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Photonic Communications and Quantum Information Storage Capacities

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ABSTRACT

This paper presents photonic communications and data storage capacitates for classical and quantum communications over a quantum channel. These capacities represent a generalization of Shannon's classical channel capacity and coding theorem in two ways. First, it extends classical results for bit communication transport to all frequencies in the electromagnetic spectrum. Second, it extends the results to quantum bit (qubit) transport as well as a hybrid of classical and quantum communications. Nature's limits on the rate at which classical and/or quantum information can be sent error-free over a quantum channel using classical and/or quantum error-correcting codes are presented as a function of the thermal background light level and Einstein zero-point energy. Graphical results are given as well as numerical results regarding communication rate limits using Planck's natural frequency and time-interval units!

Keywords: Quantum Communications; Quantum Information Storage; Quantum Error-correcting Coding; Nature's Photonic Limits

1. Introduction

Photonic modulation can be used, respectively, to reliably transport classical information bits as well as quantum information qubits, see **Figure 1**. Using the "second-quantization" of the electromagnetic field, quantum mechanical models for coherent photonic states and Shannon's sphere-packing argument, the quantized analog of Shannon's classical channel capacity and coding theorem is derived when classical or quantum information bits are transported over a quantum channel. Using this result, the unit information metric between a classical bit and a quantum bit, the qubit, is established from which the quantum channel capacity and spectral efficiency, quantum information storage density and quantum information storage capacity are developed. It is shown that the quantized capacities (signal energy discrete) reduces to Shannon's classical results when the energy in the field is assumed to be continuous and the channel center frequency ν_c° Hz/°K is less than the partitioning frequency ν_p° Hz/°K, i.e., $\nu_c^\circ \le \nu_p^\circ = 0.4k/h$ Hz/°K, where k is Boltzman's and h is Planck's constant.

Nature's limits on the rate at which classical and quantum information can be sent error-free over a quantum channel using classical and/or quantum error-correcting codes are presented as functions of the thermal background light level and Einstein's zero-point energy (ZPE). For system engineering design, numerous graphical results are plotted for both the quantized and quantum channel capacities, spectral efficiencies and the information storage capacities. The results demonstrate the feasibility of Terabit per sec to Petabit per sec data rates and Petabyte information storage capacities of 2/ln2 bits/photon or one qubit per photon. In this regard, it is shown that the qubit information unit equals two nats/qubit or 2/ln 2 bits/qubit, i.e., 1 qubit = 2 nats = 2/ln 2 bits! Finally, it is shown that error-free quantum communications can be asymptotically approached in a wideband pristine environment using a minimum of 0.345 photons per bit or one photon per qubit. In a low temperature environment, it is shown that classical or quantum information, storage density and communications capacity do not depend upon energy but upon the ratio of two integers, viz., the ratio of the number of photons per message, N_s to the number of dimension per message, N, or equivalently, the interialess photon - time bandwidth product BT_p. By setting a fundamental bandwidth limitation on the quantum channel bandwidth B using Planck's natural frequency and time - interval units at boundary $BT_p = 1$, it is shown that Planck's quantum communications capacity approximately equals 1043 qubits/sec or (2.9)1043 bits/sec.

It is further shown that there exists a quantum error correcting code that achieves zero MEP if and only if the code rate $\Re_{N_s} \le 2\ln 2 = E_{qb} Z_0 \ln 2$ where $Z_0 = h\nu/2$ is Einstein's zero-point energy level. From this we obtain the energy per bit to noise condition $E_{qb}/Z_0 \ge \ln 2$, or equivalently, bit rate $\Re_b \le S/Z_0 \ln 2$ for all $\nu \in [B,\infty]$. This compares with the classical results of Shannon where $E_{qb}/Z_0 \ge \ln 2$ and $\Re_b \le S/N_0 \ln 2$ for $\nu \le \nu_p$. Finally, information quanta are identified and related to Planck - Einstein energy quanta.

2. System Functional Architectures

In this section, functional architectures for quantized-classical and quantum communication systems are presented along with system parameters and performance metrics. System parameters include: time intervals, channel bandwidth, bit and qubit signal energies and associated thermal background light levels. Performance metrics include: channel capacities, information storage densities and information storage capabilities. Relationships connecting these performance metrics are established together with those that relate quantum assets to their classical counterparts.

2.1 Classical-Quantum Communication System Models

For transmission, classical bits are encoded into coded-digits using a classical $[N,K]$ error-correcting code; see **Figure 1**.

More specifically, assume *M*equiprobable messages containing $K = \log_2 M$ bits per message, see **Figure 1**. Each message is assumed to last for $T = (\log_2 M)T_b$ sec per message; T_b is the time per bit. Each *K*-tuple is encoded into *N* coded-digits using a classical [*N, K*] error-correcting code of code rate $R_N = K/N = (\log_2 M)/N$ bits per dimension. Each coded digit lasts for T_N sec such that $T = NT_N$ seconds per message are used to modulate the polarization of a coherent photon source of rate $\mathcal{R}_p = T_p^{-1}$ photons per second and center frequency $\nu_c \epsilon [B,\infty]$ Hz at code rate $R_{N_s} = \log_2 M/N_s$ bits per photon; here N_s is the average number of signal photons in each message. Thus every *T* sec an *N*-dimensional photonic signal containing energy $E = N_s E_{qb} = N_s h\nu_c$ is transmitted with energy per qubit $E_{qb} = h\nu_c$. Here *h* is Planck's constant.

Consider now the extended classical-quantum system shown in **Figure 1**. Here a classical error-correcting code [*N, K*] is concatenated with a quantum error-correcting $\lfloor\lfloor N, K_{qb} \rfloor\rfloor$ code containing K_{qb} qubits per message with quantum code rate $Q_N^h = N/K_{qb}$ dimensions per qubit. The notation $\lfloor\lfloor N, K_{qb} \rfloor\rfloor$ implies an *N*-dimensional quantum code protecting K_{qb}-qubits. Since the systems in **Figure 1** contain both classical and quantum subsystems, we refer to this system as a hybrid system. In this regard, we can write the time interval-coding equation for **Figure 1**, viz.,

$$T = KT_b = (\log_2 \log_2 M)T_b = NT_N = K_{qb}T_{qb} = N_s T_p \quad (1)$$

where T_{qb} is the time per qubit and $K \le N \le K_{qb} \le N_s$. Dividng both sides of (1) by $(\log_2 M)T_p$, the bit rate can be related to the quantum code rate $R_N^q = N/K_{qb}$ dimensions per qubit, the quantum-modulator rate $Q_M^h = K_{qb}/N_s$ qubits per photon. The combined transmit code rate $R_{N_s} = R_N R_N^q Q_M^h$ is related to the channel photon rate $\mathcal{R}_p$ and bit rate $\mathcal{R}_b$, *i.e.*,

$$\mathcal{R}_b = R_{N_s}\mathcal{R}_p = R_N Q_N^h Q_M^h \mathcal{R}_p \text{bits/sec.} \quad (2)$$

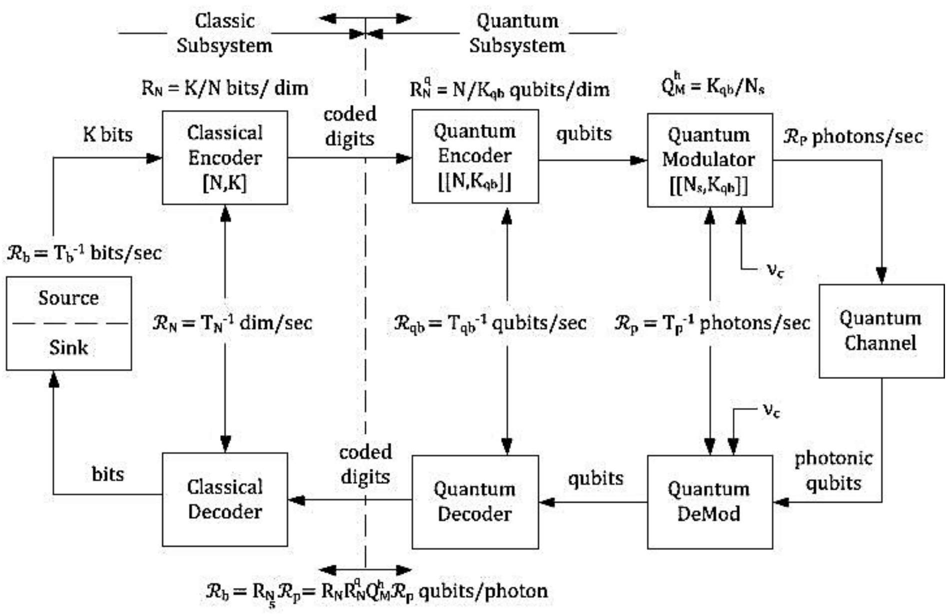

Figure 1. Quantum communication system functional architecture with concatenated classical and quantum error correcting codes.

Furthermore, (1) and (2) allow us to connect all rates to the number of classical messages

$$M = 2^K = 2^{\mathcal{R}_b T} = 2^{NR_N} = 2^{N_s R_{N_s}} = 2^{R_{N_s}\mathcal{R}_p T} \quad (3)$$

in the hybrid system transmit alphabet. The various energy packages are related to the energy per message E to the energy per photonic qubit $E_{qb} = h\nu_c$ through

$$E = (\log_2 M)E_b = NE_N = N_s E_{qb} = N_s h\nu_c \quad (4)$$

where E_b is the energy per bit and E_N is the energy per dimension. The corresponding power-energy relationship is

$$E = ST = (\log_2 M) ST_b = NST_N = K_{qb} S = N_s ST_p \quad (5)$$

where S watts is the average power per message. From (5), we note that the number of photons per bit (qubits/bit) is given by the ratio $P_b = E_b / E_{qb} = R_{N_s}^{-1}$ which may be used as a measure of the energy efficiency of a quantum communication system to transport classical information.

The time interval-coding equations, code rates, energy-power relationships and the alphabet sizes will be useful when the performance metrics of the quantized-classical systems are compared to their quantum counterparts of **Figure 1**. In particular, the quantized channel capacity C bits per sec and C_N bits per dimension (spectral efficiency in (bits/sec)/Hz) and the quantized information storage capacity C_{N_s} bits per photon are related to the average information stored in bandwidth W Hz through

$$I = CT = NC_N = N_s C_{N_s} \text{ bits / message.} \quad (6)$$

The quantum counterparts are the quantum channel capacity of Q qubits per second and Q_N qubits per dimension and the quantum information storage density Q_{N_s} qubits per sec are related to the average quantum information I_Q storage in bandwidth W Hz through

$$I_Q = QT = NQ_N = N_s Q_{N_s} \text{ qubits / message.} \quad (7)$$

With all parameters of our system models defined and connecting relationships introduced, we are in a position to present the quantized classical capacities, the quantum capacities, their connections and nature's limits regarding error-free transmission. Before doing so, we present the quantum channel model.

3. Quantum Communications Channel Model

From elementary quantum mechanics, the vibrational states of an atomic harmonic oscillator have energies that depend on frequency $E(n) = (n + 1/2)h\nu$, $n = 0,1,2\ldots$ and the probability of finding the "oscillator" in vibrational state n is

$$P(n) = \exp\exp(-n\nu/\nu_0)[1 - \exp(-\nu/\nu_0)] \quad (8)$$

where $\nu_0 = kT^\circ / h$ is the "natural frequency" of the quantum channel, k is Boltzmann's constant and T° is the background temperature in degrees Kelvin.

Thus using this notation one can show that [3]

$$\begin{aligned} E_0(\nu)/N_0 &= (\nu/2\nu_0)\coth(\nu/2\nu_0) \\ &= (Z_0/N_0)\coth(Z_0/N_0) \end{aligned} \quad (9)$$

where $N_0 = kT^\circ$ eJoules characterizes the energy level of thermal noise defined in classical systems and $Z_0 = h\nu/2$. In limit as ν/ν_0 approaches zero, $E_0(\nu) = N_0$ while limit as ν/ν_0 approaches infinity, $E_0(\nu) = Z_0$; Z_0 is Einstein's zero-point energy (ZPE) found in quantum mechanics where all thermal energy in the background light vanishes. We will use this condition to partition the electromagnetic spectrum into a classical region and a quantum region, see **Figure 2**.

In the classical region $\nu/\nu_0 \le \nu_p/\nu_0 \le 0.4$ and $\tanh x \cong x$. In this region, $E_0(\nu) \cong N_0$ and energy may be treated as a continuous variable (photon energy levels are small and infinitesimally close together) while for $\nu/\nu_0 > \nu_p/\nu_0 = 0.4$ we may consider this to be the "quantum region." For $\nu/\nu_0 \le 0.4$, we will show that all quantized capacity results reduce to Shannon's classical results [1]. **Figure 3** depicts the notion of our quantum channel of bandwidth $W = 2B$ Hz, note $\nu_p = 0.4\nu_0$. At room temperature, $T^\circ = 300^\circ K$ and $\nu_p = 2.5$ THz. Further, as T° approaches zero, ν_0 approaches zero and all thermal energy vanishes. By letting h approach

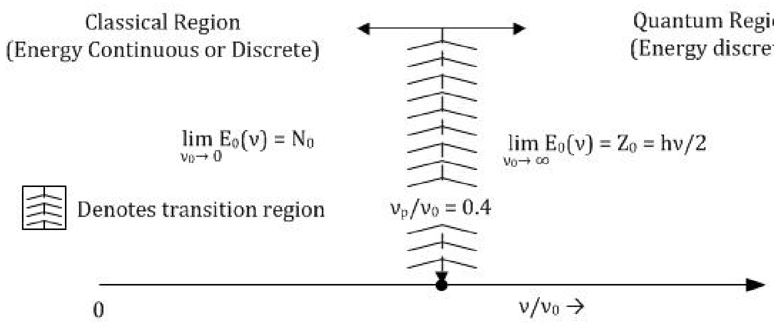

Figure 2. Partitioning the classical and quantum regions.

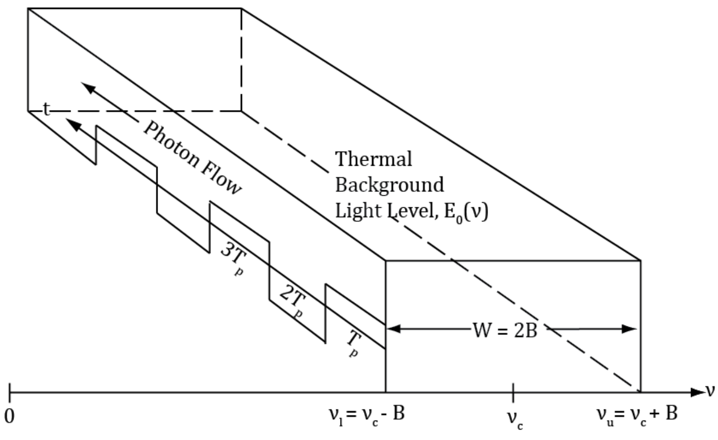

Figure 3. Quantum channel concept.

zero (or ν_0 approaches infinity), all quantum mechanical effects are eliminated and the channel model reduces to the classical white noise model.

4. Quantized Shannon Communication Channel and Information Storage Capacities

We are now in a position to develop the quantized version of Shannon's classical channel capacity for all frequencies in the electromagnetic spectrum. We will show that the quantized results reduce to the continuous energy case of Shannon in the frequency region where $\nu \le \nu_p$.

Based upon the quantum mechanical results derived in [2] and the use of Shannon's sphere packing argument [1], there exists a classical [*N, K*] code for the system, **Figure 1**, and a concatenated classical code [*N*, *K*] with a hybrid quantum error-correcting code $\lfloor\lfloor N, K_{qb} \rfloor\rfloor$ for the system of **Figure 1**, such that the message error probability (MEP) can be made arbitrarily small when the number of bits in *M* equally messages, are less than the average information storage *I*, *i.e.*,

$$\log_2 \log_2 M \le I = \left(1/W \right) \int_{\nu_l}^{\nu_u} I(\nu)\, d\nu \,\text{bits/message.} \quad (10)$$

On the other hand, for $\log_2 M > I$, then the MEP approaches one for all codes [3].

$$I(\nu) = (N/2)\log_2[1 + 4D\tanh(\nu/2\nu_0)] \quad (11)$$

where $D = N_s / N = \mathcal{R}_p / W$ and the limits ν_l and ν_u define the quantum channel band edges, see **Figure 3**.

5. Graphical Results

As we have seen, the parameter $D = N_s / N$ in (11) plays a key role in establishing values for all quantum capacities. Since $WT_N = 1$, the parameter *D* satisfies

$$D = N_s / N = \mathcal{R}_p / W = \left(WT_p\right)^{-1} = T_N / T_p$$

Thus *D* can be viewed as one of photon density per dimension or as the inverse of the photon-time bandwidth product. The condition $\nu_p \le 0.4\nu_0$ serves to partition the electromagnetic spectrum into two disjoint regions. The region $\nu \le 0.4\nu_0$ holds for classical communications (quantized or unquantized) in that quantum effects do not manifest themselves and Planck's constant is absent from all performance results. In addition, in this frequency region the photonic energy in the communication signal may be assumed continuous. For all $\nu > \nu_p$, quantum effects in the background light begin to manifest themselves.

Figures 4 and **5** demonstrate quantum communications capacity-bandwidth tradeoffs versus E_{qb} / N_0. **Figure 4** plots quantum communication storage capacity $R_{N_s} \le Q_{N_s}$ in qubits/photon versus energy per qubit to noise ratio for various photon time duration-bandwidth product BT_p. **Figure 5** plots quantum communications capacity $R_N \le Q_N$ in qubits per dimension versus qubit energy-to-noise ratio for various values of BT_p.

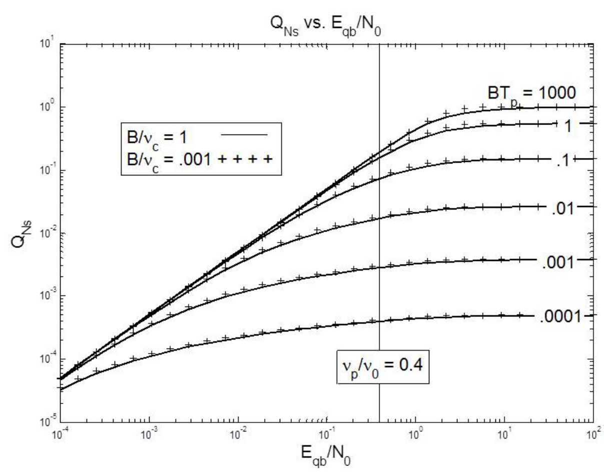

Figure 4. Quantum communications storage capacity-bandwidth (qubits/photon) tradeoff versus energy-per-qubit to thermal noise ratio

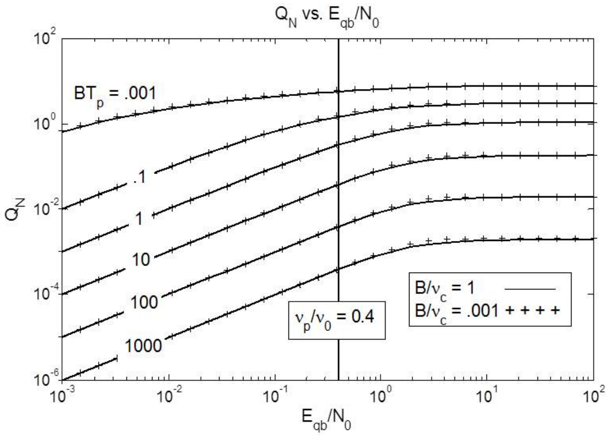

Figure 5. Quantum communications capacity Q_N in qubits per dimension versus E_{qb}/N_0 for various photon-time bandwidth products, BT_p.

From these curves we see that performance is, for all practical purposes, insensitivity to the normalized bandwidth parameter B/ν_c. **Figure 6** plots $P_b^q = 1/Q_{N_s}$ which is the minimum number of photons per qubit to achieve quantum communications capacity Q_{N_s}. From **Figures 4** and **6** we observe the limit of one photon per qubit is theoretically achievable.

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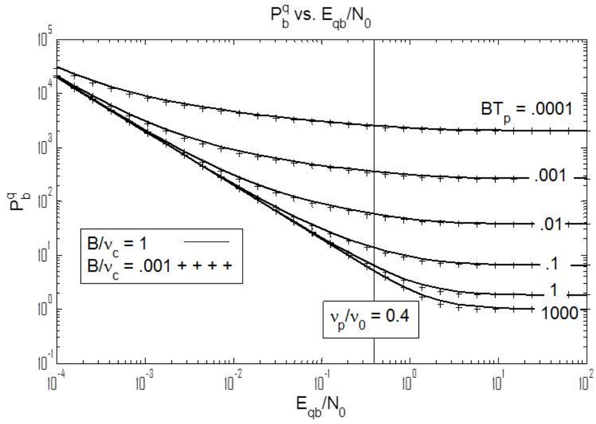

Figure 6. Minimum number of photons per qubit required to achieve capacity Q_{N_s}; ($P_b^q = 1/Q_{N_s}$ photons/qubit).

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