf{R}_2^H$$

needs to hold. Herein $\mathbf{B}$ denotes the matrix $\mathbf{B} = \mathbf{I} - \tilde{\mathbf{G}}\tilde{\mathbf{G}}^+$. Thus real-valued coefficients require $\mathbf{R}_1\mathbf{T}_{12}\mathbf{R}_2^H$ to be real-valued. According to Equation (6) this expression can be rewritten as $\mathbf{R}_1\mathbf{R}_1^H\mathbf{H}\mathbf{P}_2\mathbf{P}_2^H$, which is real by construction. This proves that both Υ_1 and Υ_{2+q} are purely real-valued coefficients.

The solution for $\mathbf{\Phi}$ in the method proposed by [5] again depends on the question whether it satisfies an explicit transversality condition or not. [3] reveals that in the second case there are no additional conditions on $\mathbf{\Phi}$ except for the fact that it does need to be of appropriate dimensions. Hence any matrix which produces a real $\mathbf{R}_1\mathbf{\Phi}\mathbf{R}_2^H$, qualifies as a solution. Due to the fact that $\mathbf{R}_1$

[1]If the result of the generalized Schur decomposition is presented as a pair of triangular matrices and the appropriate pair of unitary matrices, the real-valued upper Hessenberg form can be reconciled by simple unitary transformations.

[2]Details on this topic can be found in [5], [8] and [10].

[3]This fact stems from the possibility presented in [11] to express both unitary matrices as a product of symmetric orthogonal factors.

[4][10] finds a solution for $\mathbf{\Phi}$ by solving $\mathbf{\Phi}\mathbf{R}_2^H\mathbf{G} = \mathbf{R}_1^H\mathbf{G}$. If no consistent solution exists, the model is supposed to have no solution at all.

has full column rank and $\mathbf{R}_2^H$ has full row rank, any solution involving nullspaces reduces to the trivial solution $\mathbf{\Phi} = \mathbf{0}$. Additional solutions are given by any matrix of the form $\mathbf{R}_1^H \mathbf{\Psi} \mathbf{R}_2$ where $\mathbf{\Psi}$ is an arbitrary quadratic matrix in $\mathbb{R}^{n \times n}$. On the other hand an explicit transversality condition implies

$$\mathbf{R}_1\mathbf{\Phi}\mathbf{R}_2^H = \mathbf{R}_1\mathbf{T}_{12}\mathbf{P}_2^H\mathbf{H}^+\mathbf{R}_2\mathbf{R}_2^H + \mathbf{R}_1\mathbf{R}_1^H\mathbf{G}\mathbf{P}_1(\ker(\mathbf{WP}))_1 \ker((\ker(\mathbf{WP}))_2)\mathbf{J}\mathbf{P}_2^H\mathbf{H}^+ \cdot \mathbf{R}_2\mathbf{R}_2^H.$$

In order to guarantee real-valued entries for this expression, $\mathbf{P}_1(\ker(\mathbf{WP}))_1 \ker((\ker(\mathbf{WP}))_2)\mathbf{J}\mathbf{P}_2^H$ needs to contain exclusively real numbers as well. Some properties of the nullspace of any submatrix of a unitary matrices help to establish this property. Therefore we notice that $\ker(\mathbf{WP}) = \mathbf{P}^H \ker(\mathbf{W})$. Using this result yields $\ker((\ker(\mathbf{WP}))_i) = \ker(\mathbf{P}_i^H \ker(\mathbf{W}))$ where $i = 1$ ($i = 2$) denotes the set of rows of $\mathbf{P}^H$ associated with the stable (unstable) eigenvalues. For $i = 2$, this expression is denoted as $\mathbf{\Theta}$ below. Thus the result for $\mathbf{R}_1\mathbf{\Phi}\mathbf{R}_2^H$ simplifies to

$$\mathbf{R}_1\mathbf{\Phi}\mathbf{R}_2^H = \mathbf{R}_1\mathbf{T}_{12}\mathbf{P}_2^H\mathbf{H}^+\mathbf{R}_2\mathbf{R}_2^H + \mathbf{R}_1\mathbf{R}_1^H\mathbf{G}\mathbf{P}_1\mathbf{P}_1^H \ker(\mathbf{W})\mathbf{\Theta}\mathbf{J}\mathbf{P}_2^H\mathbf{H}^+\mathbf{R}_2\mathbf{R}_2^H.$$

This matrix's elements belong to the set of real numbers, iff $\mathbf{\Theta}\mathbf{J}\mathbf{P}_2^H$ is real-valued. A sufficient condition for this is

$$\begin{pmatrix} \Re(\mathbf{\Theta}) & \Im(\mathbf{\Theta}) \end{pmatrix} \begin{pmatrix} \Re(\mathbf{J}) & \Im(\mathbf{J}) \\ \Im(\mathbf{J}) & \Re(\mathbf{J}) \end{pmatrix} \begin{pmatrix} \Im(\mathbf{P}_2^H) \\ \Re(\mathbf{P}_2^H) \end{pmatrix} = \mathbf{0}. \quad (12)$$

Potentially this equation has several solutions. The most obvious is the trivial solution $\mathbf{J} = \mathbf{0}$, which holds always. The potential alternative solutions build on the right-hand (left-hand) nullspace of the first (third factor) within Equation (12). If the row-space of the first factor's right-hand nullspace includes the row-space of the third factor, *i.e.*

$$\operatorname{rank}\left(\ker\begin{pmatrix} \Re(\mathbf{\Theta}) & \Im(\mathbf{\Theta}) \end{pmatrix}\right) = \operatorname{rank}\left(\left(\left(\ker\begin{pmatrix} \Re(\mathbf{\Theta}) & \Im(\mathbf{\Theta}) \end{pmatrix}\Sigma_1\right)^T \ \left(\Im(\mathbf{P}_2^H)\right)^T \ \left(\Re(\mathbf{P}_2^H)\right)^T\right)^T\right),$$

there is a set of additional solutions of the form

$$\mathbf{J} = \left(\ker\begin{pmatrix} \Re(\mathbf{\Theta}) & \Im(\mathbf{\Theta}) \end{pmatrix}\sum\nolimits_1 \begin{pmatrix} \Im(\mathbf{P}_2^H) \\ \Re(\mathbf{P}_2^H) \end{pmatrix}^+ + \mathbf{D}\left(\mathbf{I} - \begin{pmatrix} \Im(\mathbf{P}_2^H) \\ \Re(\mathbf{P}_2^H) \end{pmatrix}\begin{pmatrix} \Im(\mathbf{P}_2^H) \\ \Re(\mathbf{P}_2^H) \end{pmatrix}^+\right)\right)_1 \begin{pmatrix} \mathbf{I} \\ i\mathbf{I} \end{pmatrix}, \quad (13)$$

in which $\sum_1$ is an arbitrary real-valued matrix of appropriate dimensions with at least the same rank as $\ker\begin{pmatrix} \Re(\mathbf{\Theta}) & \Im(\mathbf{\Theta}) \end{pmatrix}$ and $\mathbf{D}$ is an appropriately dimensioned, but otherwise arbitrary matrix. A similar solution exists if

$$\operatorname{rank}\left(\ker\left(\left((\Im(\mathbf{P}_2^H))^T \ (\Re(\mathbf{P}_2^H))^T\right)\right)\right) = \operatorname{rank}\left(\begin{pmatrix} \Re(\mathbf{\Theta}) & \Im(\mathbf{\Theta}) \end{pmatrix}\left(\left(\ker\begin{pmatrix} (\Im(\mathbf{P}_2^H))^T \\ (\Re(\mathbf{P}_2^H))^T \end{pmatrix}\sum\nolimits_2\right)^T\right)\right)^T$$

holds. This solution turns out to be

$$\mathbf{J} = \left(\begin{pmatrix} \Re(\mathbf{\Theta}) & \Im(\mathbf{\Theta}) \end{pmatrix}^+ \sum\nolimits_2^T \left(\ker\left(\begin{pmatrix} \Im(\mathbf{P}_2^H) \\ \Re(\mathbf{P}_2^H) \end{pmatrix}^T\right)\right)^T + \left(\mathbf{I} - \begin{pmatrix} \Re(\mathbf{\Theta}) & \Im(\mathbf{\Theta}) \end{pmatrix}^+\begin{pmatrix} \Re(\mathbf{\Theta}) & \Im(\mathbf{\Theta}) \end{pmatrix}\right)\mathbf{E}\right)_1 \begin{pmatrix} \mathbf{I} \\ i\mathbf{I} \end{pmatrix}. \quad (14)$$

Herein $\sum_2^T$ is a arbitrary matrix of appropriate dimensions with at least the same rank as the factor with which it is postmultiplied, while $\mathbf{E}$ is arbitrary, but of appropriate dimensions. For all three solutions discussed the coefficients Υ_1 and Υ_{2+q} are purely real-valued coefficients.

Hence, we can conclude this section by emphasizing that the structural properties of the generalized Schur decomposition guarantee the existence of real-valued coefficients for any solution obtained by the method of [10], while for solutions derived along the arguments of [3,5] those properties are not sufficient, because the solution's degrees of indeterminacy allow for complex-vaued coefficients as well. In order to avoid those, the degrees of freedoms need to be restricted in the sense that the indeterminacy is limited to linear subspaces, which render the mentioned coefficients to real-valued matrices. Nevertheless, the existence of a real-valued solution is in any case asserted by the possibility of the trivial solution for $\mathbf{J}$, which holds under any circumstances.

4. Conditions for a Real-Valued Coefficient of the Contemporary Shock Variables

In the process of establishing the conditions for a real-valued coefficient of the contemporaneous shock variables the paper follows the same classification as used in the last section. Thus again the solution are decided into one balancing the expectational errors within the unstable part of the model and one explaining those as functions of the shock terms. In addition in both solution methods the cases in which the transversality condition is or is not used explicitly are distinguished as well.

The first method produces the coefficient

$$\Upsilon_2 \equiv \mathbf{P}_1\mathbf{P}_1^H\mathbf{G}^+\mathbf{R}_1(\mathbf{R}_1^H - \mathbf{\Phi}\mathbf{R}_2^H)\mathbf{C}. \quad (15)$$

This coefficient appears to be real-valued whenever

$\mathbf{R}_1\mathbf{\Phi}\mathbf{R}_2^H$ is real. According to the results of the last section this condition holds always. Hence, the model's solution is real-valued by construction and no further conditions are required.

For the second method the coefficient Υ_2 has the general form

$$\Upsilon_2 \equiv \mathbf{P}_1\mathbf{P}_1^H\mathbf{G}^+\mathbf{R}_1\mathbf{R}_1^H\begin{pmatrix}\mathbf{C}-\tilde{\mathbf{G}}\tilde{\mathbf{G}}^+\mathbf{R}_2\mathbf{R}_2^H\mathbf{C}\\ -\tilde{\mathbf{G}}(\mathbf{I}-\tilde{\mathbf{G}}^+\mathbf{R}_2\mathbf{R}_2^H\tilde{\mathbf{G}})\mathbf{Z}\end{pmatrix}. \tag{16}$$

If the shock term explains the expectational error completely, the pseudoinverses in Equation (16) are rendered to inverses and the matrix $\mathbf{Z}$ is a zeromatrix. Hence the coefficient is real-valued. But if the shock term explains the expectational error only partially, there remain indeterminacies within the solution. In this situation the matrix $\mathbf{Z}$ takes on two different values depending on whether a specific transversality condition is integrated into the model's solution or not. For the case that no specific transversality condition applies, there is no additional restriction on $\mathbf{Z}$ because all other terms in Equation (16) are real-valued by construction. Thus any matrix $\mathbf{Z}$ belonging to $\mathbb{R}^{n\times m}$ generates purely real-valued solutions for the economic model. For the opposite case the transversality condition implies

$$\begin{aligned}\mathbf{Z} = {} & (\mathbf{I}-\tilde{\mathbf{G}}^+\mathbf{R}_2\mathbf{R}_2^H\tilde{\mathbf{G}})^+\tilde{\mathbf{G}}^+\mathbf{R}_1\mathbf{R}_1^H\mathbf{G}\mathbf{P}_1\mathbf{P}_1^H\\ & (\mathbf{G}^+\mathbf{R}_1\mathbf{R}_1^H(\mathbf{I}-\tilde{\mathbf{G}}\tilde{\mathbf{G}}^+\mathbf{R}_2\mathbf{R}_2^H)\mathbf{C}-\ker(\mathbf{W})\mathbf{\Theta}\mathbf{F})\end{aligned}$$

where $\mathbf{F}$ is left undetermined, but appropriately dimensioned. Inserting this expression into Equation (16) demonstrates that a purely real solution requires

$$\begin{aligned}&\Im(\mathbf{P}_1\mathbf{P}_1^H\mathbf{G}^+\mathbf{R}_1\mathbf{R}_1^H\tilde{\mathbf{G}}(\mathbf{I}-\tilde{\mathbf{G}}^+\mathbf{R}_2\mathbf{R}_2^H\tilde{\mathbf{G}})\cdot\\ &(\mathbf{I}-\tilde{\mathbf{G}}^+\mathbf{R}_2\mathbf{R}_2^H\tilde{\mathbf{G}})^+\tilde{\mathbf{G}}^+\mathbf{R}_1\mathbf{R}_1^H\mathbf{G}\mathbf{P}_1\mathbf{P}_1^H\ker(\mathbf{W})\mathbf{\Theta}\mathbf{F})=\mathbf{0}.\end{aligned} \tag{17}$$

Denoting the factor in front of $\mathbf{F}$ as Ξ and solving Equation (17) for all terms involving on $\mathbf{F}$ as the nullspace of this factor yields

$$\begin{pmatrix}\Im(\mathbf{F})\\ \Re(\mathbf{F})\end{pmatrix}=\ker\begin{pmatrix}\Re(\Xi) & \Im(\Xi)\end{pmatrix}\mathbf{E}, \tag{18}$$

where $\mathbf{E}$ is arbitrary, but real-valued and of appropriate dimensions. Thus the solutions for $\mathbf{F}$ are given by

$$\mathbf{F}=\begin{pmatrix}\mathrm{i}\mathbf{I} & \mathbf{I}\end{pmatrix}\ker\begin{pmatrix}\Re(\Xi) & \Im(\Xi)\end{pmatrix}\mathbf{E}. \tag{19}$$

Inserting any of these solutions into $\mathbf{Z}$ delivers finally a real-valued coefficient Υ_2. Again, similarly as in the last section, it should be pointed out that Equation (17) can be fulfilled independently of the rank of Ξ by the trivial solution for $\mathbf{F}$. Hence a real-valued coefficient Υ_2 can be asserted under any circumstances.

5. Conclusions

In this paper we have established the conditions for the existence of a purely real-valued solution path to any linear rational expectations model which is based on the generalized Schur decomposition. For solutions, which explain the influence of the expectational error on the model's stable part as a function of its influence on the model's unstable part, the structural properties of the generalized Schur decomposition suffice to guarantee that the entire solution path is a real-valued one. But for solutions explaining the expectational error directly by the exogenous shock term those properties are not sufficient to exclude complex-valued solution paths. Nevertheless, existing degrees of freedom allow to establish additional constraints which force the solution paths into the real domain. The paper shows also that the trivial solution for the degrees of freedom generate a real-valued solution path under any circumstances. Thus the existence of at least one of those paths is asserted.

These results imply that a direct explanation of the expectational error by the exogenous shock term generates the potential of complex-valued solutions. Certainly this can be interpreted as a disadvantage compared to the methods which force the unstable variables to balance each other and use those to explain the expectational error's influence on the stable part. On the other hand the first approach allows for a solution of models which have been unsolvable with the second approach. Since it is always possible to find solutions obtained by the first approach which take one pure real values, this paper concludes that the mentioned disadvantages of this approach are balanced by its advantages. Hence this paper supports a more wide-spread application of the method presented first in [5] for the solution of macroeconomic models.

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