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Best Equivariant Estimator of Extreme Quantiles in the Multivariate Lomax Distribution

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Abstract

The minimum risk equivariant estimator of a quantile of the common marginal distribution in a multivariate Lomax distribution with unknown location and scale parameters under Linex loss function is considered.

Keywords

Best Affine Equivariant Estimator, Quantile Estimation, Lomax (Pareto II) Distributions, Linex Loss Function

1. Introduction

In the analysis of income data, lifetime contexts, and business failure data the univariate Lomax (Pareto II) distribution with density $\frac{r}{\sigma}\left(1+\frac{x-\mu}{\sigma}\right)^{-1-r}$; $x>\mu$, is a useful model [1]. The lifetime of a decreasing failure rate component may be describe by this distribution. It has been recommended by [2] as a heavy tailed alternative to the exponential distribution. The interested reader can see [3] and [4] for more details.

A multivariate generalization of the Lomax distribution has been proposed by [5] and studied by [6]. It may be obtained as a gamma mixture of independent exponential random variables in the following way. Consider a system of n components. It is then reasonable to suppose that the common operating environment shared by all components induces some kind of correlation among them. If for a given environment τ, the component lifetimes $X_1, X_2, \cdots, X_n$ are independently exponentially distributed $E\left(\mu, \sigma/\tau\right)$ with density $\frac{\tau}{\sigma}\exp\left\{-\frac{\tau}{\sigma}\left(x-\mu\right)\right\}$, $x>\mu$, and the changing nature of the environment is accounted by a distribution function

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F(.), then the unconditional joint density of $X_1, X_2, \cdots, X_n$ is

$$f_0\left(x_1, x_2, \cdots, x_n; \mu, \sigma\right) = \int_0^\infty \frac{\tau^n}{\sigma^n} \exp\left\{-\frac{\tau}{\sigma}\sum_{i=1}^n \left(x_i - \mu\right)\right\} I_{(\mu,\infty)}\left(x_{(1)}\right) \mathrm{d}F\left(\tau\right). \tag{1}$$

where $x_{(1)} = \min\left\{x_1, x_2, \cdots, x_n\right\}$. Furthermore, if $F(\cdot)$ is a gamma distribution $G(r,1)$ with density $\frac{1}{\Gamma(r)}\tau^{r-1}\mathrm{e}^{-\tau}; r > 0$, then (1) become

$$f_1\left(x_1, x_2, \cdots, x_n; \mu, \sigma\right) = \frac{\Gamma(n+r)}{\Gamma(r)\sigma^n} \frac{1}{\left(1 + \frac{1}{\sigma}\sum_{i=1}^n \left(x_i - \mu\right)\right)} I_{(\mu,\infty)}\left(x_{(1)}\right). \tag{2}$$

This is called multivariate Lomax $ML_n(r, \mu, \sigma)$ with location parameter μ and scale parameter σ. The same distribution is referred to as Mardia's multivariate Pareto II distribution, see [3] and [7]. If take $\mu = 0$ and assign a different scale parameter, σ_i to each X_i we have

$$f_2\left(x_1, x_2, \cdots, x_n; \sigma_i\right) = \frac{\Gamma(n+r)}{\Gamma(r)\prod_{i=1}^n \sigma_i} \cdot \frac{1}{\left(1 + \sum_{i=1}^n \frac{x_i}{\sigma_i}\right)^{n+r}} I_{(\mu,\infty)}\left(x_{(1)}\right). \tag{3}$$

For more information about the work on this distribution, the reader can see [8].

2. Best Affine Equivarient Estimator

Let $X_1, X_2, \cdots, X_n; n \ge 2$ are from a multivariate Lomax distribution $ML_n(r, \mu, \sigma)$ with unknown μ and σ and known *r*. We consider the linear function $\theta = \mu + k\sigma$ for given $k \ge 0$. When $k = p^{-1/r} - 1;\ 0 < p < 1$, θ is the 100(1 − *p*) th quantile of the marginal distribution of X_i. Quantile estimation is of interest in reliability theory and lifetesting. [9] generalized results in [10] to a strictly Convex loss.

In this paper we consider the Linex loss function

$$L(\theta, \delta) = \mathrm{e}^{a\left(\frac{\delta - \theta}{\sigma}\right)} - a\left(\frac{\delta - \theta}{\sigma}\right) - 1 \tag{4}$$

where $a \ne 0$ is the shape parameter, which was introduced by [11] and was extensively used by [12].

The minimal sufficient statistic in the model (2) is (*S*, *X*) where, $S = \sum_{i=1}^n \left(X_i - X_{(1)}\right)$ and $X = X_{(1)}$. Conditional on τ, a random variable with $G(r,1)$ distribution, *S* and *X* are independent with

$$S \mid \tau \sim G\left(n-1, \frac{\sigma}{\tau}\right), \quad X \mid \tau \sim E\left(\mu, \frac{\sigma}{n\tau}\right). \tag{5}$$

So, the density of (*S*, *X*) is

$$\begin{aligned} f\left(s, x; \mu, \sigma\right) &= \int_0^\infty \frac{1}{(n-2)!} \frac{\tau^{n-1}}{\sigma^{n-1}} s^{n-2} \mathrm{e}^{-\frac{\tau s}{\sigma}} \frac{n\tau}{\sigma} \mathrm{e}^{\frac{n\tau}{\sigma}(x-\mu)} \frac{1}{\Gamma(r)} \tau^{r-1} \mathrm{e}^{-\tau} \mathrm{d}t \\ &= \frac{n\Gamma(n+r)}{(n-2)!\Gamma(r)\sigma^n} \cdot \frac{s^{n-2}}{\left[1 + \frac{1}{\sigma}\left\{s + n(x-\mu)\right\}\right]^{n+r}}; \quad x > \mu, s > 0 \end{aligned} \tag{6}$$

The problem of estimating $\theta = \mu + k\sigma$; $k \ge 0$ under the loss (4) is invariant under the affine group of transformations $(S, X) \to (cS, cX + b)$ and the equivariant estimator have the form $\delta = X + cS$ where c is a real constant.

Following [13], we study scale equivariant estimators of the form $\delta = \phi(Z)S$, where $Z = \frac{X}{S}$ and $\phi(.)$ is a measurable function. Thus the equivariant estimator is of the form $\phi(Z)S$, where $\phi(Z)=Z+c$. Now, consider the risk of the estimator $X+cS$ for estimating $\mu+k\sigma$ when the loss is (4).

$$\begin{aligned}R(\theta,\delta)&=E\left\{\mathrm{e}^{a\left(\frac{X+cS-\mu-K\sigma}{\sigma}\right)}-a\left(\frac{X+cS-\mu-K\sigma}{\sigma}\right)-1\right\}\\&=\mathrm{e}^{-a\frac{\mu+K\sigma}{\sigma}}E\left[\mathrm{e}^{a\left(\frac{X+cS}{\sigma}\right)}\right]-\frac{a}{\sigma}E(X)-\frac{ac}{\sigma}E(S)+\frac{a\mu}{\sigma}+ak-1\\&=\mathrm{e}^{-\frac{a\mu}{\sigma}-ak}E\left\{E\left[\mathrm{e}^{\frac{a}{\sigma}(X+cS)}\middle|\tau\right]\right\}-\frac{a}{\sigma}E\left\{E\left(X|\tau\right)\right\}-\frac{ac}{\sigma}E\left[E\left(S|\tau\right)\right]+\frac{a\mu}{\sigma}+ak-1.\end{aligned}\tag{7}$$

Now, since $\tau\sim G(r,1)$ and $X|\tau\sim E\left(\mu,\frac{\sigma}{n\tau}\right)$ and $S|\tau\sim G\left(n-1,\frac{\sigma}{\tau}\right)$ we have

$$R(\theta,\delta)=n\mathrm{e}^{-ak}E_\tau\left\{\frac{\tau^n}{(n\tau-a)(\tau-ac)^{n-1}}\right\}-\frac{a}{\sigma}\left\{\frac{nr\mu}{\sigma}+1\right\}-\frac{acr(n-1)}{\sigma}+\frac{a\mu}{\sigma}+ak-1\tag{8}$$

which is finite if $r>ac$. By the invariant property of the problem we can take $(\mu,\sigma)=(0,1)$ and the risk becomes

$$R((0,1),\delta)=n\mathrm{e}^{-ak}E_\tau\left\{\frac{\tau^n}{(n\tau-a)(\tau-ac)^{n-1}}\right\}-a-acr(n-1)+ak-1\tag{9}$$

Differentiate the risk with respect to c and equating to zero, the minimizing c must satisfies the following equation

$$E_\tau\left\{\frac{\tau^n}{(n\tau-a)(\tau-ac_0)^n}\right\}=r,\tag{10}$$

Yielding the best affine equivariant estimator $\delta_{\text{equivariant}}=\delta_0=\phi_0(Z)S$, where

$$\phi_0(Z)=Z+c_0.$$

3. Improved Estimator

For improving upon δ_0, we study scale equivariant estimator $\delta=\phi(Z)S$. The risk of δ depends on (μ,σ) through $\frac{\mu}{\sigma}$, so without loss of generality one can take $\sigma=1$ and write

$$R(\delta,\mu)=E_\mu\left\{E_\mu\left[L(\phi(Z)S,\theta)\middle|Z=z\right]\right\}.\tag{11}$$

The minimization of $R(\delta,\mu)$ leads to the following equation

$$E_\mu\left[S\,\mathrm{e}^{acS}\middle|Z=z\right]=\mathrm{e}^{-a(\mu+k)}E_\mu\left[S|Z=z\right].\tag{12}$$

let $z>0$, then the conditional density of S given $Z=z>0$ is proportional to

$$\frac{S^{n-1}}{\left(1+S(1+nz)-n\mu\right)^{n+r}};\ S>\max\left\{0,\frac{\mu}{z}\right\}.\tag{13}$$

Consider now $\mu\le 0$ and fix $z>0$, then setting

$$q(S;\mu)=\frac{S^n}{\left(1+S(1+nz)-n\mu\right)^{n+r}}. \tag{14}$$

From (12) we compute the following expectations as follows

$$E_\mu\left(S\,|\,Z=z\right)=\int_0^\infty q(s;\mu)\,\mathrm{d}s=\frac{1}{(1+nz)^{n+1}(1-n\mu)^{r-1}}\int_0^1 u^n(1-u)^{r-2}\,\mathrm{d}u$$

and

$$E_\mu\left(S\,\mathrm{e}^{acS}\,|\,Z=z\right)=\int_0^\infty \mathrm{e}^{acS}q(s;\mu)\mathrm{d}s=\frac{1}{(1+nz)^{n+1}(1-n\mu)^{r-1}}$$

$$\int_0^1 \mathrm{e}^{ac\frac{1-n\mu}{1+nz}\cdot\frac{u}{1-u}}u^n(1-u)^{r-2}\,\mathrm{d}u,$$

where $u=\dfrac{S(1+nz)}{1+S(1+nz)-n\mu}$. Hence (12) becomes

$$\int_0^1 \mathrm{e}^{ac\frac{1-n\mu}{1+nz}\cdot\frac{u}{1-u}}u^n(1-u)^{r-2}\,\mathrm{d}u=\mathrm{e}^{-a(\mu+k)}\frac{\Gamma(r-1)n!}{\Gamma(n+r)} \tag{15}$$

any $c=\phi(Z)$ satisfying (15) minimizes $R(\delta,\mu)=E\left[E\left(L(\delta,\theta)\,|\,Z\right)\right]$, for $\mu\le 0$ and $Z>0$. Now, let $\mu>0$ and fix again $Z>0$, then $S>\dfrac{\mu}{Z}$, $q(S,\mu)=\dfrac{S^n}{\left[1+S(1+nZ)-n\mu\right]^{n+r}}$.

So we have

$$E_\mu\left[S\,|\,Z=z\right]=\int_{\mu/z}^\infty q(S;\mu)\,\mathrm{d}s=\frac{1}{(1+nz)^{n+1}(1-n\mu)^{r-1}}$$

$$\int_{\frac{\mu(1+nz)}{z+\mu(1+nz)-n\mu z}}^1 u^n(1-u)^{r-2}\,\mathrm{d}u$$

and

$$E_\mu[S\,\mathrm{e}^{acS}\,|\,Z=z]=\int_{\mu/z}^\infty \mathrm{e}^{acS}q(S;\mu)\,\mathrm{d}s=\frac{1}{(1+nz)^{n+1}(1-n\mu)^{r-1}}$$

$$\int_{\frac{\mu(1+nz)}{z+\mu(1+nz)-n\mu z}}^1 \mathrm{e}^{acS}u^n(1-u)^{r-2}\,\mathrm{d}u$$

and hence (7) becomes

$$\int_{\frac{\mu(1+nz)}{z+\mu(1+nz)-n\mu z}}^1 u^n(1-u)^{r-2}\,\mathrm{d}u=\int_{\frac{\mu(1+nz)}{z+\mu(1+nz)-n\mu z}}^1 \mathrm{e}^{acS}u^n(1-u)^{r-2}\mathrm{d}u \tag{16}$$

any $c=c(\mu)$ satisfying (16) minimizes $R(\delta,\mu)=E\left[E\left[L(\delta,\theta)\,|\,Z\right]\right]$ for $\mu>0$ and $Z>0$ [14]. Now for deriving an improved equivariant estimator upon this we must find a bound for c in formula (15) and (16). As we can not derive c from Equations (15) and (16) explicitely, this would not be achieved.

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