Scientific
Research

A Qualitative Perstective on Idempotency Defect of Two Level System Interacting with Laser and Quantized Field

Sayed Abdel-Khalek[1,2], Mohammed Mohammed Ali Ahmed[1,4], Waleed Nabeel Razek[3], Abdel Shafy Obada[4]

[1]Mathematics Department, Faculty of Science, Taif University, Taif, Saudi Arabia; [2]Department of Mathematics, Faculty of Science, Sohag University, Sohag, Egypt; [3]Mathematics Department, Physics Division, National Research Centre, Cairo, Egypt; [4]Mathematics Department, Faculty of Science, Al-Azher University, Nassr City, Cairo, Egypt.
Email: abotalb2010@yahoo.com

ABSTRACT

Entanglement due to the interaction of a two level atom with a laser and quantized field is investigated. The role of the nonlinearity due to these interactions is discussed. It is found that the nonlinearity changes strongly the behavior of the entanglement also the detuning parameters have important role in the structure of the measure of entanglement.

Keywords: *Entanglement, Idempotency Defect, Two-Level System, Quantized*

1. Introduction

Quantum information processing provides different way for manipulating information other than the classical one. This is related to entanglement, which plays an essential role in the quantum information such as quantum computing [1], teleportation [2], cryptography [3], dense coding [4] and entanglement swapping [5]. Thus intensive efforts have been done to understand theoretically and experimentally the entanglement in quantum systems. For instance, the entanglement between two qubits in an arbitrary pure state has been quantified by the concurrence [6] and Peres-Horodecki measure [7]. However, that of the mixed states is quantified by the average concurrence over all possible pure state ensemble decomposition. Additionally, the entropic relations are used in investigating the entanglement in quantum system. In this regard von Neumann entropy (NE) [8], linear entropy (LE) and the Shannon information entropy (SE) [9] have been frequently used in treating entanglement in the quantum systems. The NE [10] and the LE [11] have been applied to the JCM. It is worth mentioning that the SE involves only the diagonal elements of the density matrix and in some cases it can give information similar to that obtained from the NE and LE.

The paper is prepared in the following order: In Section 2 we define the system Hamiltonian. In Section 3, we derive the time evolution operator and density matrix. In

Section 4 we investigate the atomic inversion and idempotency defect (ID). In Section 5 we summarize the main results.

2. The System Hamiltonian

The scheme, we are going to discuss exploits the passage of a single atom through a quantized cavity [12]. We wish to underline the relevance of this aspect from an experimental point of view. Preparing and controlling a single atom is certainly much easier to achieve with respect to the case when the manipulation of many atoms is required. In addition, taking into consideration the low efficiency of the atomic state detectors today used in laboratory, conditional measurement procedures involving one atom only instead of many ones, have to be preferred. The dynamics of several Hamiltonian models describing such systems is exactly treatable and, in most cases, testable in the laboratory. The point to be appreciated is indeed that, studying such systems, one has the opportunity to induce entanglement and control its evolution in a multipartite physical system. In this paper: we study a two-level atom injected into a cavity field, where the dipole-allowed transitions between the lower level and the upper levels are nonresonant with the cavity mode. Furthermore, we assume that the interaction including an arbitrary form of nonlinearity of the intensity-dependent coupling. In the rotating wave approximation (RWA), the interaction of the cavity mode with the injected atom is

described by the Hamiltonian ($\hbar = 1$):

$$H = H_A + H_F + H_{AF} + H_{AL,} \qquad (1)$$

where $H_A = \omega_0 S_z$ is the atom Hamiltonian, $H_F = \omega_c \hat{a}^\dagger \hat{a}$ is the field Hamiltonian, $H_{AF} = \frac{\xi}{2}(S_+ \hat{a} + \hat{a}^\dagger S_-)$ is the atom-field Hamiltonian, and

$$H_{AL} = \frac{r}{2}(S_+ e^{-i(\omega_L t + \psi_L)} + e^{i(\omega_L t + \psi_L)} S_-)$$

is the atom-laser Hamiltonian.

The atomic transition frequency is denoted by ω_o, the frequencies of the cavity and laser fields are denoted by ω_c and ω_L receptivity, and the laser field is assigned by the phase ψ_L, the operator $\hat{a}(\hat{a}^\dagger)$ is the annihilation (creation) operator of the cavity field and obey the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, ξ and r are the coupling constants associated with the cavity field and the classically described laser field respectively which are assumed real. The S operators are the coherence operators for the atom, satisfying the following angular momentum commutation relations $[S_{ij}, S_{lm}] = S_{im}\delta_{jl} - S_{lj}\delta_{mi}$ where $i,j,l,m = 1,2,...$

3. Time Evolution Operator and Density Matrix

To obtain the time evolution operator, we must eliminate the explicit time dependence of the previous Hamiltonian 1), we use the following unitary operator [13]

$$T(t) = e^{-i(\omega_L t + \psi_L)(S_z + \hat{a}^\dagger \hat{a})}, \qquad (2)$$

we redefine the Hamiltonian as :

$$H_1 = T^\dagger(t)HT(t) - iT^\dagger(t)\frac{dT^\dagger(t)}{dt}. \qquad (3)$$

Then the Hamiltonian 1) will be in the following form:

$$H_1 = \Delta_c \hat{a}^\dagger \hat{a} + \Delta_L S_z + \frac{\xi}{2}(S_+ \hat{a} + \hat{a}^\dagger S_-) + \frac{r}{2}(S_+ + S_-), \quad (4)$$

where $\Delta_c = \omega_c - \omega_l$ and $\Delta_l = \omega_0 - \omega_l$, if we consider the resonant pumping case $\omega_0 = \omega_l$ and rearrange the Hamiltonian terms, the Hamiltonian (4) becomes.

$$H_1 = rS_z' + \Delta_C \hat{a}^+ \hat{a} + \frac{\xi}{2}(\hat{a}^+ + \hat{a})S_z'$$
$$+ \frac{\xi}{2}(\hat{a} - \hat{a}^+)(S_+' + S_-') \qquad (5)$$

The new atomic operators set

$$(S_z' = S_x, S_+' = S_z + iS_y, S_-' = S_z - iS_y)$$

which obey the angular momentum commutation relations among themselves. The eigenvector of $S_z' = S_x$ is the dressed states of the atom in the laser field alone and the operators S_+' and S_-' are the corresponding raising and lowering operators.

To associate the third and the fourth terms of H_1 we use the following unitary transformation:

$$\tilde{H} = P^\dagger H_1 P, \qquad (6)$$

where P is an atomic state dependent displacement operator of the cavity field state which is defined as :

$$P = e^{\frac{\lambda}{2\Delta_c}(\hat{a} - \hat{a}^\dagger)S_z'} \qquad (7)$$

Then the Hamiltonian 4) takes the following form:

$$\tilde{H} = rS_z' + \Delta_c \hat{a}^+ \hat{a} + \frac{\xi}{2}(\hat{a} - \hat{a}^+)(S_+' v + S_-' v^+) - \frac{\lambda^2}{16\Delta_c}, \quad (8)$$

where

$$v = e^{\frac{\xi}{2\Delta_c}(\hat{a}^\dagger - \hat{a})} = e^{\frac{1}{2}(\frac{\xi}{2\Delta_c})^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(\frac{\xi}{2\Delta_c}\right)^{n+m} \frac{(-1)^n}{n!m!} \hat{a}^m \hat{a}^{\dagger n}. (9)$$

We can drop $\frac{\lambda^2}{16\Delta_c}$ which has no effect in the dynamics of the system.

This Hamiltonian describes a two level system with energy levels separated by r coupled to a single quantized mode of radiation frequency Δ_c and there is an infinite sequence of probe resonance zones in the neighborhood of the Rabi sub harmonic resonances at

$$\Delta_c = \frac{\varepsilon}{m}, m = \pm 1, \pm 2, ...$$

If we consider of the RWA, the slowly varying terms are identified and retained, in the m th resonance zone S_+' has the zero order time dependence $e^{-i\varepsilon t}$ that is approximately canceled by the zero order time dependence $e^{-i\frac{\Delta_c}{m}}$ of all field operators of the form $\hat{a}^{\dagger m}\hat{a}^{n+m}$, for any n.

The Hamiltonian for the mth resonance zone is:

$$H_{eff} = \Delta_c \hat{a}^\dagger \hat{a} + rS_z' + \frac{\xi}{4}(S_+' f_m + S_-' f_m^\dagger) \qquad (10)$$

In this case, we recognize as the mth-order multiphoton Jaynes Cumming Hamiltonian where we consider

$$f_m = f(n)\hat{a}^m, \quad f(n) = e^{\left(\frac{\xi}{2\Delta_c}\right)^2} \sum_{\ell=0}^{\infty} \left(\frac{\xi}{2\Delta_c}\right)^{2\ell} \frac{(-1)^\ell}{(\ell!)^2} \hat{a}^\ell \hat{a}^{+\ell}$$

destroys m photons of frequency Δ_c where H_{eff} can be further simplified by means of another rotating wave approximation (RWA),(as discussed in detail in [12-14]).

The time evolution operator is given by:

$$U(t) = T(t)Pe^{-iH_{eff}}P^\dagger T^\dagger(t)$$
$$= e^{-i\theta(\hat{a}^\dagger \hat{a})}\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}e^{i\psi_L(\hat{a}^\dagger \hat{a})}, \qquad (11)$$

$$\theta = (\omega_L t + \psi_L)$$

where

$$U_{11} = \frac{1}{2}e^{i(\psi_L - \theta)}(A + B + D + E),$$

$$U_{12} = \frac{1}{2}e^{-i(\psi_L + \theta)}(A - B + D - E),$$

$$U_{21} = \frac{1}{2}e^{i(\psi_L + \theta)}(A + B - D - E), \qquad (12)$$

$$U_{22} = \frac{1}{2}e^{-i(\psi_L - \theta)}(A - B - D + E),$$

and

$$A = Ge^{-i\Delta_C\left(n + \frac{m}{2}\right)t}F_1(t)G^+,$$

$$B = iG\xi e^{-i\Delta_C\left(n + \frac{m}{2}\right)t}f(m)E_1(t)G, \qquad (13)$$

$$D = iG\xi e^{-i\Delta_C\left(n - \frac{m}{2}\right)t}f(n)E_2(t)G^+,$$

$$E = iG^+\xi e^{-i\Delta_C\left(n - \frac{m}{2}\right)t}F_2(t)G,$$

where

$$F_j = \cos(\mu_j t) - i\delta\frac{\sin(\mu_j t)}{\mu_j}, E_j = -\frac{\sin(\mu_j t)}{\mu_j}, j = 1,2.$$

Also,

$$\mu_1(n) = \frac{1}{4}\sqrt{16\delta^2 + (\lambda v_1(n))^2},$$

$$\mu_2(n) = \mu_1(n-m), v_1(n) = L_n^m(\xi^2)\sqrt{\frac{(n+k)!}{n!}},$$

where, $v_2(n) = v_1(n-m), \delta = \frac{1}{2}(\varepsilon - m\Delta_c)$, $G = v^2$ and $L_n^m(\xi^2)$ is the Laguerre function. At $t = 0$ the wave function of the system can be written as:

$$\left|\Psi(0)\right\rangle = \left[\cos\left(\frac{\vartheta}{2}\right)|e\rangle + \sin\left(\frac{\vartheta}{2}\right)|g\rangle\right]|\alpha\rangle \qquad (14)$$

where ϑ is the superposition state parameter

$$|\alpha\rangle = \sum_n \exp(-\frac{|\alpha|^2}{2})\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \qquad (15)$$

with the mean photon number $\bar{n} = |\alpha|^2$, then the wave function at any time $t > 0$ is given by

$$\left|\Psi(t)\right\rangle = U(t)\left|\Psi(0)\right\rangle$$
$$= e^{-i(\omega_L + \psi_L)(\hat{a}^+\hat{a})}\{U_{12}|e,\alpha\rangle + U_{22}|g,\alpha\rangle\} \qquad (16)$$

In the pure state case it is well known that the density matrix of the atom-field interaction can be written as:

$$\rho(t) = \left|\Psi(t)\right\rangle\left\langle\Psi(t)\right| \qquad (17)$$

In order to analyze what happens to the two level sys-tems interacting with laser and quantized field, we trace out field variables from the state $\hat{\rho}(t)$ and get the re-duced atomic density matrix of the system given by

$$\rho_A(t) = tr_F\left|\psi_{AF}(t)\right\rangle\left\langle\psi_{AF}(t)\right|$$
$$= \sum_L\langle L|\rho|L\rangle = \rho_{ee}(t)|e\rangle\langle e| + \rho_{eg}(t)|e\rangle\langle g|$$
$$+ \rho_{ge}(t)|g\rangle\langle e| + \rho_{gg}(t)|g\rangle\langle g|,$$

$$(18)$$

where

$$\rho_{ee}(t) = \langle n|e^{-i(\theta n - \psi_L)}U_{12}|\alpha\rangle\langle\alpha|U^+_{12}e^{i(\theta n - \psi_L)}|n\rangle,$$

$$\rho_{eg}(t) = \langle n|e^{-i(\theta n - \psi_L)}U_{12}|\alpha\rangle\langle\alpha|U^+_{22}e^{i(\theta n - \psi_L)}|n\rangle,$$

$$\rho_{ge}(t) = \langle n|e^{-i(\theta n - \psi_L)}U_{22}|\alpha\rangle\langle\alpha|U^+_{12}e^{i(\theta n - \psi_L)}|n\rangle,$$

$$\rho_{gg}(t) = \langle n|e^{-i(\theta n - \psi_L)}U_{22}|\alpha\rangle\langle\alpha|U^+_{11}e^{i(\theta n - \psi_L)}|n\rangle.$$

$$(19)$$

4. Atomic Inversion and Idempotency Defect

Atomic inversion can be considered as the simplest im-portant quantity. It is defined as the difference between the probability of finding the atom in the exited state and in the ground state, the time dependent atomic inversion in the mth resonance $(\varepsilon = m\Delta_c)$ is given by:

$$W(t) = \rho_{ee}(t) - \rho_{gg}(t) \qquad (20)$$

Finally the idempotency defect as a measure of entan-glement is written as:

$$\varepsilon_t^{(J)} = Tr_f\{\rho(t)(1 - \rho(t))\}$$
$$= 1 - Tr_A(\rho(t))^2 \qquad (21)$$
$$= 2\rho_{ee}\rho_{gg} - 2|\rho_{eg}|^2$$

In many cases of quantum information processing, one requires a state with high purity and large amount of en-tanglement. Therefore, it is necessary to consider the purity of the state and its relation with entanglement.

Here we use the idempotency defect, defined by linear entropy, as a measure of the degree of purity for a state $\rho(t)$, in analogy to what is done for the calculation of the entanglement in terms of von Neumann entropy [15] which has similar behavior in the same cases.

[In all our figures we have plotted the atomic inversion $W(t)$ and the entanglement ID $\varepsilon_t^{(J)}$ as a function of the scaled time ξt for the case of two-photon processes]. In **Figure 1**, we plot the atomic inversion and ID as a func-tion of the scaled time. It is seen that, when we neglect the Laguerre function term, the atomic inversion shows fast Rabi oscillations, and the periodic behavior of the ID is seen. In this case the usual two-photon features are ob-served. Once the Laguerre term is considered the general

The Laguerre function term is ignored

The Laguerre function term is considered
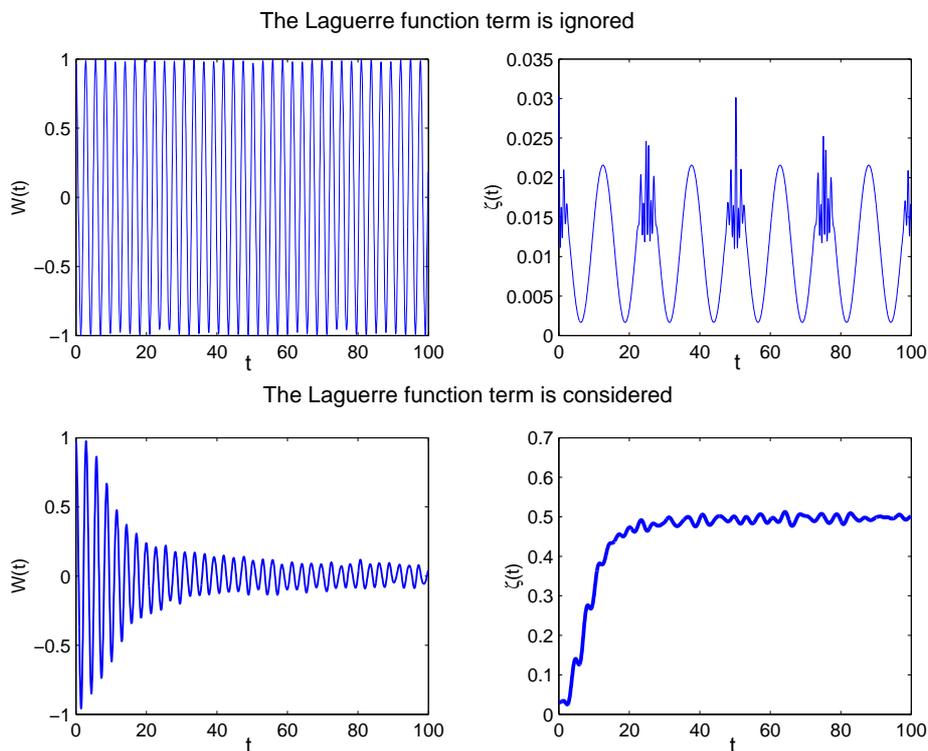
Figure 1. The evolution of the (a) atomic inversion and idempotency defect against the scaled time rt **for the parameters** α = 5 , ξ = 0.5, k = 2 **the laser detuning parameter** $\Delta_C = 1$, θ = 0 **and** δ = 0
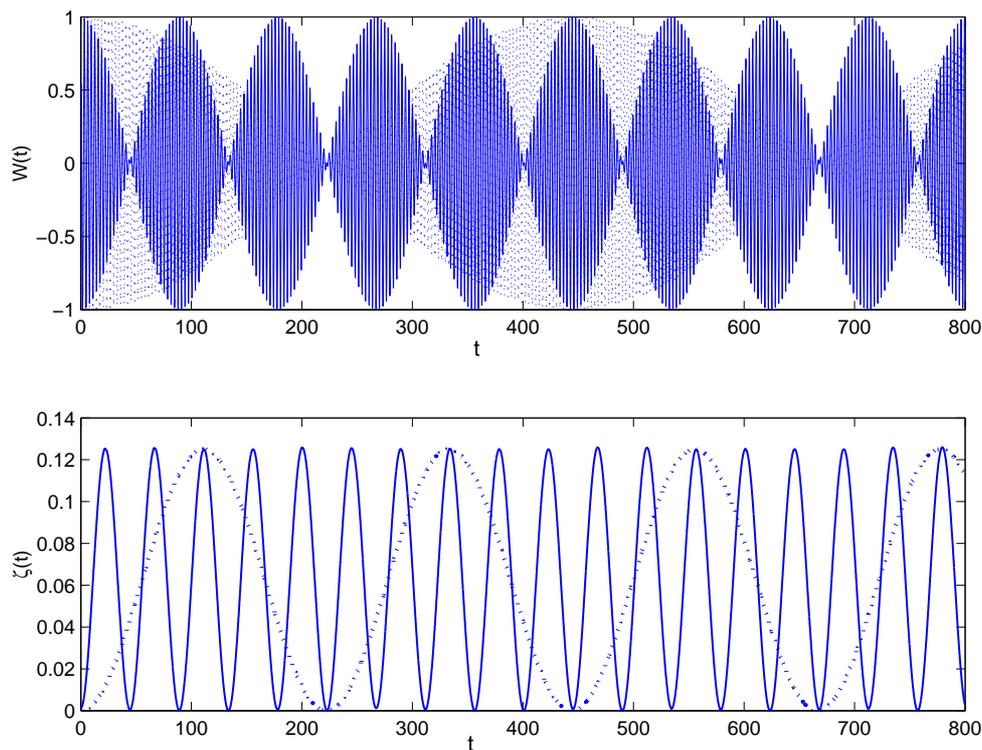
Figure 2. The evolution of the (a) atomic inversion and (b) idempotency defect against the scaled time rt **for the parameters** α = 0 , ξ = 0.02 **(dotted line),** ξ = 0.1 **(solid line),** k = 2 **the laser detuning parameter** $\Delta_C = 1$, θ = 0 **and** δ = 0

SGRE

Figure 3. The evolution of the (a) atomic inversion and (b) idempotency defect against the scaled time for the parameters $\alpha = \sqrt{10}$, $\xi = 0.05$ (dotted line), $\xi = 0.01$ (solid line), $k = 2$ the laser detuning parameter $\Delta_C = (2n\text{-}0.5)\pi\,/\,t$, $\theta = 0$ and $\delta = 0$
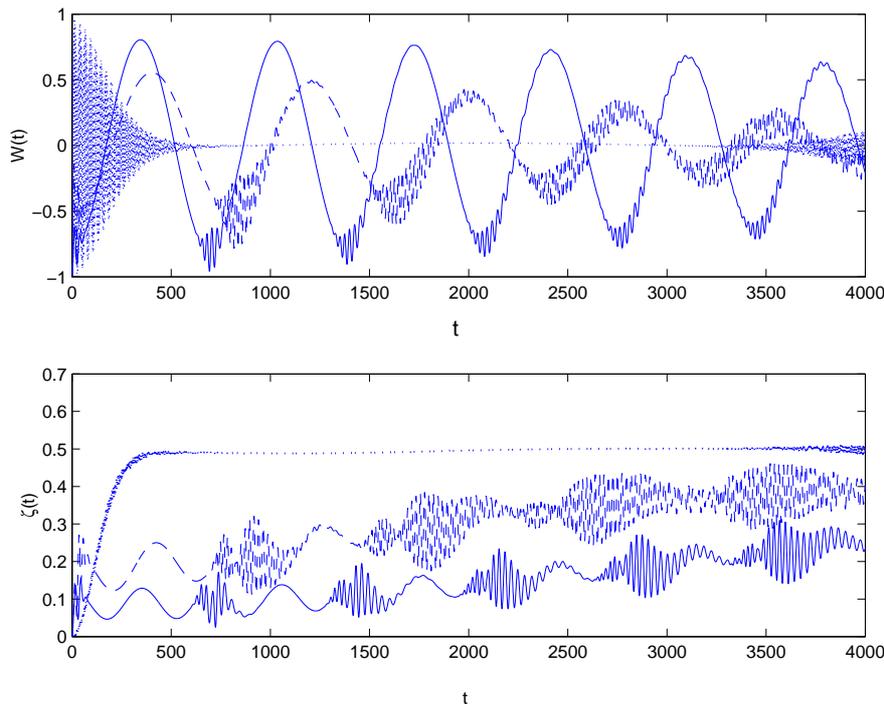
Figure 4. The evolution of the (a) atomic inversion and (b) idempotency defect against the scaled time rt for the parameters $\alpha = \sqrt{10}$, $k = 2$ the laser detuning parameter $\Delta_C = 1$, $\theta = 0$ and with different values of the detuning $\delta = 0$ (solid line), $\delta = 0.01$ (dashed line), $\xi = 0.1$ (dotted line)

SGRE
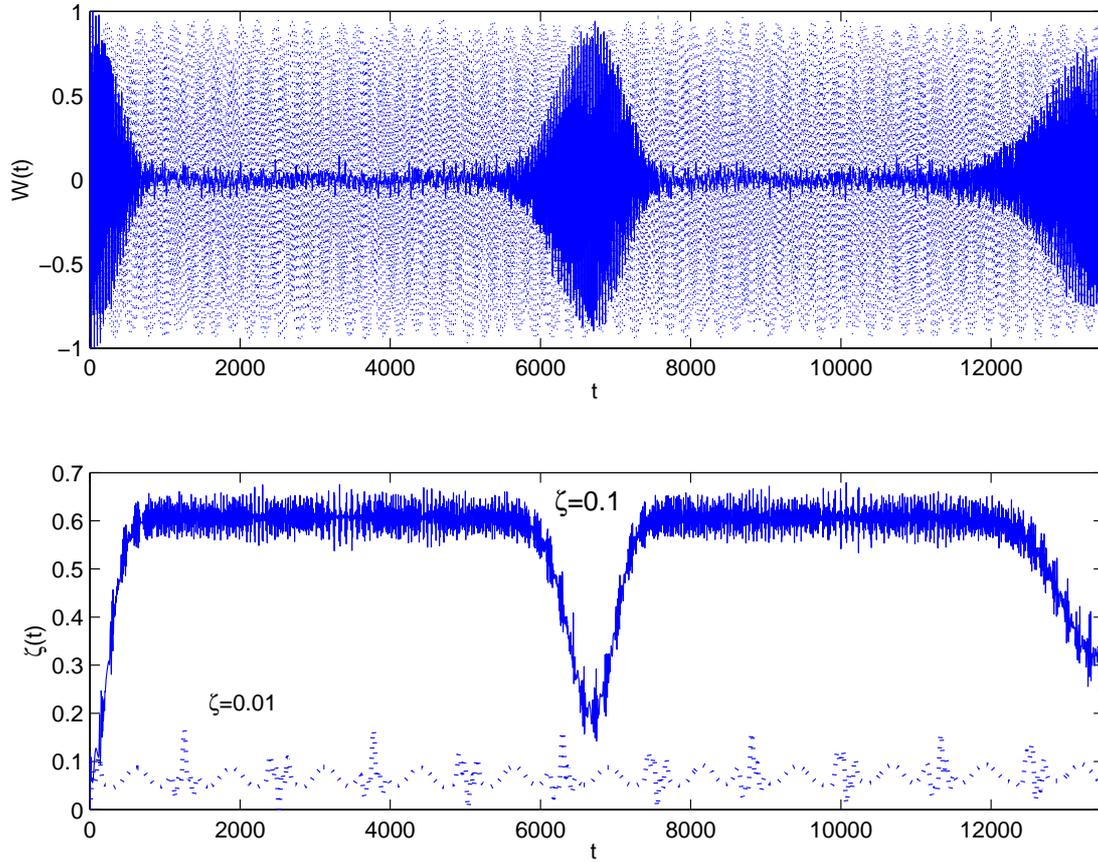
Figure 5. The evolution of the (a) Atomic inversion and (b) Idempotency defect against the scaled time, rt for the parameters $\alpha = \sqrt{10}$, $\xi = 0.01$ **(dotted line),** $\xi = 0.1$ **(solid line),** $k = 2$ **the laser detuning parameter** $\Delta_C = 1$, $\theta = 0$ **and** $\delta = 0.1$

features of both the atomic inversion and ID are washed out, where as the time goes on the atomic inversion shows small amplitude of the oscillations and steady state of the ID (see **Figure 1**). As the time increases further the maximum entangled state can be obtained, where $W(t) = 0 \Rightarrow$ $\rho_{ee}(t) = \rho_{gg}(t)$ at this moment the entanglement reaches its maximum value. An interesting case is seemed in **Figure 2**, where we have considered small values of the parameter ξ . In this case the collapse-revival phenomenon is shown. Also, the periodic oscillation occurs each 60π for atomic inversion and the zero ID at 15π for $\xi = 0.1$.

Now we shed the some light on the affect of the different parameters on the both of the atomic inversion and the corresponding ID. For example the effect of the detuning Δ_C is considered in **Figure 3**. In this case we see that the amplitude of the value of the detuning increased as the value of the detuning increased. The ID decreases and reaches steady state as Δ_C increases further.

On the other hand, the detuning of the quantized field δ, plays the opposite role, where the ID shows steady state at maximum instead of lowing its value due to the detuning of the laser field.

As δ increases further (say $\delta = 0.5$) we obtain a long

lived entanglement, as to show the fixed value of the ID ($\xi = 0.5$). In **Figure 5**, we see that the collapse-revival phenomenon is clearly seen. Also, a perfect correspondence between the atomic inversion revival and the local maxima corresponds to the collapse periods. Also, we see that the collapse-revival phenomenon is clearly seen. Also, a perfect correspondence between the atomic inversion revival and the local maxima corresponds to the collapse periods. we have plotted the atomic inversion and the corresponding field Idempotency for a state of mean photon number of the coherent field as $\bar{n} = 10$, fixed the detuning parameter $\Delta_c = 1$ and the parameter $\delta = 0$ and changed the parameter ξ with values (0.01 and 0.1), when $\xi = 0.01$, oscillated with period equal $\xi t = 2000\pi$ and $W(t)$ oscillated with same interval and $\varepsilon_t^{(J)}$ has a long-lived entanglement in each interval. When $\xi = 0.01$, then $\varepsilon_t^{(J)}$ has a weak oscillation and $W(t)$ oscillated with a very small interval between maximum and minimum values.

5. Conclusions

In the present paper we show that the on idempotency defect can be used to quantify entanglement of a two level

system interacting with laser and quantized field. We obtain the long living entanglement due to the idempotency defects of the field. Our results show that the nonlinearity changes strongly the behavior of the entanglement also the detuning parameters have important role in the structure of the measure of entanglement. Also, all the interaction parameters play an important role on the behavior of idempotency defect.

REFERENCES

[1] G. Benenti, G. Casati and G. Strini "Principles of Quantum Computation and Information, Vol. 1," *World Scientific*, 2004.

[2] C. Bennet, G. Brassard, C. Crepeau, R. Jozsa, A. Peres and W. K. Wootersm, "Teleporting an Unknown Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels," *Physical Review Letter*, Vol. 70, No. 13, 1993, pp. 1895-1899 .

[3] A. Ekert, "Quantum Cryptography Based on Bell's Theorem," *Physical Review Letter*, Vol. 67, No. 6, 1991, pp. 661-663.

[4] L. Ye and G.-C. Guo, "Scheme for Implementing Quantum Dense Coding in Cavity QED," *Physical Review Letter A*, Vol. 346, No. 5-6, 2005, pp. 330-336.

[5] O. Glöckl, S. Lorenz, C. Marquardt, J. Heersink, M. Brownnutt, C. Silberhorn, Q. Pan, N. V. Loock, N. Korolkova and G. Leuchs, "Experiment towards Continuous-Variable Entanglement Swapping: Highly Correlated Four-Partite Quantum State," *Physical Review Letter A*, Vol. 68, No. 1, 2003, pp. 012319-012327.

[6] W. K. Wootters, "Entanglement of Formation of an Arbitrary State of Two Qubits," *Physical Review Letter*, Vol. 80, No. 10, 1998, pp. 2245-2248.

[7] A. Peres, "Separability Criterion for Density Matrices," *Physical Review Letter*, Vol. 77, No. 8, 1996, pp. 1413-14150.

[8] J. von Neumann, "Mathematical Foundations of Quantum Mechanics," Princeton University Press, Princeton, 1955.

[9] C. E. Shannon and W. Weaver, "The Mathematical Theory of Communication," Urbana University Press, Chicago, 1949.

[10] S. J. D. Phoenix and P. L. Knight, "Fluctuations and Entropy in Models of Quantum Optical Resonance," *Annals of Physics*, Vol. 186, No. 2, pp. 381-407, 1988.

[11] F. A. A. El-Orany and A.-S. Obada, "On the Evolution of Superposition of Squeezed Displaced Number States with the Multiphoton Jaynes–Cummings Model," *Jounal of Optics B*: *Quantum Semiclass Optics*, Vol. 5, No. 1, 2003, pp. 60-72.

[12] M. Lewenstein and T. W. Mossberg, "Spectral and Statistical Properties of Strongly Driven Atoms Coupled to Frequency-Dependent Photon Reservoirs," *Physical Review Letter A*, Vol. 37, No. 2048, 1988, pp. 2048-2062.

[13] C. K. Law and J. H. Eberly, "Response of a Two-Level Atom to a Classical Field and a Quantized Cavity Field of Different Frequencies," *Physical Review Letter A*, Vol. 43, No. 11, 1991, pp. 6337-6344..

[14] J. H. Eberly and V. D. Popov, "Phase-Dependent Pump-Probe Line-Shape Formulas," *Physical Review Letter A*, Vol. 37, No. 6, 1988, pp. 2012-2016.

[15] G. M. A. Al-Kader and M. M. A. Ahmad, "On the Interaction of Two-Level Atoms with Squeezed Coherent State," *Chinese Jounal of Physical*, Vol. 39, No. 1, 2001, pp. 50-63.

[16] A. Ekert and P. L. Knight, "Entangled Quantum Systems and the Schmidt Decomposition," *American Journal of Physics*, Vol. 63, No. 5, 1994, pp. 415-423.