Open Journal of Statistics, 2016, 6, 381-386
Published Online June 2016 in SciRes. http://www.scirp.org/journal/ojs
http://dx.doi.org/10.4236/ojs.2016.63033

Decomposition of Point-Symmetry Using Ordinal Quasi Point-Symmetry for Ordinal Multi-Way Tables

Yusuke Saigusa, Kouji Tahata, Sadao Tomizawa

Department of Information Sciences, Tokyo University of Science, Chiba, Japan
Email: saigusaysk@gmail.com, kouji_tahata@is.noda.tus.ac.jp, tomizawa@is.noda.tus.ac.jp

Received 12 April 2016; accepted 5 June 2016; published 8 June 2016

Abstract

For multi-way tables with ordered categories, the present paper gives a decomposition of the point-symmetry model into the ordinal quasi point-symmetry and equality of point-symmetric marginal moments. The ordinal quasi point-symmetry model indicates asymmetry for cell probabilities with respect to the center point in the table.

Keywords

Decomposition, Multi-Way Table, Ordinal Quasi Point-Symmetry, Point-Symmetry

1. Introduction

Consider an $R_1 \times R_2 \times \cdots \times R_T$ table with ordered categories. Let $i = (i_1, \cdots, i_T)$ for $i_k = 1, \cdots, R_k$ and $k = 1, \cdots, T$, and let p_i denote the probability that an observation will fall in *i*th cell of the table. Let X_k denote the *k*th variable of the table for $k = 1, \cdots, T$. Denote the *h*th-order ($h = 1, \cdots, T-1$) marginal probability $P\left(X_{k_1} = i_{k_1}, \cdots, X_{k_h} = i_{k_h}\right)$ by $p^{(k_1, \cdots, k_h)}_{i_{k_1}, \cdots, i_{k_h}}$ with $1 \le k_1 < \cdots < k_h \le T$.

In the case of $R_1 = \cdots = R_T \; (= R)$, the symmetry ($\mathrm{S}^T$) model is defined by

$$p_i = \psi_i \quad \text{for any } i,$$

where $\psi_i = \psi_j$ for any permutation $j = (j_1, \cdots, j_T)$ of *i* (Bhapkar and Darroch, [1]; Agresti, [2], p. 439). We may also refer to this model as the permutation-symmetry model.

The *h*th-order marginal symmetry (MS^T_h) model is defined by, for a fixed *h* ($h = 1, \cdots, T-1$),

How to cite this paper: Saigusa, Y., Tahata, K. and Tomizawa, S. (2016) Decomposition of Point-Symmetry Using Ordinal Quasi Point-Symmetry for Ordinal Multi-Way Tables. *Open Journal of Statistics*, **6**, 381-386.
http://dx.doi.org/10.4236/ojs.2016.63033

$$p_{i_1,\cdots,i_h}^{(s_1,\cdots,s_h)} = p_{j_1,\cdots,j_h}^{(s_1,\cdots,s_h)} = p_{i_1,\cdots,i_h}^{(t_1,\cdots,t_h)} \qquad \text{for any } (i_1,\cdots,i_h),$$

where $(j_1,\cdots,j_h)$ is any permutation of $(i_1,\cdots,i_h)$, and for any $(s_1,\cdots,s_h)$ and $(t_1,\cdots,t_h)$ (Bhapkar and Darroch, [1]). The *h*th-order quasi symmetry (QS_h^T) model is defined by, for a fixed *h* ($h=1,\cdots,T-1$),

$$p_i = \mu\left(\prod_{k=1}^{T}\alpha_k(i_k)\right)\left(\prod\prod_{1\le k_1<k_2\le T}\alpha_{k_1k_2}(i_{k_1}i_{k_2})\right)\cdots\left(\prod\cdots\prod_{1\le k_1<\cdots<k_h\le T}\alpha_{k_1,\cdots,k_h}(i_{k_1},\cdots,i_{k_h})\right)\psi_i \qquad \text{for any } i,$$

where $\psi_i=\psi_j$ for any permutation *j* of *i* (Bhapkar and Darroch, [1]). Bhapkar and Darroch [1] gave the theorem that:

1) *For the* R^T *table and a fixed h* ($h=1,\cdots,T-1$), *the* S^T *model holds if and only if both the* QS_h^T *and* MS_h^T *models hold.*

Tahata, Yamamoto and Tomizawa [3] considered the *h*th-linear ordinal quasi symmetry (LQS_h^T) model, which was defined by, for a fixed *h* ($h=1,\cdots,T-1$),

$$p_i = \mu\left(\prod_{k=1}^{T}\alpha_k^{i_k}\right)\left(\prod\prod_{1\le k_1<k_2\le T}\alpha_{k_1k_2}^{i_{k_1}i_{k_2}}\right)\cdots\left(\prod\cdots\prod_{1\le k_1<\cdots<k_h\le T}\alpha_{k_1,\cdots,k_h}^{i_{k_1}\cdots i_{k_h}}\right)\psi_i \quad \text{for any } i,$$

where $\psi_i=\psi_j$ for any permutation *j* of *i*. This model is a special case of the QS_h^T model. The LQS_h^T model is the ordinal quasi symmetry model when $h=1$ (Agresti, [4], p. 244). Tahata *et al*. [3] also considered the *h*th-order marginal moment equality (MME_h^T) model, which was expressed as, for a fixed *h* ($h=1,\cdots,T-1$),

$$\mu_{k_1,\cdots,k_l} = \mu_{1,\ldots,l} \quad (l=1,\cdots,h),$$

where $\mu_{k_1,\cdots,k_l}=E\left(X_{k_1}\cdots X_{k_l}\right)$ for $1\le k_1<\cdots<k_l\le T$. Tahata *et al*. [3] obtained the theorem that:

2) *For the* R^T *table and a fixed h* ($h=1,\cdots,T-1$), *the* S^T *model holds if and only if both the* LQS_h^T *and* MME_h^T *models hold.*

Various decompositions of the symmetry model are given by several statisticians, e.g. Caussinus [5], Bishop, Fienberg and Holland ([6], Ch.8), Read [7], Kateri and Papaioannou [8], and Tahata and Tomizawa [9].

For the $R_1\times R_2\times\cdots\times R_T$ table, the point-symmetry (P^T) model is defined by

$$p_i=\gamma_i \quad \text{for any } i,$$

where $\gamma_i=\gamma_{i^*}$ and $i^*=\left(i_1^*,\cdots,i_T^*\right)$ with $i_k^*=R_k+1-i_k$ for $k=1,\cdots,T$ (Wall and Lienert, [10]; Tomizawa, [11]). This model indicates the point-symmetry of cell probabilities with respect to the center point of multi-way table.

For the R^T table, Tahata and Tomizawa [12] considered the *h*th-order marginal point-symmetry (MP_h^T) model defined by, for a fixed *h* ($h=1,\cdots,T-1$),

$$p_{i_{k_1},\cdots,i_{k_h}}^{(k_1,\cdots,k_h)} = p_{i_{k_1}^*,\cdots,i_{k_h}^*}^{(k_1,\cdots,k_h)} \quad \left(1\le k_1<\cdots<k_h\le T; i_l=1,\cdots,R_l; l=k_1,\cdots,k_h\right).$$

Tahata and Tomizawa [12] also considered the *h*th-order quasi point-symmetry (QP_h^T) model defined by, for a fixed *h* ($h=1,\cdots,T-1$),

$$p_i = \mu\left(\prod_{k=1}^{T}\alpha_k(i_k)\right)\left(\prod\prod_{1\le k_1<k_2\le T}\alpha_{k_1k_2}(i_{k_1}i_{k_2})\right)\cdots\left(\prod\cdots\prod_{1\le k_1<\cdots<k_h\le T}\alpha_{k_1,\cdots,k_h}(i_{k_1},\cdots,i_{k_h})\right)\gamma_i \quad \text{for any } i,$$

where $\gamma_i=\gamma_{i^*}$. Tahata and Tomizawa [12] gave the theorem that:

3) *For the* R^T *table and a fixed h* ($h=1,\cdots,T-1$), *the* P^T *model holds if and only if both the* QP_h^T *and* MP_h^T *models hold.*

Theorem 3) is Theorem 1) with structures in terms of permutation-symmetry, *i.e.* the S^T, QS_h^T and MS_h^T models, replaced by structures in terms of point-symmetry, *i.e.* the P^T, QP_h^T and MP_h^T models. However, a theorem in terms of point-symmetry corresponding to Theorem 2) is not obtained yet. So we are now interested in the decomposition of the P^T model.

In the present paper, Section 2 proposes three models. Section 3 gives a new decomposition of the P^T model. Section 4 provides the concluding remarks.

2. Models

Let $S=\left\{h \mid h=2m-1, m=1,\cdots,\left\lfloor \frac{T}{2} \right\rfloor\right\}$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to *x*. Consider the model defined by, for a fixed odd number *h* ($h\in S$),

$$\mu_{k_1k_2,\cdots,k_l}=\mu^*_{k_1k_2,\cdots,k_l} \quad \left(1\le k_1<k_2<\cdots<k_l\le T; l=1,3,\cdots,h\right),$$

where

$$\mu_{k_1k_2,\cdots,k_l}=E\left(X_{k_1}X_{k_2}\cdots X_{k_l}\right), \quad \mu^*_{k_1k_2,\cdots,k_l}=E\left(X^*_{k_1}X^*_{k_2}\cdots X^*_{k_l}\right),$$

and $X^*_k=R_k+1-X_k$ for $k=1,\cdots,T$. We shall refer to this model as the *h*th-order marginal moment point-symmetry (MMP_h^T) model. Note that if the MP_h^T model holds then the MMP_h^T model holds. Under the MMP_1^T model, we see, for any *k* ($k=1,\cdots,T$),

$$\mu_k=\frac{R_k+1}{2}.$$

Then we obtain, for any k_1 and k_2 ($1\le k_1<k_2\le T$),

$$\begin{aligned}\mu_{k_1k_2}-\mu^*_{k_1k_2}&=\sum_{i_{k_1}=1}^{R_{k_1}}\sum_{i_{k_2}=1}^{R_{k_2}}\left(i_{k_1}i_{k_2}-i^*_{k_1}i^*_{k_2}\right)p^{(k_1,k_2)}_{i_{k_1}i_{k_2}}\\&=-\left(R_{k_1}+1\right)\left(R_{k_2}+1\right)+\left(R_{k_1}+1\right)\mu_{k_2}+\left(R_{k_2}+1\right)\mu_{k_1}=0.\end{aligned}$$

Under the MMP_3^T model, we see, for any k_1, k_2 and k_3 ($1\le k_1<k_2<k_3\le T$),

$$\mu_{k_1k_2k_3}=-\frac{1}{2}\left(\frac{1}{2}\left(R_{k_1}+1\right)\left(R_{k_2}+1\right)\left(R_{k_3}+1\right)-\left(R_{k_1}+1\right)\mu_{k_2k_3}-\left(R_{k_2}+1\right)\mu_{k_1k_3}-\left(R_{k_3}+1\right)\mu_{k_1k_2}\right).$$

Then we obtain, for any k_1, k_2, k_3 and k_4 ($1\le k_1<k_2<k_3<k_4\le T$),

$$\begin{aligned}\mu_{k_1k_2k_3k_4}-\mu^*_{k_1k_2k_3k_4}&=\sum_{i_{k_1}=1}^{R_{k_1}}\sum_{i_{k_2}=1}^{R_{k_2}}\sum_{i_{k_3}=1}^{R_{k_3}}\sum_{i_{k_4}=1}^{R_{k_4}}\left(i_{k_1}i_{k_2}i_{k_3}i_{k_4}-i^*_{k_1}i^*_{k_2}i^*_{k_3}i^*_{k_4}\right)p^{(k_1,k_2,k_3,k_4)}_{i_{k_1}i_{k_2}i_{k_3}i_{k_4}}\\&=\left(R_{k_1}+1\right)\left(R_{k_2}+1\right)\left(R_{k_3}+1\right)\left(R_{k_4}+1\right)-\left(R_{k_1}+1\right)\left(R_{k_2}+1\right)\mu_{k_3k_4}\\&\quad-\left(R_{k_1}+1\right)\left(R_{k_3}+1\right)\mu_{k_2k_4}-\left(R_{k_1}+1\right)\left(R_{k_4}+1\right)\mu_{k_2k_3}-\left(R_{k_2}+1\right)\left(R_{k_3}+1\right)\mu_{k_1k_4}\\&\quad-\left(R_{k_2}+1\right)\left(R_{k_4}+1\right)\mu_{k_1k_3}-\left(R_{k_3}+1\right)\left(R_{k_4}+1\right)\mu_{k_1k_2}+\left(R_{k_1}+1\right)\mu_{k_2k_3k_4}\\&\quad+\left(R_{k_2}+1\right)\mu_{k_1k_3k_4}+\left(R_{k_3}+1\right)\mu_{k_1k_2k_4}+\left(R_{k_4}+1\right)\mu_{k_1k_2k_3}\\&=0.\end{aligned}$$

Thus we are not interested in the MMP_h^T model with *h* being even. Therefore we shall consider the MMP_h^T model with *h* being odd.

Consider the model defined by

$$p_i=\mu\left(\prod_{k=1}^{T}\alpha_k^{i_k}\right)\gamma_i \quad \text{for any } i,$$

where $\gamma_i=\gamma_{i^*}$. We shall refer to this model as the ordinal quasi point-symmetry (OQP^T) model. In the case of $T=2$, this model is identical to the model proposed by Tahata and Tomizawa [13]. The special case of the OQP^T model obtained by putting $\alpha_1=\cdots=\alpha_T=1$ is the P^T model. Also the OQP^T model is the special case of the QP_1^T model obtained by putting $\left\{\alpha_k\left(i_k\right)=\alpha_k^{i_k}\right\}$. The OQP^T model may be expressed as

$$\log\frac{p_i}{p_{i^*}}=\beta_0+\sum_{k=1}^{T}i_k\beta_k \quad \text{for any } i,$$

with $\beta_0=-\sum_k\left(R_k+1\right)\log\alpha_k$ and $\beta_k=\log\alpha_k^2$. From this equation, we can see the log-odds that an observation falls in ith cell instead of in the point-symmetric i^*th cell, *i.e.* $\log\left(p_i/p_{i^*}\right)$, is described as a linear combination with intercept β_0 and slope β_k for the category indicator i_k under the OQP^T model. Thus the parameter β_k can be interpreted as the effect of a unit increase in the kth variable on the log-odds.

Consider the model being more general than the OQP^T model as follows, for a fixed odd number h ($h \in S$),

$$p_i = \mu\left(\prod_{k=1}^{T}\alpha_k^{i_k}\right)\left(\prod\prod\prod_{1\le k_1<k_2<k_3\le T}\alpha_{k_1k_2k_3}^{i_{k_1}i_{k_2}i_{k_3}}\right)\cdots\left(\prod_{1\le k_1<\cdots<k_h\le T}\cdots\prod\alpha_{k_1,\cdots,k_h}^{i_{k_1}\cdots i_{k_h}}\right)\gamma_i \quad \text{for any } i,$$

where $\gamma_i = \gamma_{i^*}$. We shall refer to this model as the hth-linear ordinal quasi point-symmetry (LQP_h^T) model. Especially, when $h=1$, the LQP_h^T model is identical to the OQP^T model. Also the LQP_h^T model is the special case of the QP_h^T model obtained by putting $\left\{\alpha_k\left(i_k\right)=\alpha_k^{i_k}\right\}, \left\{\alpha_{k_1k_2k_3}\left(i_{k_1}i_{k_2}i_{k_3}\right)=\alpha_{k_1k_2k_3}^{i_{k_1}i_{k_2}i_{k_3}}\right\},\cdots,\left\{\alpha_{k_1,\cdots,k_h}\left(i_{k_1},\cdots,i_{k_h}\right)=\alpha_{k_1,\cdots,k_h}^{i_{k_1}\cdots i_{k_h}}\right\}$, and $\left\{\alpha_{k_1k_2}\left(i_{k_1}i_{k_2}\right)=1\right\}$, $\left\{\alpha_{k_1k_2k_3k_4}\left(i_{k_1}i_{k_2}i_{k_3}i_{k_4}\right)=1\right\},\cdots,\left\{\alpha_{k_1,\cdots,k_{h-1}}\left(i_{k_1},\cdots,i_{k_{h-1}}\right)=1\right\}$.

Figure 1 shows the relationships among models.

3. Decomposition of Point-Symmetry

We obtain the following theorem:

Theorem 1. *For the* $R_1\times R_2\times\cdots\times R_T$ *table and a fixed odd number h* ($h\in S$)*, the* P^T *model holds if and only if both the* LQP_h^T *and* MMP_h^T *models hold.*

Proof. If the P^T model holds, then both the LQP_h^T and MMP_h^T models hold. Assuming that both the LQP_h^T and MMP_h^T models hold, then we shall show the P^T model holds. Let $q=\{q_i\}$ denote cell probabilities which satisfy both the LQP_h^T and MMP_h^T models. The LQP_h^T model is expressed as

$$\log q_i = \log\mu\gamma_i + \sum_{k=1}^{T} i_k\log\alpha_k + \sum\sum\sum_{1\le k_1<k_2<k_3\le T} i_{k_1}i_{k_2}i_{k_3}\log\alpha_{k_1k_2k_3} + \cdots + \sum_{1\le k_1<\cdots<k_h\le T}\cdots\sum i_{k_1}\cdots i_{k_h}\log\alpha_{k_1,\cdots,k_h},$$

where $\gamma_i=\gamma_{i^*}$. Let

$$c=\sum_{i_1=1}^{R_1}\cdots\sum_{i_T=1}^{R_T}\gamma_i, \quad \pi_i=\frac{\gamma_i}{c}.$$

Note that $\pi=\{\pi_i\}$ satisfy $0<\pi_i<1$, $\sum_{i_1}\cdots\sum_{i_T}\pi_i=1$ and $\pi_i=\pi_{i^*}$. Then the LQP_h^T model is also expressed as

$$\begin{aligned}\log\left(\frac{q_i}{\pi_i}\right) &= \log\mu c + \sum_{k=1}^{T} i_k\log\alpha_k + \sum\sum\sum_{1\le k_1<k_2<k_3\le T} i_{k_1}i_{k_2}i_{k_3}\log\alpha_{k_1k_2k_3} \\ &\quad + \cdots + \sum_{1\le k_1<\cdots<k_h\le T}\cdots\sum i_{k_1}\cdots i_{k_h}\log\alpha_{k_1,\cdots,k_h}.\end{aligned} \tag{1}$$

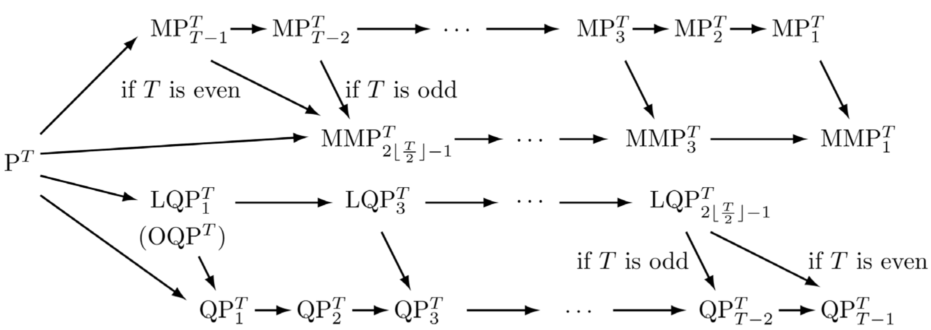

Figure 1. Relationships among various models. Note: "$M_1 \to M_2$" indicates that model M_1 implies model M_2.

The MMP_h^T model is expressed as

$$\mu^{q}_{k_1k_2,\cdots,k_l} = \mu^{q*}_{k_1k_2,\cdots,k_l} \quad \left(1 \le k_1 < k_2 < \cdots < k_l \le T; l = 1,3,\cdots,h\right), \tag{2}$$

where

$$\mu^{q}_{k_1k_2,\cdots,k_l} = \sum_{i_{k_1}=1}^{R_{k_1}} \cdots \sum_{i_{k_l}=1}^{R_{k_l}} i_{k_1} \cdots i_{k_l} q^{(k_1,\cdots,k_l)}_{i_{k_1},\cdots,i_{k_l}}, \qquad \mu^{q*}_{k_1k_2,\cdots,k_l} = \sum_{i_{k_1}=1}^{R_{k_1}} \cdots \sum_{i_{k_l}=1}^{R_{k_l}} i^*_{k_1} \cdots i^*_{k_l} q^{(k_1,\cdots,k_l)}_{i_{k_1},\cdots,i_{k_l}}.$$

Then we denote $\mu^{q}_{k_1k_2,\cdots,k_l}$ $(= \mu^{q*}_{k_1k_2,\cdots,k_l})$ by $\mu^{0}_{k_1k_2,\cdots,k_l}$.

Consider arbitrary cell probabilities $p = \{p_i\}$ which satisfy the MMP_h^T model and

$$\mu^{p}_{k_1k_2,\cdots,k_l} = \mu^{p*}_{k_1k_2,\cdots,k_l} = \mu^{0}_{k_1k_2,\cdots,k_l} \quad \left(1 \le k_1 < k_2 < \cdots < k_l \le T; l = 1,3,\cdots,h\right), \tag{3}$$

where

$$\mu^{p}_{k_1k_2,\cdots,k_l} = \sum_{i_{k_1}=1}^{R_{k_1}} \cdots \sum_{i_{k_l}=1}^{R_{k_l}} i_{k_1} \cdots i_{k_l} p^{(k_1,\cdots,k_l)}_{i_{k_1},\cdots,i_{k_l}}, \qquad \mu^{p*}_{k_1k_2,\cdots,k_l} = \sum_{i_{k_1}=1}^{R_{k_1}} \cdots \sum_{i_{k_l}=1}^{R_{k_l}} i^*_{k_1} \cdots i^*_{k_l} p^{(k_1,\cdots,k_l)}_{i_{k_1},\cdots,i_{k_l}}.$$

From (1), (2) and (3),

$$\sum_{i_1=1}^{R_1} \cdots \sum_{i_T=1}^{R_T} \left(p_i - q_i\right) \log\left(\frac{q_i}{\pi_i}\right) = 0. \tag{4}$$

Let $K(\cdot;\cdot)$ denote the Kullback-Leibler information, e.g., it between q and π is

$$K(q;\pi) = \sum_{i_1=1}^{R_1} \cdots \sum_{i_T=1}^{R_T} q_i \log\left(\frac{q_i}{\pi_i}\right).$$

From (4),

$$K(p;\pi) = K(p;q) + K(q;\pi).$$

Thus, for fixed π,

$$K(q;\pi) = \min_p K(p;\pi),$$

and then q uniquely minimize $K(p;\pi)$ (see Darroch and Ratcliff, [14]).

Let $q^* = \{q_{i^*}\}$. Then, in a similar way as described above, we obtain

$$K(q^*;\pi) = \min_p K(p;\pi),$$

and then q^* uniquely minimize $K(p;\pi)$, hence $q = q^*$. Namely q satisfy the P^T model. The proof is completed.

For the analysis of data, the test of goodness-of-fit of the LQP_h^T model is achieved based on, e.g., the likelihood ratio chi-square statistic which has a chi-square distribution with the number of degrees of freedom

$$\begin{cases} \dfrac{1}{2}\left(\displaystyle\prod_{k=1}^{T} R_k - 1\right) - \displaystyle\sum_{i=1}^{\frac{h+1}{2}} \binom{T}{2i-1} & \left(R_k : \text{odd for } k = 1,\cdots,T\right), \\ \dfrac{1}{2}\displaystyle\prod_{k=1}^{T} R_k - \displaystyle\sum_{i=1}^{\frac{h+1}{2}} \binom{T}{2i-1} & (\text{otherwise}). \end{cases}$$

Also the number of degrees of freedom for the MMP_h^T model is

$$\sum_{i=1}^{\frac{h+1}{2}} \binom{T}{2i-1}.$$

We point out that, for a fixed *h*, the number of degrees of freedom for the P^T model is equal to sum of those for the LQP_h^T and MMP_h^T models.

4. Concluding Remarks

For multi-way contingency tables, we have proposed the MMP_h^T, OQP^T and LQP_h^T models. Under the OQP^T model, the log-odds that an observation falls in a cell instead of in its point-symmetric cell is described as a linear combination of category indicators. For a fixed odd number *h* ($h \in S$), the LQP_h^T model implies the QP_h^T model.

We have gave the theorem that the P^T model holds if and only if both the LQP_h^T and MMP_h^T models. For the analysis of data, the decomposition given in the present paper may be useful for determining the reason when the P^T model fits data poorly.

Acknowledgements

The authors thank the editor and the referees for their helpful comments.

References

[1] Bhapkar, V.P. and Darroch, J.N. (1990) Marginal Symmetry and Quasi Symmetry of General Order. *Journal of Multivariate Analysis*, **34**, 173-184. http://dx.doi.org/10.1016/0047-259X(90)90034-F

[2] Agresti, A. (2013) Categorical Data Analysis. 3rd Edition, Wiley, Hoboken.

[3] Tahata, K., Yamamoto, H. and Tomizawa, S. (2011) Linear Ordinal Quasi-Symmetry Model and Decomposition of Symmetry for Multi-Way Tables. *Mathematical Methods of Statistics*, **20**, 158-164. http://dx.doi.org/10.3103/S1066530711020050

[4] Agresti, A. (2010) Analysis of Ordinal Categorical Data. 2nd Edition, Wiley, Hoboken. http://dx.doi.org/10.1002/9780470594001

[5] Caussinus, H. (1965) Contribution à l'analyse statistique des tableaux de corrélation. *Annales de la Faculté des Sciences de l'Université de Toulouse*, **29**, 77-182. http://dx.doi.org/10.5802/afst.519

[6] Bishop, Y.M.M., Fienberg, S.E. and Holland, P.W. (1975) Discrete Multivariate Analysis: Theory and Practice. MIT Press, Cambridge.

[7] Read, C.B. (1977) Partitioning Chi-Square in Contingency Tables: A Teaching Approach. *Communications in Statistics, Theory and Methods*, **6**, 553-562. http://dx.doi.org/10.1080/03610927708827513

[8] Kateri, M. and Papaioannou, T. (1997) Asymmetry Models for Contingency Tables. *Journal of the American Statistical Association*, **92**, 1124-1131. http://dx.doi.org/10.1080/01621459.1997.10474068

[9] Tahata, K. and Tomizawa, S. (2014) Symmetry and Asymmetry Models and Decompositions of Models for Contingency Tables. *SUT Journal of Mathematics*, **50**, 131-165.

[10] Wall, K. and Lienert, G.A. (1976) A Test for Point-Symmetry in J-Dimensional Contingency-Cubes. *Biometrical Journal*, **18**, 259-264.

[11] Tomizawa, S. (1985) The Decompositions for Point Symmetry Models in Two-Way Contingency Tables. *Biometrical Journal*, **27**, 895-905. http://dx.doi.org/10.1002/bimj.4710270811

[12] Tahata, K. and Tomizawa, S. (2008) Orthogonal Decomposition of Point-Symmetry for Multiway Tables. *Advances in Statistical Analysis*, **92**, 255-269. http://dx.doi.org/10.1007/s10182-008-0070-5

[13] Tahata, K. and Tomizawa, S. (2015) Ordinal Quasi Point-Symmetry and Decomposition of Point-Symmetry for Cross-Classifications. *Journal of Statistics: Advances in Theory and Applications*, **14**, 181-194.

[14] Darroch, J.N. and Ratcliff, D. (1972) Generalized Iterative Scaling for Log-Linear Models. *Annals of Mathematical Statistics*, **43**, 1470-1480. http://dx.doi.org/10.1214/aoms/1177692379