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Improvement of the Preliminary Test Estimator When Stochastic Restrictions are Available in Linear Regression Model

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ABSTRACT

Ridge type estimators are used to estimate regression parameters in a multiple linear regression model when multicolinearity exists among predictor variables. When different estimators are available, preliminary test estimation procedure is adopted to select a suitable estimator. In this paper, two ridge estimators, the Stochastic Restricted Liu Estimator and Liu Estimator are combined to define a new preliminary test estimator, namely the Preliminary Test Stochastic Restricted Liu Estimator (PTSRLE). The stochastic properties of the proposed estimator are derived, and the performance of PTSRLE is compared with SRLE in the sense of mean square error matrix (MSEM) and scalar mean square error (SMSE) for the two cases in which the stochastic restrictions are correct and not correct. Moreover the SMSE of PTSRLE based on Wald (WA), Likelihood Ratio (LR) and Lagrangian Multiplier (LM) tests are derived, and the performance of PTSRLE is compared using WA, LR and LM tests as a function of the shrinkage parameter d with respect to the SMSE. Finally a numerical example is given to illustrate some of the theoretical findings.

Keywords: Preliminary Test Estimator; Mean Square Error Matrix; Scalar Mean Square Error; Stochastic Restricted Liu Estimator; Liu Estimator; Wald Test; Likelihood Ratio Test; Lagrangian Multiplier Test

1. Introduction

A common problem in a multiple linear regression model is a multicolllinearity. Some biased estimators are proposed to solve this problem such as the Ordinary Ridge Estimator (ORE) by Hoerl and Kennard [1], the Restricted Ridge Estimator (RRE) by Sarkar [2], the Liu Estimator (LE) by Liu [3], the Restricted Liu Estimator (RLE) by Kaçiranlar, *et al*. [4] and the Stochastic Restricted Liu Estimator (SRLE) by Hubert and Wijekoon [5]. When different estimators are available the preliminary test estimation procedure is adopted to select a suitable estimator. The preliminary test approach was first proposed by Bancroft [6] and then has been studied by many researchers, such as Judge and Bock [7], Wijekoon and Trenkler [8] and Saleh and Kibria [9]. Later Kibria and Saleh [10] have discussed the performance of preliminary test ridge estimators based on WA [11], the LR [12] and the LM [13] tests. Then Yang and Xu [14] have introduced the preliminary test Liu estimators based on these three tests by combining the Restricted Liu Estimator (RLE) and the Liu Estimator.

In this paper, two ridge estimators, the Stochastic Restricted Liu Estimator and Liu Estimator are combined to define a new preliminary test estimator. The new PTSRLE is introduced and derives its stochastic properties in Section 2. The mean square error and scalar mean square error comparisons between PTSRLE and SRLE are carried out in Section 3. In Section 4 the SMSE of the PTSRLE based on WA, LR and LM tests are derived and the performance of the PTSRLE is compared using WA, LR and LM tests as a function of the shrinkage parameter d with respect to the Scalar Mean Square Error. Finally in Section 5, we illustrated these comparisons with a numerical example.

2. Model Specification and Stochastic Properties of the Proposed Estimator

First we consider the multiple linear regression model

$$Y = X\beta + \varepsilon,\ \varepsilon \sim N\left(0, \sigma^2 I\right), \tag{1}$$

where Y is an $n \times 1$ observable random vector, X is an $n \times p$ known design matrix of rank p, β is a $p \times 1$ vector of unknown parameters and ε is an $n \times 1$ vector of disturbances.

In addition to sample Model (1), let us be given some prior information about β in the form of a set of m independent stochastic linear restrictions as follows;

$$r = R\beta + \delta + \upsilon,\ \upsilon \sim N\left(0, \sigma^2 \Omega\right) \tag{2}$$

where r is an $m \times 1$ stochastic known vector R is a $m \times p$ of full row rank $m \le p$ with known elements, δ is non zero $m \times 1$ unknown vector and υ is an $m \times 1$ random vector of disturbances and Ω is assumed to be known and positive definite. Further it is assumed that υ is stochastically independent of ε, *i.e.*,

$$E\left(\varepsilon\upsilon'\right) = 0.$$

Let us now turn to the question of the statistical evaluation of the compatibility of sample and stochastic information. The classical procedures is to test the hypothesis

$$H_0 : \delta = 0 \quad \text{against} \quad H_1 : \delta \neq 0 \tag{3}$$

under linear Model (1) and stochastic prior information (2).

The Ordinary Least Squares Estimator (OLSE) for the Model (1) and mixed estimator [15] due to a stochastic prior restriction (2) are given by

$$\hat{\beta} = S^{-1}X'\,Y \ \text{ and } \ \hat{\beta}_m = \hat{\beta} + S^{-1}R'\left(\Omega + RS^{1}R'\right)^{-1}\left(r - R\hat{\beta}\right) \tag{4}$$

respectively, where $S = X'X$

The Ordinary Stochastic Pre Test Estimator (OSPE) of β [8] is defined as

$$\hat{\beta}_{OSPE} = \begin{cases} \hat{\beta}_m & \text{if } H_0 : \delta = 0 \\ \hat{\beta} & \text{if } H_1 : \delta \neq 0 \end{cases} \tag{5}$$

Further, we can write (5) as follows

$$\hat{\beta}_{OSPE} = \hat{\beta}_m I_{\left[0, F_{m,n-p}(\alpha)\right)}\left(F\right) + \hat{\beta} I_{\left[F_{m,n-p}(\alpha), \infty\right)}\left(F\right) \tag{6}$$

where,

$$F = \frac{\left(r - R\hat{\beta}\right)'\left(\Omega + RS^{-1}R'\right)^{-1}\left(r - R\hat{\beta}\right)}{m\hat{\sigma}^2} \tag{7}$$

which has a non-central $F_{m,n-p,\lambda}$ distribution under $H_1 : \delta \neq 0$, with non-centrality parameter

$$\lambda = \frac{\delta'\left(\Omega + RS^{-1}R'\right)^{-1}\delta}{2\sigma^2} \text{ with } \hat{\sigma}^2 = \frac{\left(Y - X\hat{\beta}\right)'\left(Y - X\hat{\beta}\right)}{n-p}, \tag{8}$$

$$I_{\left[0, F_{m,n-p}(\alpha)\right)}\left(F\right)$$

and

$$I_{\left[F_{m,n-p}(\alpha), \infty\right)}\left(F\right)$$

are indicator functions which take the value one if F falls in the subscripted interval and zero otherwise.

$$F_{m,n-p}\left(\alpha\right)$$

is the upper α-level critical value from the central F distribution

$$F_{m,n-p,0}.$$

When different estimators are available for the same parameter vector β in the linear regression model one must solve the problem of their comparison. Usually as a simultaneous measure of covariance and bias, the mean square error matrix is used, and is defined by

$$M\left(\hat{\beta}, \beta\right) = E\left[\left(\hat{\beta} - \beta\right)\left(\hat{\beta} - \beta\right)'\right] = D\left(\hat{\beta}\right) + B\left(\hat{\beta}\right)B\left(\hat{\beta}\right)' \tag{9}$$

where $D\left(\hat{\beta}\right)$ is the dispersion matrix and

$$B\left(\hat{\beta}\right) = E\left(\hat{\beta}\right) - \beta$$

denotes the bias vector. We recall that the Scalar Mean Square Error

$$SMSE\left(\hat{\beta}, \beta\right) = trace\left(MSE\left(\hat{\beta}, \beta\right)\right).$$

Now the Liu estimator

$$\hat{\beta}_{LE} = F_d\hat{\beta} \tag{10}$$

and stochastic restricted Liu estimator

$$\hat{\beta}_{srd} = F_d\hat{\beta}_m \tag{11}$$

are combined to define the new preliminary test estimator (Preliminary Test Stochastic Restricted Liu Estimator (PTSRLE)) as

$$\tilde{\beta}_{PTSRLE}\left(d\right) = \begin{cases} F_d\hat{\beta}_m & \text{if } H_0 : \delta = 0 \\ F_d\hat{\beta} & \text{if } H_1 : \delta \neq 0 \end{cases} \tag{12}$$

where,

$$F_d = \left(S + I\right)^{-1}\left(S + dI\right)$$

with $0 < d < 1$ and d is the shrinkage parameter.

Then we can write (12) as follows

$$\begin{aligned} &\tilde{\beta}_{PTSRLE}\left(d\right) \\ &= F_d\hat{\beta}_m I_{\left[0, F_{m,n-p}(\alpha)\right)}\left(F\right) + F_d\hat{\beta} I_{\left[F_{m,n-p}(\alpha), \infty\right)}\left(F\right). \\ &= F_d\hat{\beta}_{OSPE} \end{aligned} \tag{13}$$

Wijekoon [8] derived the stochastic properties of OSPE. By using those results the expectation vector, bias vector, dispersion matrix, MSEM and SMSE of $\tilde{\beta}_{PTSRLE}\left(d\right)$

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can be shown as follows

$$E\left[\tilde{\beta}_{PTSRLE}(d)\right]=F_dE\left[\hat{\beta}_{OSPE}\right]=F_d\beta+h_\lambda(2)F_dH\delta \quad (14)$$

$$B\left[\tilde{\beta}_{PTSRLE}(d)\right]=F_d\left[(d-1)(S+dI)^{-1}\beta+h_\lambda(2)H\delta\right] \quad (15)$$

$$\begin{aligned}&D\left[\tilde{\beta}_{PTSRLE}(d)\right]=F_dD\left[\hat{\beta}_{OSPE}\right]F_d'\\&=\sigma^2F_dS^{-1}F_d'-\sigma^2h_\lambda(2)F_dGF_d'\\&\quad+\left[2h_\lambda(2)-h_\lambda(4)-h_\lambda^2(2)\right]F_dH\delta\delta'H'F_d'\end{aligned} \quad (16)$$

$$\begin{aligned}&MSE\left[\tilde{\beta}_{PTSRLE}(d)\right]\\&=\sigma^2F_dS^{-1}F_d'-\sigma^2h_\lambda(2)F_dGF_d'\\&+\left[2h_\lambda(2)-h_\lambda(4)-h_\lambda^2(2)\right]F_dH\delta\delta'H'F_d'\\&+F_d\left[(d-1)(S+dI)^{-1}\beta+h_\lambda(2)H\delta\right]\\&\times\left[(d-1)(S+dI)^{-1}\beta+h_\lambda(2)H\delta\right]'F_d'\end{aligned} \quad (17)$$

and

$$\begin{aligned}&SMSE\left[\tilde{\beta}_{PTSRLE}(d)\right]\\&=\sigma^2tr\left(F_dS^{-1}F_d'\right)-\sigma^2h_\lambda(2)tr\left(F_dGF_d'\right)\\&+\left[2h_\lambda(2)-h_\lambda(4)\right]\eta'F_d'F_d\eta\\&+2(1-d)h_\lambda(2)\beta'(S+I)^{-1}F_d\eta+(1-d)^2\beta'(S+I)^{-2}\beta\end{aligned} \quad (18)$$

respectively, where,

$$G=S^{-1}R'\left(\Omega+RS^{-1}R'\right)^{-1}R'S^{-1},$$

$$H=S^{-1}R'\left(\Omega+RS^{-1}R'\right)^{-1},$$

$$\delta=E(r)-R\beta,\quad \eta=H\left[R\beta-E(r)\right]$$

and

$$h_\lambda(\ell)=\Pr\left(\frac{\chi^2_{m+\ell,\lambda}}{\chi^2_{n-p}}\le\frac{mF_{m,n-p}(\alpha)}{n-p}\right)\ \text{for}\ \ell\in\mathrm{N}.$$

Hubert and Wijekoon [5] have given the MSE and SMSE for SRLE as

$$\begin{aligned}&MSE\left(\hat{\beta}_{srd}\right)=\sigma^2F_dS^{-1}F_d'-\sigma^2F_dGF_d'\\&+F_d\left[(d-1)(S+dI)^{-1}\beta+H\delta\right]\\&\times\left[(d-1)(S+dI)^{-1}\beta+H\delta\right]'F_d'\end{aligned} \quad (19)$$

$$\begin{aligned}&SMSE\left(\hat{\beta}_{srd}\right)=\sigma^2tr\left(F_dS^{-1}F_d'\right)-\sigma^2tr\left(F_dGF_d'\right)\\&+\eta'F_d'F_d\eta+2(1-d)\beta'(S+I)^{-1}F_d\eta\\&+(1-d)^2\beta'(S+I)^{-2}\beta\end{aligned} \quad (20)$$

Now we will see some properties of $\tilde{\beta}_{PTSRLE}(d)$,

- Note that the PTSRLE reduces to the OSPE when $d=1$.
- If $\alpha=1$ then $h_\lambda(2)=h_\lambda(4)=0$ and hence the MSE matrix of $\tilde{\beta}_{PTSRLE}(d)$ reduces to

$$MSE\left[\hat{\beta}_{LE}\right]=\sigma^2F_dS^{-1}F_d'+(1-d)^2(S+I)^{-1}\beta\beta'(S+I)^{-1}$$

which is the MSE matrix of Liu estimator.

- If $\alpha=0$ then $h_\lambda(2)=h_\lambda(4)=1$ and hence the MSE matrix of $\tilde{\beta}_{PTSRLE}(d)$ reduces to

$$\begin{aligned}&MSE\left[\hat{\beta}_{srd}\right]=\sigma^2F_dS^{-1}F_d'-\sigma^2F_dGF_d'\\&+F_d\left[(d-1)(S+dI)^{-1}\beta+H\delta\right]\\&\times\left[(d-1)(S+dI)^{-1}\beta+H\delta\right]'F_d'\end{aligned}$$

which is the MSE matrix of SRLE.

- If $\lambda\to\infty$ then $h_\lambda(\ell)\to0$, and hence from (17), the MSE matrix of the PTSRLE tends towards that of the LE.

3. Performance of the Proposed Estimator

In this section, we will compare the PTSRLE with the SRLE in the sense of mean square error matrix and scalar mean square error when stochastic restrictions are correct and not correct.

Definition: (MSEM Superiority of Estimators)

Let two alternative estimators $\hat{\beta}_1$ and $\hat{\beta}_2$ of β be given. Then $\hat{\beta}_2$ is said to be superior to $\hat{\beta}_1$ with respect to the MSEM criterion if and only if

$$M\left(\hat{\beta}_1,\beta\right)-M\left(\hat{\beta}_2,\beta\right)\ge0. \quad (21)$$

3.1. Comparison between the PTSRLE and SRLE under MSE Criterion

In this subsection, we will compare the PTSRLE with SRLE under MSE criterion when the stochastic restrictions are correct and not correct.

Consider the MSE difference between the PTSRLE and SRLE,

$$\begin{aligned}&MSE\left[\tilde{\beta}_{PTSRLE}(d)\right]-MSE\left[\hat{\beta}_{srd}\right]\\&=F_d\left[D+d_1d_1'-d_2d_2'\right]F_d'\end{aligned} \quad (22)$$

where,

$$D=\sigma^2\left(1-h_\lambda(2)\right)G+\xi H\delta\delta'H'$$

$$d_1=(d-1)(S+dI)^{-1}\beta+h_\lambda(2)H\delta,$$

$$d_2=(d-1)(S+dI)^{-1}\beta+H\delta$$

and

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$$\xi = 2h_\lambda(2) - h_\lambda(4) - h_\lambda^2(2) \geq 0 .$$

3.1.1. Theorem 3.1:

1) If the stochastic restrictions are true (*i.e.*, $\delta = 0$); the SRLE is always superior to the PTSRLE in the mean squared error matrix sense.

2) Under the assumption

$$(d-1)(S+dI)^{-1}\beta \in \Re(D),$$

the SRLE is not worse than the PTSRLE if and only if:

$$\begin{aligned}
&\left\{\left[(d-1)(S+dI)^{-1}\beta + h_\lambda(2)H\delta\right]' D^- \right.\\
&\quad \left.\times\left[(d-1)(S+dI)^{-1}\beta + h_\lambda(2)H\delta\right]+1\right\}\\
&\times\left\{\left[(d-1)(S+dI)^{-1}\beta + H\delta\right]' D^- \right.\\
&\quad \left.\times\left[(d-1)(S+dI)^{-1}\beta + H\delta\right]-1\right\}\\
&\leq\left\{\left[(d-1)(S+dI)^{-1}\beta + h_\lambda(2)H\delta\right]' D^- \right.\\
&\quad \left.\times\left[(d-1)(S+dI)^{-1}\beta + H\delta\right]\right\}^2
\end{aligned}$$

in the mean square error matrix sense when the stochastic restriction are not true (*i.e.*, $\delta \neq 0$). Here $\Re(.)$ denotes the column space of the corresponding matrix.

3.1.2. Proof:

If the stochastic restrictions are correct then

$$E(r) - R\beta = \delta = 0,$$

and consequently the Equation (22) reduced to

$$MSE\left[\tilde{\beta}_{PTSRLE}(d)\right] - MSE\left[\hat{\beta}_{srd}\right] = F_d D F_d' \qquad (23)$$

Since $0 \leq h_\lambda(2) \leq 1$ the D matrix is clearly nonnegative definite.

Therefore the mean square difference in (23) is clearly nonnegative definite matrix since D is nonnegative definite matrix. Hence the SRLE is always superior to the PTSRLE in the mean square error matrix sense when $\delta = 0$.

If the stochastic restriction are not correct then

$$E(r) - R\beta = \delta \neq 0,$$

and consequently with respect to the MSE matrix criterion $\hat{\beta}_{srd}$ is superior to $\tilde{\beta}_{PTSRLE}(d)$ if and only if ($D + d_1 d_1' - d_2 d_2'$) is nonnegative definite. Since D is nonnegative definite, we can apply the lemma of [16] (see Appendix) to analyze the MSE matrix superiority of $\hat{\beta}_{srd}$ over $\tilde{\beta}_{PTSRLE}(d)$.

According to [17] (Theorem A.76, p. 514) we can derive the generalized inverse of D as

$$\begin{aligned}
D^- &= \frac{1}{\sigma^2(1-h_\lambda(2))}\\
&\times\left[G^- - \frac{\xi}{\sigma^2(1-h_\lambda(2)) + \xi\delta' H' G^- H\delta} G^- H\delta\delta' H' G^-\right]
\end{aligned} \qquad 24)$$

After some straightforward calculation we can show that

$$\delta' H' G^- H\delta = 2\sigma^2\lambda \qquad (25)$$

Using (24) and (25) we can easily prove that $DD^- H\delta = H\delta$. This implies that $H\delta \in \Re(D)$. If $(d-1)(S+dI)^{-1}\beta \in \Re(D)$ then we have $d_1 \in \Re(D)$ and $d_2 \in \Re(D)$.

To establish condition (1) in the lemma (see Appendix), we find $f_{ij} = d_i' D^- d_j$ for $i = 1,2, j = 1,2$ such that

$$\begin{aligned}
f_{11} &= \left\{\left[(d-1)(S+dI)^{-1}\beta + h_\lambda(2)H\delta\right]' D^- \right.\\
&\quad \left.\times\left[(d-1)(S+dI)^{-1}\beta + h_\lambda(2)H\delta\right]\right\},
\end{aligned}$$

$$\begin{aligned}
f_{22} &= \left\{\left[(d-1)(S+dI)^{-1}\beta + H\delta\right]' D^- \right.\\
&\quad \left.\times\left[(d-1)(S+dI)^{-1}\beta + H\delta\right]\right\}
\end{aligned}$$

and,

$$\begin{aligned}
f_{12} &= \left\{\left[(d-1)(S+dI)^{-1}\beta + h_\lambda(2)H\delta\right]' D^- \right.\\
&\quad \left.\times\left[(d-1)(S+dI)^{-1}\beta + H\delta\right]\right\}
\end{aligned}$$

Hence, according to the lemma the mean square error matrix difference

$$MSE\left[\tilde{\beta}_{PTSRLE}(d)\right] - MSE\left[\hat{\beta}_{srd}\right]$$

is nonnegative definite if and only if

$$\begin{aligned}
&\left\{\left[(d-1)(S+dI)^{-1}\beta + h_\lambda(2)H\delta\right]' D^- \times\left[(d-1)(S+dI)^{-1}\beta + h_\lambda(2)H\delta\right]+1\right\}\times\left\{\left[(d-1)(S+dI)^{-1}\beta + H\delta\right]' D^- \right.\\
&\left.\times\left[(d-1)(S+dI)^{-1}\beta + H\delta\right]-1\right\} \leq \left\{\left[(d-1)(S+dI)^{-1}\beta + h_\lambda(2)H\delta\right]' D^- \times\left[(d-1)(S+dI)^{-1}\beta + H\delta\right]\right\}^2
\end{aligned}$$

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This completes the proof of theorem.

3.2. Comparison between the PTSRLE and SRLE under SMSE Criterion

In this subsection, we will compare the PTSRLE with the SRLE under SMSE criterion when stochastic restricttions are correct and not correct.

If the stochastic restrictions are correct then

$$E(r)-R\beta=\delta=0,$$

and consequently the SMSE difference between $\tilde{\beta}_{PTSRLE}(d)$ and $\hat{\beta}_{srd}$ can be written as

$$\begin{aligned}&SMSE\left[\tilde{\beta}_{PTSRLE}(d)\right]-SMSE\left[\hat{\beta}_{srd}\right],\\&=\sigma^2\left[1-h_\lambda(2)\right]tr\left(F_dGF_d'\right)\end{aligned}$$

which is nonnegative definite as

$$0\le h_\lambda(2)\le 1.$$

Hence $\hat{\beta}_{srd}$ is always superior to $\tilde{\beta}_{PTSRLE}(d)$ when $\delta=0$.

If the stochastic restrictions are not correct then

$$E(r)-R\beta=\delta\neq 0,$$

and consequently since the matrix S is positive definite, there exist an orthogonal matrix P and a positive definite diagonal matrix

$$\Lambda=\operatorname{diag}\left(\lambda_1,\lambda_2,\cdots,\lambda_p\right)$$

such that $P'SP=\Lambda$, with $P'P=PP'=I$. Then the SMSE difference between SRLE and PTSRLE can be written as

$$\begin{aligned}&SMSE\left[\hat{\beta}_{srd}\right]-SMSE\left[\tilde{\beta}_{PTSRLE}(d)\right]=\sum_{i=1}^{p}\frac{(\lambda_i+d)}{(\lambda_i+1)^2}\left\{\sigma^2\left(h_\lambda(2)-1\right)(\lambda_i+d)\tilde{a}_{ii}+\left(1-2h_\lambda(2)+h_\lambda(4)\right)(\lambda_i+d)\tilde{\eta}_i^2\right.\\&\left.+2(1-d)\left(1-h_\lambda(2)\right)\tilde{\eta}_i\gamma_i\right\}=\sum_{i=1}^{p}\frac{(\lambda_i+d)}{(\lambda_i+1)^2}\left\{\left(1-2h_\lambda(2)+h_\lambda(4)\right)\lambda_i\tilde{\eta}_i^2+2\left(1-h_\lambda(2)\right)\tilde{\eta}_i\gamma_i-\sigma^2\left(1-h_\lambda(2)\right)\tilde{a}_{ii}\lambda_i\right.\\&\left.-d\left[\sigma^2\left(1-h_\lambda(2)\right)\tilde{a}_{ii}-\left(1-h_\lambda(2)+h_2(4)\right)\tilde{\eta}_i^2+2\left(1-h_\lambda(2)\right)\tilde{\eta}_i\gamma_i\right]\right\}\end{aligned}\tag{26}$$

where,

$$\gamma=P'\beta=\left(\gamma_1,\gamma_2,\cdots,\gamma_p\right)',\quad \tilde{\eta}=P'\eta=\left(\tilde{\eta}_1,\tilde{\eta}_2,\cdots,\tilde{\eta}_p\right)'$$

and $\tilde{a}_{ii}\ge 0$ is the i^{th} diagonal element of the matrix $\tilde{A}=P'AP$. Therefore, the SMSE difference in (26) is nonnegative definite if and only if $0<d\le d^*$, where,

$$d^*=\frac{\min\limits_{1\le i\le p}\left\{\left[1-2h_\lambda(2)+h_\lambda(4)\right]\lambda_i\tilde{\eta}_i^2+2\left[1-h_\lambda(2)\right]\tilde{\eta}_i\gamma_i-\sigma^2\left[1-h_\lambda(2)\right]\tilde{a}_{ii}\lambda_i\right\}}{\max\limits_{1\le i\le p}\left\{\sigma^2\left[1-h_\lambda(2)\right]\tilde{a}_{ii}-\left[1-2h_\lambda(2)+h_\lambda(4)\right]\tilde{\eta}_i^2+2\left[1-h_\lambda(2)\right]\tilde{\eta}_i\gamma_i\right\}}\tag{27}$$

Now we summarize our findings:

Theorem 3.2:

1) If the stochastic restrictions are true (*i.e.* $\delta=0$); the SRLE is always superior to the PTSRLE in the scalar mean squared error sense.

2) If the stochastic restrictions are not true (*i.e.* $\delta\neq 0$); the Preliminary Test Stochastic restricted Liu Estimator has Smaller SMSE than the Stochastic Restricted Liu Estimator if and only if $0\le d\le d^*$, where d^* is given in (27).

4. PTSRLE Based on WA, LR and LM Tests

In general, the finite sample test such as t or F was used to define the preliminary test estimator. Since these finite sample tests are not always available it is very useful to consider the preliminary test estimators based on the three tests WA, LR and LM. The WA test offers the advantage of only requiring estimates of the unrestricted model, whereas LR test requires estimates of both unrestricted and the restricted model. The LM test only requires estimates of the restricted model. In different situations, we may find one or the other of these tests which is easier to compute. Judge and Bock [7] have rewritten the model given in (1) and (2) to obtain the F statistics for testing the hypothesis in (3). Using the same model we can derive the test statistics for the WA, the LR and the LM tests which are well employed for testing the Hypothesis (3) and given by

$$\begin{aligned}&\varepsilon_{WA}=\frac{(n+m)mF}{n-p},\\&\varepsilon_{LR}=(n+m)\ln\left[1+\frac{mF}{n-p}\right]\\&\text{and }\ \varepsilon_{LM}=\frac{(n+m)mF}{\left[n-p+mF\right]}\end{aligned}\tag{28}$$

respectively [18].

It's known that under the null hypothesis H_0, the

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three test statistics have the same asymptotic chi-square distribution with m degrees of freedom [18]. When the exact distribution is approximated by the asymptotic chi-square distribution, the critical value for an α-level test of H_o is approximated by the central chi-square critical value $\chi_m^2(\alpha)$ for large sample tests. This asymptotic chi-square distribution has wide application in the field of Econometrics. Based on the above tests, the PTSRLE takes the form [10] as

$$\tilde{\beta}_{PTSRLE}(d,\varepsilon_*) = F_d\hat{\beta}_m I_{\left[0,\chi_m^2(\alpha)\right)}(\varepsilon_*) + F_d\hat{\beta}\, I_{\left[\chi_m^2(\alpha),\infty\right)}(\varepsilon_*) \tag{29}$$

where (*) stands for either WA, LR or LM tests values, and $\chi_m^2(\alpha)$ is the upper percentiles of the central χ^2 distribution with *m* degrees of freedom.

By using the equation in (18), now we can obtain the SMSE of the PTSRLE based on WA, LR and LM tests.

$$\begin{aligned} &SMSE\left[\tilde{\beta}_{PTSRLE}(\varepsilon_*,d)\right] \\ &= \sigma^2 tr\left(F_d S^{-1} F_d'\right) - \sigma^2 h_\lambda^*(2)\, tr\left(F_d G F_d'\right) \\ &\quad + \left[2h_\lambda^*(2) - h_\lambda^*(4)\right]\eta' F_d' F_d \eta \\ &\quad + 2(1-d)h_\lambda^*(2)\beta'(S+I)^{-1}F_d\eta \\ &\quad + (1-d)^2\beta'(S+I)^{-2}\beta \end{aligned} \tag{30}$$

where,

$$h_\lambda^*(\ell) = \Pr\left(\frac{\chi_{m+\ell,\lambda}^2}{\chi_{n-p}^2} \le \frac{mc^*}{n-p}\right)$$

for $\ell \in N$, and c^* takes the value for WA, LR and LM tests as

$$c^{WA} = \frac{(n-p)\chi_m^2(\alpha)}{(n+m)m}, \quad c^{LR} = \frac{(n-p)\left(e^{\chi_m^2(\alpha)/(n+m)} - 1\right)}{m}$$

and

$$c^{LM} = \frac{(n-p)\chi_m^2(\alpha)}{m\left[n+m-\chi_m^2(\alpha)\right]}$$

respectively.

We consider the SMSE difference between WA and LR tests of the PTSRLE

$$\begin{aligned} &SMSE\left[\tilde{\beta}_{PTSRLE}(\varepsilon_{WA},d)\right] - SMSE\left[\tilde{\beta}_{PTSRLE}(\varepsilon_{LR},d)\right] \\ &= \sigma^2 tr\left(F_d G F_d'\right)\psi_1 - \eta' F_d' F_d \eta\left[2\psi_1 - \psi_1^*\right] \\ &\quad - 2(1-d)\psi_1\beta'(S+I)^{-1}F_d\eta \end{aligned} \tag{31}$$

where:

$$\psi_1 = h_\lambda^{LR}(2) - h_\lambda^{WA}(2)$$

and

$$\psi_1^* = h_\lambda^{LR}(4) - h_\lambda^{WA}(4).$$

Now we consider the SMSE difference between LR and LM tests of the PTSRLE

$$\begin{aligned} &SMSE\left[\tilde{\beta}_{PTSRLE}(\varepsilon_{LR},d)\right] - SMSE\left[\tilde{\beta}_{PTSRLE}(\varepsilon_{LM},d)\right] \\ &= \sigma^2 tr\left(F_d G F_d'\right)\psi_2 - \eta' F_d' F_d \eta\left[2\psi_2 - \psi_2^*\right] \\ &\quad - 2(1-d)\psi_2\beta'(S+I)^{-1}F_d\eta \end{aligned} \tag{32}$$

where:

$$\psi_2 = h_\lambda^{LM}(2) - h_\lambda^{LR}(2)$$

and

$$\psi_2^* = h_\lambda^{LM}(4) - h_\lambda^{LR}(4)$$

***Case I*:** If the stochastic restrictions are true then $\delta = 0$.

Note that $\psi_1 \ge 0$ as $c^{LR} \ge c^{WA}$ and $\psi_2 \ge 0$ as $c^{LM} \ge c^{LR}$ then the SMSE difference in (31) reduced to $\sigma^2 tr\left(F_d G F_d'\right)\psi_1$ which is nonnegative definite as $\psi_1 \ge 0$. Similarly the SMSE difference in (32) reduced to $\sigma^2 tr\left(F_d G F_d'\right)\psi_2$ which is nonnegative definite as $\psi_2 \ge 0$.

***Case II*:** If the stochastic restrictions are not true then $\delta \ne 0$.

We can rewrite the SMSE difference in (31) as follows

$$\begin{aligned} &SMSE\left[\tilde{\beta}_{PTSRLE}(\varepsilon_{WA},d)\right] - SMSE\left[\tilde{\beta}_{PTSRLE}(\varepsilon_{LR},d)\right] \\ &= \sum_{i=1}^{p}\frac{(\lambda_i+d)}{(\lambda_i+1)^2}\left\{\sigma^2\lambda_i\tilde{a}_{ii}\psi_1 - \tilde{\eta}_i^2\lambda_i\left(2\psi_1 - \psi_1^*\right) - 2\gamma_i\tilde{\eta}_i\psi_1\right. \\ &\quad \left. + d\left[\sigma^2\tilde{a}_{ii}\psi_1 - \tilde{\eta}_i^2\left(2\psi_1 - \psi_1^*\right) + 2\gamma_i\tilde{\eta}_i\psi_1\right]\right\} \end{aligned} \tag{33}$$

Therefore the SMSE difference in (33) is nonnegative definite if $d_1^* \le d < 1$, where

$$d_1^* = \frac{\max\limits_{1\le i\le p}\left\{\tilde{\eta}_i^2\lambda_i\left(2\psi_1 - \psi_1^*\right) + 2\gamma_i\tilde{\eta}_i\psi_1 - \sigma^2\lambda_i\tilde{a}_{ii}\psi_1\right\}}{\min\limits_{1\le i\le p}\left\{\sigma^2\tilde{a}_{ii}\psi_1 - \tilde{\eta}_i^2\left(2\psi_1 - \psi_1^*\right) + 2\gamma_i\tilde{\eta}_i\psi_1\right\}} \tag{34}$$

Hence, $\tilde{\beta}_{PTSRLE}(\varepsilon_{LR},d)$ will dominate $\tilde{\beta}_{PTSRLE}(\varepsilon_{WA},d)$ if $d_1^* \le d < 1$ and we can similarly get that $\tilde{\beta}_{PTSRLE}(\varepsilon_{WA},d)$ will dominate $\tilde{\beta}_{PTSRLE}(\varepsilon_{LR},d)$ whenever $0 < d \le d_2^*$ where

$$d_2^* = \frac{\min\limits_{1\le i\le p}\left\{\tilde{\eta}_i^2\lambda_i\left(2\psi_1 - \psi_1^*\right) + 2\gamma_i\tilde{\eta}_i\psi_1 - \sigma^2\lambda_i\tilde{a}_{ii}\psi_1\right\}}{\max\limits_{1\le i\le p}\left\{\sigma^2\tilde{a}_{ii}\psi_1 - \tilde{\eta}_i^2\left(2\psi_1 - \psi_1^*\right) + 2\gamma_i\tilde{\eta}_i\psi_1\right\}} \tag{35}$$

We can rewrite the SMSE difference in (32) as follow:

$$\begin{aligned} &SMSE\left[\tilde{\beta}_{PTSRLE}(\varepsilon_{LR},d)\right] - SMSE\left[\tilde{\beta}_{PTSRLE}(\varepsilon_{LM},d)\right] \\ &= \sum_{i=1}^{p}\frac{(\lambda_i+d)}{(\lambda_i+1)^2}\left\{\sigma^2\lambda_i\tilde{a}_{ii}\psi_2 - \tilde{\eta}_i^2\lambda_i(2\psi_2 - \psi_2^*) - 2\gamma_i\tilde{\eta}_i\psi_2\right. \\ &\quad \left. + d\left[\sigma^2\tilde{a}_{ii}\psi_2 - \tilde{\eta}_i^2(2\psi_2 - \psi_2^*) + 2\gamma_i\tilde{\eta}_i\psi_2\right]\right\} \end{aligned} \tag{36}$$

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Therefore, the SMSE difference in (36) is nonnegative definite if $d_3^* \le d < 1$, where

$$d_3^* = \frac{\max\limits_{1\le i\le p}\left\{\tilde{\eta}_i^2\lambda_i\left(2\psi_2-\psi_2^*\right)+2\gamma_i\tilde{\eta}_i\psi_2-\sigma^2\lambda_i\tilde{a}_{ii}\psi_2\right\}}{\min\limits_{1\le i\le p}\left\{\sigma^2\tilde{a}_{ii}\psi_2-\tilde{\eta}_i^2\left(2\psi_2-\psi_2^*\right)+2\gamma_i\tilde{\eta}_i\psi_2\right\}} \quad (37)$$

Hence, $\tilde{\beta}_{PTSRLE}\left(\varepsilon_{LM},d\right)$ will dominate $\tilde{\beta}_{PTSRLE}\left(\varepsilon_{LR},d\right)$ if $d_3^* \le d < 1$ and we can similarly get that $\beta_{PTSRLE}\left(\varepsilon_{LR},d\right)$ will dominate $\tilde{\beta}_{PTSRLE}\left(\varepsilon_{LM},d\right)$ whenever $0 < d \le d_4^*$ where

$$d_4^* = \frac{\min\limits_{1\le i\le p}\left\{\tilde{\eta}_i^2\lambda_i\left(2\psi_2-\psi_2^*\right)+2\gamma_i\tilde{\eta}_i\psi_2-\sigma^2\lambda_i\tilde{a}_{ii}\psi_2\right\}}{\max\limits_{1\le i\le p}\left\{\sigma^2\tilde{a}_{ii}\psi_2-\tilde{\eta}_i^2\left(2\psi_2-\psi_2^*\right)+2\gamma_i\tilde{\eta}_i\psi_2\right\}} \quad (38)$$

Now the performance of the PTSRLE estimator based on WA, LR and LM tests are compared with respect to the SMSE, and the following theorem can be stated.

Theorem 4.1:

1) The stochastic restrictions are true (*i.e.* $\delta = 0$); then

$$SMSE\left[\tilde{\beta}_{PTSRLE}\left(\varepsilon_{LM},d\right)\right] \le SMSE\left[\tilde{\beta}_{PTSRLE}\left(\varepsilon_{LR},d\right)\right] \le SMSE\left[\tilde{\beta}_{PTSRLE}\left(\varepsilon_{WA},d\right)\right]$$

2) The stochastic restrictions are not true (*i.e.* $\delta \ne 0$) then

a) If $\max\left\{d_1^*,d_3^*\right\} \le d < 1$ then

$$SMSE\left[\tilde{\beta}_{PTSRLE}\left(\varepsilon_{LM},d\right)\right] \le SMSE\left[\tilde{\beta}_{PTSRLE}\left(\varepsilon_{LR},d\right)\right] \le SMSE\left[\tilde{\beta}_{PTSRLE}\left(\varepsilon_{WA},d\right)\right]$$

b) If $0 < d < \min\left\{d_2^*,d_4^*\right\}$ then

$$SMSE\left[\tilde{\beta}_{PTSRLE}\left(\varepsilon_{WA},d\right)\right] \le SMSE\left[\tilde{\beta}_{PTSRLE}\left(\varepsilon_{LR},d\right)\right] \le SMSE\left[\tilde{\beta}_{PTSRLE}\left(\varepsilon_{LM},d\right)\right]$$

where, d_1^*, d_2^*, d_3^* and d_4^* are given in Equations (34), (35), (37) and (38), respectively.

From Theorem 4.1(2b) and according to [14] we can say that when d is small, WA test has the smallest SMSE than the other tests. Similarly according to the results stated in (2a), the LM test has the smallest SMSE than the other tests when d becomes large.

5. Numerical Example

To illustrate our theoretical results, we consider the following data set on Portland cement originally due to Woods, Steinour and Starke [19]. This data set came from an experimental investigation of the heat evolved during the setting and hardening of Portland cements of varied composition and the dependence of this heat on the percentages of four compounds in the clinkers from which the cement was produced. The four compounds considered by Woods, Steinour and Starke [19] are tricalium aluminate: $3CaO{\cdot}Al_2O_3$, tricalcium silicate: $3CaO{\cdot}SiO_2$, tetracalcium aluminaferrite: $4CaO{\cdot}Al_2O_3{\cdot}Fe_2O_3$, and beta-dicalcium silicate: $2CaO{\cdot}SiO_2$, which we will denote by X_1, X_2, X_3 and X_4, respectively. The dependent variable Y is the heat evolved in calories per gram of cement after 180 days of curing. This dataset has since then been widely used by many researchers (e.g. [4,20]).

$$X = \begin{pmatrix} 7 & 26 & 6 & 60 \\ 1 & 29 & 15 & 52 \\ 11 & 56 & 8 & 20 \\ 11 & 31 & 8 & 47 \\ 7 & 52 & 6 & 33 \\ 11 & 55 & 9 & 22 \\ 3 & 71 & 17 & 6 \\ 1 & 31 & 22 & 44 \\ 2 & 54 & 18 & 22 \\ 21 & 47 & 4 & 26 \\ 1 & 40 & 23 & 34 \\ 11 & 66 & 9 & 12 \\ 10 & 68 & 8 & 12 \end{pmatrix}, \quad Y = \begin{pmatrix} 78.5 \\ 74.3 \\ 104.3 \\ 87.6 \\ 95.9 \\ 109.2 \\ 102.7 \\ 72.5 \\ 93.1 \\ 115.9 \\ 83.8 \\ 113.3 \\ 109.4 \end{pmatrix}$$

The $\boldsymbol{X} = (\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_4)$ matrix contains $n = 13$ observations and $p = 4$ predictor variables. Since the regressor matrix $\boldsymbol{X}$ does not include a column of ones a homogeneous multiple linear regression, Model (1) without intercept is fitted to the data.

The ordinary least square estimator of regression coefficient β is

$$\hat{\beta} = S^{-1}X'Y = \left(2.1930, 1.1533, 0.7585, 0.4863\right)'$$

with

$$MSE\left[\hat{\beta},\beta\right] = 0.0638$$

and

$$\hat{\sigma}^2 = 5.8455.$$

Consider the following stochastic restrictions $r = R\beta + \delta + \nu$ where $R = (0,1,3,1)$, $r = 0$ and

$$\nu \sim N\left(0, \hat{\sigma}_{OLSE}^2 = 5.8455\right) \text{ (see [20,21]).}$$

Figures 1 and **2** are drawn by using the SMSE given in Equations (18) and (20) for different d values selected from (0, 1).

According to the **Figures 1** and **2**, we can conclude that when d is small the PTSRLE has the smallest

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SMSE value than the SRLE, OSPE and OLSE.

Figures 3 and **4** are drawn by using the SMSE given in Equation (30) for different d values selected from (0, 1). From **Figures 3** and **4**, we can notice that when d is small, the WA test has the smallest SMSE than the other tests. When d becomes large, the LM test has the smallest SMSE. Hence the data analysis supports the findings of this paper.

6. Conclusions

In this paper, we have introduced a new preliminary test estimator in a multiple linear regression model. When d is small, the PTSRLE based on WA test has the smallest SMSE than the other tests. When d becomes large, the PTSRLE based on LM test has the smallest SMSE. Moreover, for certain cases (**Figures 1** and **2**) the proposed estimator has the smallest SMSE. The results of this paper have a potential for future developments for both theoretical and practical aspects.

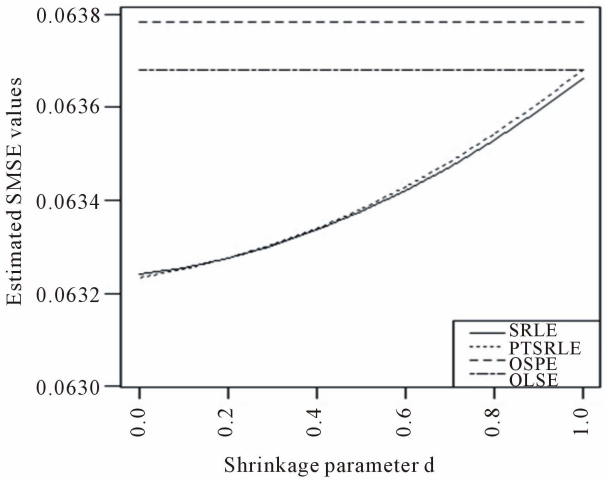

Figure 1. Estimated the SMSE values for SRLE, PTSRLE, OSPE and OLSE at $\alpha = 0.01$.

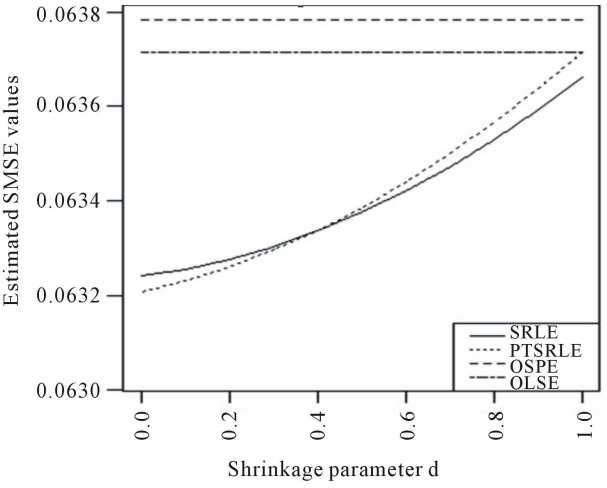

Figure 2. Estimated the SMSE values for SRLE, PTSRLE, OSPE and OLSE at $\alpha = 0.05$.

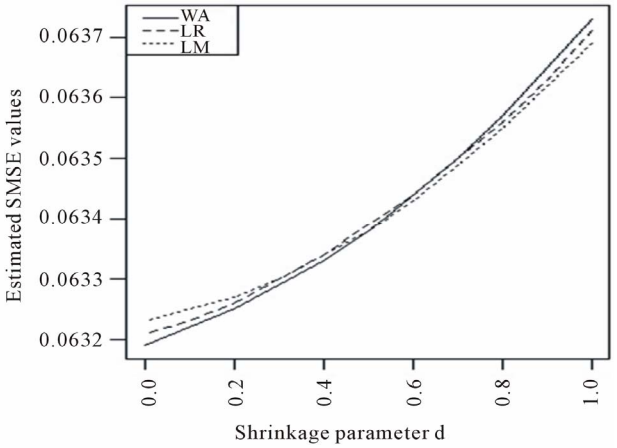

Figure 3. The SMSE of the PTSRLE based on WA, LR and LM tests for $\alpha = 0.01$.

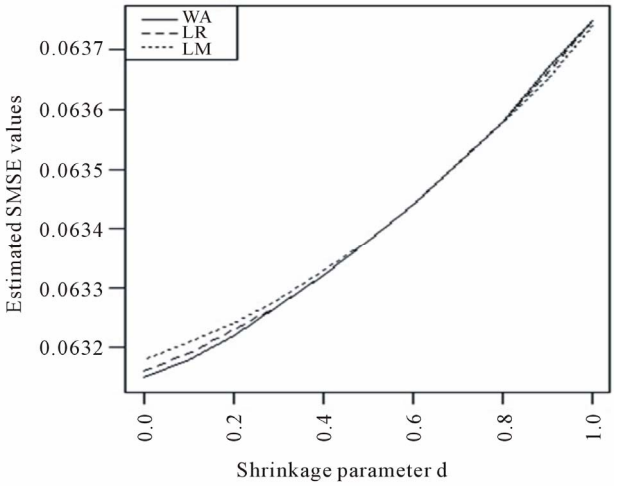

Figure 4. The SMSE of the PTSRLE based on WA, LR and LM tests for $\alpha = 0.05$.

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Appendix

Lemma: (Baksalary and Trenkler, [16])

Let C be a nonnegative definite matrix and c_1, c_2 be linearly independent vectors. Furthermore for some generalized inverse C^- of C, let $f_{ij} = c_i' C^- c_j$; $i = 1,2, j = 1,2$ and let

$$s = \frac{c_2'\left(I - CC^-\right)'\left(I - CC^-\right)c_2}{c_1'\left(I - CC^-\right)\left(I - CC^-\right)c_1}$$

where $c_1 \in \Re(C)$ and $\Re(.)$ denote the column space of the corresponding matrix. Then we have

$$C + c_1 c_1' - c_2 c_2' \geq 0$$

if and only if

1) $c_1 \in \Re(C), c_2 \in \Re(C)$ and

$$(f_{11}+1)(f_{22}+1) \leq f_{12}^2 \text{ or}$$

2) $c_1 \notin \Re(C), c_2 \in \Re(C, c_1)$ and

$$(c_2 - sc_1)' C^- (c_2 - sc_1) \leq 1 - s^2$$

and all expressions in (1) and (2) are independent of the choice of C^-.

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