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On Marginal Distributions under Progressive Type II Censoring: Similarity/Dissimilarity Properties

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Abstract

Currently, progressive censoring is intensively investigated by several researchers due to its ability to remove subjects from the experiment before the final termination point, thus saving time and cost. The closed form of marginal density of failure times under progressive type II censoring is essential to study the properties of statistical analysis under different censoring schemes. In this paper, we provide a different presentation of the marginal distribution under progressive type-II censoring and we derive closed forms for different special cases. In order to study the similarity/dissimilarity of marginal densities of order statistics for failure times, the overlap measure is used. We discovered that the overlap measure depends only on the effective size m. A numerical example based on a real life data regarding failure times of aircrafts' windshields is provided to quantify the amount of redundant information provided by the order statistics of the failure times under different progressive type-II schemes based on the overlap measure. Moreover, this data set is used as a pilot study to estimate the effective size m needed for future studies.

1. Introduction

Customer satisfaction has been a main interest for manufacturers to produce reliable products. For their products to remain desired and thus profitable, they are motivated to develop high quality and long life products. This requires having knowledge about

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censoring. If $R_1 = R_2 = \cdots = R_m = 0$, then $m = n$ which represents the complete data set. For a comprehensive recent review of progressive censoring, readers may refer to Balakrishnan & Cramer [3].

In general, the order statistics that is produced by *PTII* censoring provides more information about the underlying distribution than simple random samples (*SRS*) since their densities span over the whole range of the underlying distribution (see Figure 1).

However, different censoring schemes may provide different amount of information in *PTII* censoring due to progressively selected out different sets of $(R_1, R_2, \cdots, R_m)$ units at random. To study these properties of order statistics, it is necessary to derive the marginal distributions of the *rth* failure time and use a similarity/dissimilarity measures such as overlapping measures.

The overlapping measure (*OVL*) is a powerful tool to find the similarities/dissimilarities between any two densities. In terms of marginal densities of order statistics, the overlap can provide a good indication of the dissimilarities between two densities of two *PTII* censored failure times. This will enable us to check the amount of information that *PTII* censoring provides about the underlying distributions and its parameters.

Overlap measures are defined as the common areas under two probability density functions and have been used as measures of agreement of two income distributions and as a proportion of machines or electronic devices that have similar range of failure times. The *OVL* is used in many useful applications including, clinical trials (see Mizuno *et al.*, [4]), and in a comparison of income distributed by race (Weitzman, [5]).
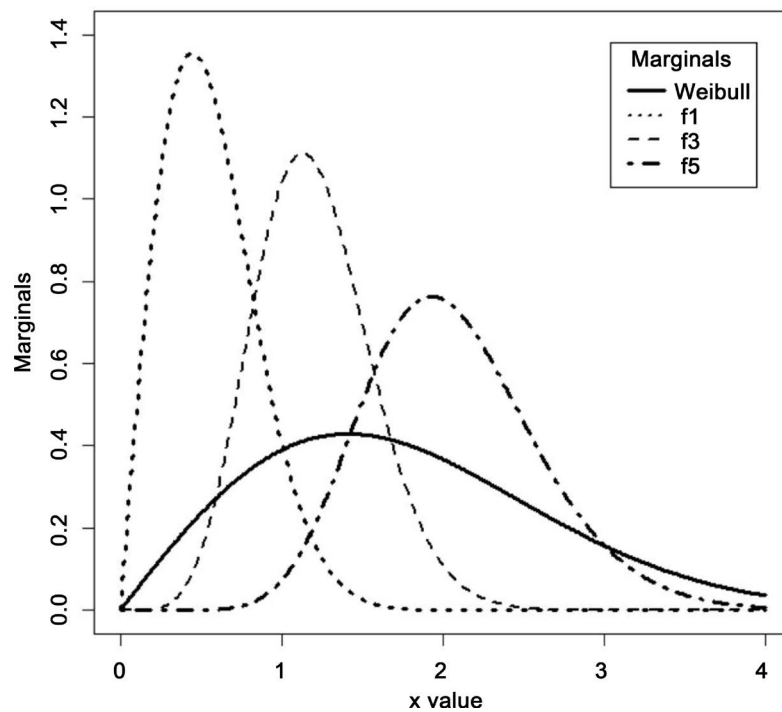

Figure 1. Overlap among densities.

The *OVL* measure (Δ) was originally introduced by Weitzman [5]. One application of Δ, was given by Ichikawa [6], who used Δ to estimate the lowest upper bound of the failure in the stress-strength model in reliability analysis. Federer *et al.*, [7] used Δ to estimate the proportion of genetic deviations in segregating populations. Moreover, Sneath [8] used Δ as a measure of clusters distinctions. Additional references of such methodology applications in ecology and other fields can be found in Mulekar and Mishra ([9] and [10]).

In this paper, we provide another presentation of the general form of the marginal density for the *rth* failure time based on *PTII* censoring which is more convenient to be used for deriving special cases of *PTII* censoring schemes such as the scheme where $(n-m)$ items are censored at the time of the first failure, the scheme where $(n-m)$ items are removed at the *mth* failure, and the equi-balanced scheme. In addition, the *OVL* coefficient is used to discriminate between two marginal densities based on *PTII* censoring. The rest of this paper is organized as follows: in Section 2, we investigate another form of the marginal density of *PTII* censoring and derive some special cases. Similarity properties of marginal densities based on *PTII* censoring using *OVL* measures are presented in Section 3. A numerical as well as a real life data examples are presented in Section 4 for illustration. Final remarks and conclusions are provided in Section 5.

2. On the Marginal Distributions Based on *PTII*

Under the *PTII* censoring for life-testing, suppose that $X_{1:m:n} < X_{2:m:n} < \cdots < X_{m:m:n}$ are the lifetimes of the completely observed units to fail, and that $\mathbf{R} = (R_1, R_2, \cdots, R_m)$ represents the numbers of units withdrawn at these failure times. If the failure times are based on an absolutely continuous distribution function F with probability density function f, the joint probability density function of the progressive censored failure times $X_{1:m:n} < \cdots < X_{m:m:n}$ (see Balakrishnan & Aggarawala [2]) is given by:

$$f_{X_{1:m:n}, \cdots, X_{m:m:n}} (x_1, \cdots, x_m) = c \prod_{i=1}^{m} f(x_i) \left[1 - F(x_i) \right]^{R_i}, \quad -\infty < x_1 < \cdots < x_m < \infty \quad (1)$$

where, $n = m + \sum_{i=1}^{m} R_i$, $m, n \in \mathbb{N}$, $R_i \in \mathbb{N}_0$, $1 \le i \le m$, $\mathbf{R} = (R_1, R_2, \cdots, R_m)$, and

$$c = n(n-1-R_1)(n-2-R_1-R_2) \cdots \left(n - \sum_{i=1}^{m} (R_i + 1) \right).$$

From the representation of the joint density function, it is obvious that progressive censoring can be embedded in the models of generalized order statistics and of sequential order statistics (Kamps [11] [12]). Moreover, Balakrishnan and Cramer [3], showed that the marginal density for the *rth* progressive type II censored order statistics from an absolutely continuous cumulative distribution function (*cdf*) *F* with probability density function (*pdf*) *f* is given by:

$$f_{X_{r:m:n}}(x) = f(x)\left(\prod_{i=1}^{r}\gamma_i\right)\sum_{j=1}^{r}a_{j,r}\left[1-F(x)\right]^{\gamma_j-1} ; x \in \mathbb{R} \qquad (2)$$

where,

$$\gamma_k = \sum_{j=k}^{m}(R_j+1) \quad \text{and} \quad a_{j,r} = \prod_{\substack{k=1\\k\neq j}}^{r}\frac{1}{\gamma_k-\gamma_j}. \quad \text{Note that} \quad \left(\prod_{i=1}^{r}\gamma_i\right)\sum_{j=1}^{r}\frac{a_{j,r}}{\gamma_j} = 1.$$

In this paper, we introduce another representation of the marginal density in (2) that can be derived from the joint density in (1) using repetitive integrals as follows:

$$f_{X_{r:m:n}}(x) = c\int_{-\infty}^{x_r}\int_{-\infty}^{x_{r-1}}\cdots\int_{-\infty}^{x_2}\int_{x_r}^{\infty}\cdots\int_{x_{m-1}}^{\infty}\prod_{i\neq r}^{m}f(x_i)\left[1-F(x_i)\right]^{R_i}\,\mathrm{d}x_m\mathrm{d}x_{m-1}\cdots\mathrm{d}x_{r+1}\mathrm{d}x_1\cdots\mathrm{d}x_{r-1}.$$

Hence,

$$f_{X_{r:m:n}}(x) = \frac{\prod_{i=1}^{r}\gamma_i f(x)\left[1-F(x)\right]^{\gamma_r-1}}{\prod_{i=r+1}^{m}\gamma_i}\times\sum_{h=1}^{r-1}\left[\frac{(-1)^{h+1}\left(1-\left[1-F(x)\right]^{\gamma_{r-h}-\gamma_r}\right)}{\prod_{i=1}^{r-h-1}\left(\gamma_{r-i-h}-\gamma_{r-h}\right)\prod_{L=1}^{h}\left(\gamma_{r-h-1}-\gamma_{r-l+1}\right)}\right]; \quad (3)$$

where, $\gamma_k = \sum_{j=k}^{m}(R_j+1)$; $r = 1,2,\cdots,m$, and $n = m + \sum_{i=1}^{m}R_i$, $m,n \in \mathbb{N}$.

The closed form in (3) is easier to use in a mathematical software in order to derive a closed form for some well known special cases. In addition, those marginal densities are important to investigate the properties needed for the statistical inferences under different censoring schemes. Using the closed form in (3) we can provide some special cases.

Special Cases

Using the new representation in (3), it is convenient to derive the following special cases.

Case 1: Ordinary order statistics (*OS*).

When $m = n$ and $R_1 = R_2 = \cdots = R_m = 0$, which represents the complete data set of order statistics, Equation (3) can be written as:

$$f_{X_{r:m:n}}(x) = m\binom{m-1}{r-1}f(x)F(x)^{r-1}\left[1-F(x)\right]^{m-r} ; x \in \mathbb{R} \qquad (4)$$

and hence,

$$\frac{1}{m}\sum_{r=1}^{m}f_{X_{r:m:n}}(x) = f(x).$$

Case 2: Equi-balanced censoring scheme.

Suppose $R_1 = R_2 = \cdots = R_m = R$, then the censoring plan with equal removal number R is called equi-balanced censoring scheme and it can be shown that:

$$f_{X_{r:m:n}}(x) = m(R+1)\binom{m-1}{r-1}f(x)\left[1-F(x)\right]^{(m-r+1)(R+1)-1}\left[1-\left[1-F(x)\right]^{(R+1)}\right]^{r-1}. \quad (5)$$

Note that, Equation (5) is simply the marginal density of the *rth* order

statistics of m observations from the cdf $G(x) = [1 - F(x)]^{(R+1)}$. Moreover,

$$\frac{1}{m}\sum_{r=1}^{m} f_{X_{r:m:n}}(x) = (R+1) f(x)(1 - F(x))^{R} = f_{1:R+1}(x), \tag{6}$$

represents the pdf of the minimum order statistics from $(R+1)$ observations.

Case 3: Type II censoring scheme.

$R_1 = R_2 = \cdots = R_{m-1} = 0$, and $R_m = n - m$ corresponds to type-II censoring, then Equation (3) can be simplified as:

$$f_{X_{r:m:n}}(x) = n \binom{n-1}{r-1} f(x) F(x)^{r-1} [1 - F(x)]^{n-r}; x \in \mathbb{R} \tag{7}$$

and hence

$$\frac{1}{m}\sum_{r=1}^{m} f_{X_{r:m:n}}(x) = f(x) \left(\frac{\sum_{r=1}^{m} n \binom{n-1}{r-1} F(x)^{r-1} [1 - F(x)]^{n-r}}{m} \right). \tag{8}$$

Note that Equation (7) is basically the pdf of the rth order statistic from a sample of size n.

To study the similarity/dissimilarity of marginal distributions of the order statistics for failure times, the OVL (Δ) measure is derived and numerated for different $PTII$ schemes to quantify the amount of information provided by the order statistics of the failure times under different schemes.

3. Similarity Properties of Marginal *pdfs* Based on *PTII* Censoring

Investigating the similarities/disemilarities among densities of order statistics based on $PTII$ is important for investigators in order to select the less costly censoring scheme with higher amount of information that this scheme provides about the underlying distribution and its parameters.

Suppose two samples of observations are drawn from two continuous distributions $f_1(x)$ and $f_2(x)$ then Weitzman's Δ is given in the following equation

$$\Delta(f_1, f_2) = \int_{-\infty}^{\infty} \min(f_1(x), f_2(x)) \mathrm{d}x.$$

The overlap measure Δ can be applied to discrete distributions by replacing the integrals with summations as well as multivariate distributions. Moreover, Δ is measured on a scale of 0 to 1; Δ value close to 0 indicates extreme dissimilarities between the two density functions and $\Delta = 1$ indicates exact similarities.

3.1. Similarity Structure between Two Consecutive Statistics from *PTII* Censoring

Using (2 or 3), the OVL between the densities f_r and f_{r+1} of two consecutive

order statistics is given by:

$$\Delta(f_r, f_{r+1}) = \int \min(f_r(x), f_{r+1}(x)) \, \mathrm{d}x,$$

where,

$$\min(f_r(x), f_{r+1}(x)) = \begin{cases} f_r(x) & \text{if } x > M_{r,r+1} \\ f_{r+1}(x) & \text{if } x \leq M_{r,r+1} \end{cases} \tag{9}$$

Thus,

$$\Delta(f_{X_{r:m:n}}, f_{X_{r+1:m:n}}) = \int_{-\infty}^{M_{r,r+1}} f_{X_{r+1:m:n}}(x) \, \mathrm{d}x + \int_{M_{r,r+1}}^{\infty} f_{X_{r:m:n}}(x) \, \mathrm{d}x. \tag{10}$$

With some algebraic manipulations using Equation (2) or (3) we can get the following results:

$$M_{r,r+1} = F^{-1}(1 - p_{r,r+1}); \quad p_{r,r+1} = {}^{\gamma_r - \gamma_{r+1}}\sqrt{\frac{\gamma_{r+1}}{\gamma_r}}. \tag{11}$$

Notice that Δ is free of the underlying distribution.

3.2. Special Cases

Case 1: Ordinary order statistics (*OS*).

Using Al-Saleh [13], when $m = n$ and $R_1 = R_2 = \cdots = R_m = 0$, then for any $r < s$, the overlapping coefficient Δ between $f_{X_{r:m:n}}$ and $f_{X_{s:m:n}}$ is given by

$$\Delta(f_r, f_s) = 1 - P(r \leq Y_{r,s} \leq s - 1) \tag{12}$$

where, $Y_{r,s} \sim Binomial(m, p_{r,s})$ with $p_{r,s} = \dfrac{\sqrt[s-r]{u_r}}{\sqrt[s-r]{u_r} + \sqrt[s-r]{u_s}}$ and $u_r = m\binom{m-1}{r-1}$, $u_s = m\binom{m-1}{s-1}$.

Case 2: Equi-balanced censoring scheme.

Suppose $R_1 = R_2 = \cdots = R_m = R$, and by using Ghahramani [14], the *OVL* measure is given by:

$$\Delta(f_r, f_s) = 1 - P(m - s + 1 \leq Y_{r,s} \leq m - r) \tag{13}$$

where, $Y_{r,s} \sim Binomial(m, p_{r,s})$ with $p_{r,s} = \dfrac{\sqrt[s-r]{v_s}}{\sqrt[s-r]{v_s} + \sqrt[s-r]{v_r}}$ and $v_r = \binom{m-1}{r-1}$, $v_s = \binom{m-1}{s-1}$.

Case 3: Type II censoring scheme.

Similarly, when $R_1 = R_2 = \cdots = R_{m-1} = 0$, and $R_m = n - m$, and applying Ghahramani [14] result, then the *OVL* measure is given by:

$$\Delta(f_r, f_s) = 1 - P(r \leq Y_{r,s} \leq s - 1) \tag{14}$$

where, $Y_{r,s} \sim Binomial(n, p_{r,s})$ with $p_{r,s} = \dfrac{\sqrt[s-r]{w_r}}{\sqrt[s-r]{w_r} + \sqrt[s-r]{w_s}}$ and $w_r = \binom{m-1}{r-1}$,

$$w_s = \begin{pmatrix} m-1 \\ s-1 \end{pmatrix}.$$

4. Illustrations Based on Simulated and Real Life Data Examples

In this section, and in order to quantify the amount of information provided by OVL for different $PTII$ censoring schemes given in Sections 3.1 & 3.2, we provide a numerical as well as a real life data examples based on failure times of aircrafts' windshields.

Example 1: The OVL for consecutive order statistics for different schemes using the general definition in Section (3.1).

Table 1 shows that the discrimination measured using $(1-\Delta)$ is higher in the schemes where $(n-m)$ items are removed at the time of the first failure, namely schemes 3, 7 and 11, compared to the remaining schemes. Moreover, the discriminations that are based on schemes 4 and 12 are close in values to ordinary ordered statistics (OS). In addition, when $R = (1,1,1,1,1)$, OS and scheme 8 have identical values.

The Δ for OS increases as the actual sample size, n, increases. Moreover, while increasing the ratio $\dfrac{m}{n}$ has no effect on the cases when censoring occurs at the time of the last failure (see schemes 2, 6 and 10), it has great effects on the remaining cases (schemes 3, 4, 7, 8, 11 and 12) where Δ decreases as the ratio $\dfrac{m}{n}$ increases.

Example 2: (Real life data)

The data set for this application is given by Blischke and Murthy [15], and

Table 1. Overlapping between two consecutive order statistics from $PTII$ censoring.

n	m	scheme	$R = (R_1, \cdots, R_m)$	$\Delta(f_r, f_s)$					
				$\Delta(f_1, f_2)$	$\Delta(f_2, f_3)$	$\Delta(f_3, f_4)$	$\Delta(f_4, f_5)$	$\Delta(f_5, f_6)$	$\Delta(f_6, f_7)$
4	4	1	$R = (0,0,0,0)$	0.578	0.704	0.750	-	-	-
10	4	2	$R = (0,0,0,6)$	0.613	0.783	0.908	-	-	-
		3	$R = (6,0,0,0)$	0.403	0.594	0.673	-	-	-
		4	$R = (2,1,2,1)$	0.565	0.707	0.712	-	-	-
5	5	5	$R = (0,0,0,0)$	0.590	0.736	0.835	0.844	-	-
10	5	6	$R = (0,0,0,5)$	0.613	0.783	0.908	0.965	-	-
		7	$R = (5,0,0,0)$	0.457	0.645	0.777	0.802	-	-
		8	$R = (1,1,1,1,1)$	0.590	0.736	0.835	0.844	-	-
7	7	9	$R = (0,0,0,0,0,0,0)$	0.603	0.766	0.885	0.942	0.962	0.945
10	7	10	$R = (0,0,0,0,0,0,3)$	0.613	0.783	0.908	0.965	0.987	0.994
		11	$R = (3,0,0,0,0,0,0)$	0.535	0.716	0.859	0.929	0.953	0.937
		12	$R = (0,1,0,1,0,1,0)$	0.613	0.760	0.887	0.937	0.960	0.911

later used by Musleh and Helu [16]. The data represent the failure times of aircrafts' windshields. The windshields consist of several layers of materials to withstand extreme temperatures and pressure. In order to maintain a regular performance of aircrafts, data on windshields are routinely collected and analyzed. The unit of measurement is 1000 h.

The *OVL* coefficient for the three special cases in Sec (3.2) using Equations 12 - 14 when $n = 70$ and $m = 6$ are presented in **Table 2** which shows that $\Delta(f_r, f_s)$ values are identical for case 1 - case 3, which means that censoring schemes have no influence on the discriminations among the *pdfs* of the order statistics. Moreover, if $r = i$ & $s = m - i + 1$; then we can easily see that as i $\left(i \leq \left[\dfrac{m}{2} \right] \right)$ increases $\Delta(f_i, f_{m-i+1})$ increases and $\Delta(f_i, f_{m-i+1})$ approaches zero as $m \to \infty$. In addition, the minimum value of $\Delta(f_r, f_s)$ is when $r = 1$ and $s = m$. Moreover, we can also express the similarity/disimilarity between the two extremes using $\Delta(f_1, f_m) = \left(\dfrac{1}{2} \right)^{m-1}$.

Since the value of Δ is a function of m only, this enables us to estimate the effective size m for any future studies using a pilot study. For Example, we can use the data in **Table 3** as a pilot study to create two clusters based on their

Table 2. Overlapping coefficients for pairs of order statistics from *PTII* censoring based on the windshield data.

(r,s)	case 1	case 2	case 3
$(1,2)$	0.5981	0.5981	0.5981
$(2,3)$	0.6708	0.6708	0.6708
$(3,4)$	0.6875	0.6875	0.6875
$(4,5)$	0.6708	0.6708	0.6708
$(5,6)$	0.5981	0.5981	0.5981
$(1,3)$	0.3467	0.3467	0.3467
$(1,4)$	0.1863	0.1863	0.1863
$(1,5)$	0.0876	0.0876	0.0876
$(1,6)$	0.0312	0.0312	0.0312

Table 3. The complete failure times of aircraft windshields.

0.301	0.309	0.557	0.943	1.070	1.124	1.248	1.281	1.281	1.303
1.432	1.480	1.505	1.506	1.568	1.615	1.619	1.652	1.652	1.757
1.795	1.866	1.876	1.899	1.911	1.912	1.914	1.981	2.010	2.038
2.085	2.089	2.097	2.135	2.154	2.190	2.194	2.223	2.224	2.229
2.300	2.324	2.349	2.385	2.481	2.610	2.625	2.632	2.646	2.661
2.688	2.823	2.890	2.902	2.934	2.962	2.964	3.000	3.103	3.114
3.117	3.166	3.344	3.376	3.385	3.443	3.467	3.478	3.578	3.595

Table 4. Progressive censored samples for the failure times of aircraft windshields.

Low Quality									
0.301	0.309	0.557	0.943	1.070	1.124	1.248	1.281	1.281	1.303
1.432	1.480	1.505	1.506	1.568	1.615	1.619	1.652	1.652	1.757
1.795	1.866	1.876	1.899	1.911	1.912	1.914	1.981	2.010	2.038
2.085	2.089	2.097							
High Quality									
2.135	2.154	2.19	2.194	2.223	2.224	2.229	2.300	2.324	2.349
2.385	2.481	2.610	2.625	2.632	2.646	2.661	2.688	2.823	2.890
2.902	2.934	2.962	2.964	3.000	3.103	3.114	3.117	3.166	3.344
3.376	3.385	3.443	3.467	3.478	3.578	3.595			

failure times: one for low quality windshields and one for high quality windshields. The new data sets are presented in Table 4.

The fit of a Weibull model for the two data sets is checked using Kolmogrov-Smirnov (KS) test, Anderson-Darling (AD) and chi-square tests. When we fit the Weibull distribution for "Low Quality" data set based on maximum likelihood estimates $\alpha_1 = 1.7947$ and $\beta = 4.6$, we observe that $KS = 0.1292$ with corresponding $p_{\text{value}} = 0.59505$, $AD = 1.40361$ and chi-square distance $= 0.69511$ with a corresponding $p_{\text{value}} = 0.87435$. Similarly when we fit the Weibull distribution for "High Quality" data set based on maximum likelihood estimates $\alpha_1 = 2.975$ and $\beta = 4.6$, we observed that $KS = 0.19536$ with $p_{\text{value}} = 0.10333$, $AD = 1.3587$ and chi-square distance $= 0.69294$ with a corresponding $p_{\text{value}} = 0.87486$. The results above indicate that Weibull model provides a good fit for the two data sets. The estimated $\hat{\Delta}$ is calculated and found to be 0.298774. Equating this value to $\left(\dfrac{1}{2}\right)^{\hat{m}-1}$, we obtain $\hat{m} \approx 3$ as an estimate of the effective size for our future study.

Moreover, we create Figure 2 to show the overlapping among densities of the order statistics $[f_r; r = 1, 2, \cdots, m; m = 6]$. Clearly, it shows that the smallest redundancy of information occurs between the densities of the extreme order statistics $(1, 6)$. In addition, it shows that densities $f_1 - f_6$ span over the whole range of the original density. In our example we choose the Weibull distribution but it can be any other distribution since $\Delta(f_r, f_s)$ is free of parameters.

5. Final Remarks and Conclusions

In the past few years, progressive censoring has received a great attention by many researchers. This is due to its advantages in reducing the cost and time of the life testing. Moreover, the availability of high speed computing resources enhances the focus on progressive censoring. In this article, we introduced a new form of the marginal distributions of the order statistics under $PTII$ censoring. In addition, we used these new forms to derive the three special cases, namely: ordinary order statistics, equi-balanced and type II censoring schemes. We derived a closed form of the OVL coefficient for any two order statistics based on $PTII$ censoring using the presented marginal distributions in Sec. 3.2.
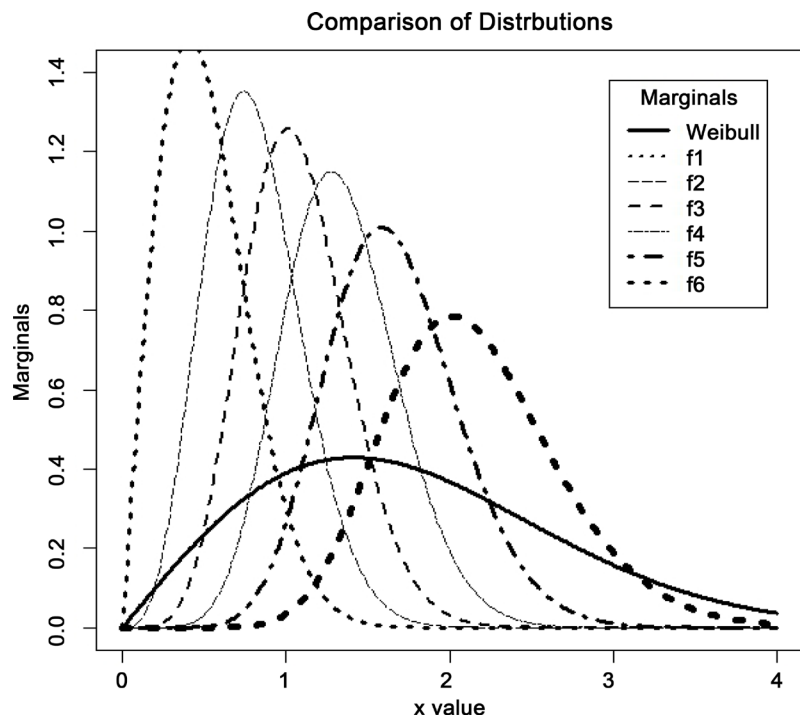
Comparison of Distrbutions

Figure 2. Overlap among densities.

Moreover, we found that the *OVL* coefficient was independent of the parent distribution and depended only on the effective size "*m*" which enabled us to estimate the effective size *m* for any future studies instead of randomly picked *m*.

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