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Complete Convergence and Weak Law of Large Numbers for $\tilde{\rho}$-Mixing Sequences of Random Variables

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ABSTRACT

In this paper, the complete convergence and weak law of large numbers are established for $\tilde{\rho}$-mixing sequences of random variables. Our results extend and improve the Baum and Katz complete convergence theorem and the classical weak law of large numbers, etc. from independent sequences of random variables to $\tilde{\rho}$-mixing sequences of random variables without necessarily adding any extra conditions.

Keywords: $\tilde{\rho}$-Mixing Sequence of Random Variables; Complete Convergence; Weak Law of Large Number

1. Introduction

Let $(\Omega,\mathcal{F},P)$ be a probability space. The random variables we deal with are all defined on $(\Omega,\mathcal{F},P)$. Let $\{X_n;n\ge1\}$ be a sequence of random variables. For each nonempty set $S\subset N$, write $\mathcal{F}_S=\sigma(X_i;i\in S)$. Given σ-algebras $\mathcal{B},\mathcal{R}$ in $\mathcal{F}$, let

$$\rho(\mathcal{B},\mathcal{R})=\sup\left\{\left|\operatorname{corr}(X,Y)\right|;X\in L_2(\mathcal{B}),Y\in L_2(\mathcal{R})\right\},$$

where $\operatorname{corr}(X,Y)=\frac{EXY-EXEY}{\sqrt{\operatorname{Var}X\operatorname{Var}Y}}$. Define the $\tilde{\rho}$-mixing coefficients by

$$\tilde{\rho}(n)=\sup\rho(\mathcal{F}_S,\mathcal{F}_T), \tag{1.1}$$

where (for a given positive integer n) this sup is taken over all pairs of nonempty finite subsets S,T of N such that $\operatorname{dist}(S,T)\ge n$.

Obviously $0\le\tilde{\rho}(n+1)\le\tilde{\rho}(n)\le1, n\ge0,$ and $\tilde{\rho}(0)=1$ except in the trivial case where all of the random variables X_i are degenerate.

Definition 1.1. A sequence of random variables $\{X_n;n\ge1\}$ is said to be a $\tilde{\rho}$-mixing sequence of random variables if there exists $k\in N$ such that $\tilde{\rho}(k)<1$.

Without loss of generality we may assume that $\{X_n;n\ge1\}$ is such that $\tilde{\rho}(1)<1$ (see [1]). Here we give two examples of the practical application of $\tilde{\rho}$-mixing.

Example 1.1. According to the proof of Theorem 2 in [2] and Remark 3 in [1], if $\{X_i;i\ge1\}$ is a strictly stationary Gaussian sequence which has a bounded positive spectral density $f(t)$, then the sequence $\{f(X_i);i\ge1\}$ has the property that $\tilde{\rho}(1)<1$. Therefore, instantaneous functions $\{f(X_i);i\ge1\}$ of such a sequence provides a class of examples for $\tilde{\rho}$-mixing sequences.

Example 1.2. If $\{X_n;n\ge1\}$ has a bounded positive spectral density $f(t)$, *i.e.*, $0<m<f(t)<M$ for every *t*, then $\tilde{\rho}(1)<1-m/M<1$. Thus, $\{X_n;n\ge1\}$ is a $\tilde{\rho}$-mixing sequence.

$\tilde{\rho}$-mixing is similar to ρ-mixing, but both are quite different. $\rho(k)$ is defined by (1.1) with index sets restricted to subsets S of $[1,n]$ and subsets T of $[n+k,\infty)$, $n,k\in N$. On the other hand, ρ-mixing sequence assume condition $\rho(k)\to0$, but $\tilde{\rho}$-mixing sequence assume condition that there exists $k\in N$ such that $\tilde{\rho}(k)<1$, from this point of view, $\tilde{\rho}$-mixing is weaker than ρ-mixing.

A number of writers have studied $\tilde{\rho}$-mixing sequences of random variables and a series of useful results have been established. We refer to [2] for the central limit theorem [1,3], for moment inequalities and the strong law of large numbers [4-9], for almost sure convergence, and [10] for maximal inequalities and the invariance principle. When these are compared with the corresponding results for sequences of independent random variables, there still remains much to be desired.

The main purpose of this paper is to study the complete convergence and weak law of large numbers of partial sums of $\tilde{\rho}$-mixing sequences of random variables and try to obtain some new results. We establish the complete convergence theorems and the weak law of large numbers. Our results in this paper extend and improve the corresponding results of Feller [11] and Baum and Katz [12].

Lemma 1.1. ([10], Theorem 2.1) *Suppose K is a positive integer,* $0 \le r < 1$, *and* $q \ge 2$. *Then there exists a positive constant* $D = D(K, r, q)$ *such that the following statement holds:*

If $\{X_i; i \ge 1\}$ *is a sequence of random variables such that* $\tilde{\rho}(K) \le r$ *and* $EX_i = 0$ *and* $E|X_i|^q < \infty$ *for all* $i \ge 1$, *then for every* $n \ge 1$,

$$E\left(\max_{1\le i\le n}|S_i|^q\right) \le D\left(\sum_{i=1}^{n} E|X_i|^q + \left(\sum_{i=1}^{n} EX_i^{\,2}\right)^{q/2}\right),$$

where $S_i = \sum_{j=1}^{i} X_i$.

Lemma 1.2. *Let* $\{X_n; n \ge 1\}$ *be a* $\tilde{\rho}$*-mixing sequence of random variables. Then for any* $x \ge 0$, *there exists a positive constant c such that for all* $n \ge 1$,

$$\left(1 - P(\max_{1\le k\le n}|X_k| > x)\right)^2 \sum_{k=1}^{n} P\left(|X_k| > x\right) \le cP\left(\max_{1\le k\le n}|X_k| > x\right).$$

Proof. Let $A_k = \left(|X_k| > x\right)$ and $\alpha_n = 1 - P\left(\bigcup_{k=1}^{n} A_k\right) = 1 - P\left(\max_{1\le k\le n}|X_k| > x\right)$. Without loss of generality, assume that $\alpha_n > 0$. By the Cauchy-Schwarz inequality and Lemma 1.2,

$$\begin{aligned}
\sum_{k=1}^{n} P(A_k) &= \sum_{k=1}^{n} P\left(A_k, \bigcup_{j=1}^{n} A_j\right) = \sum_{k=1}^{n} E\left(I_{A_k}, I_{\bigcup_{j=1}^{n} A_j}\right) \\
&= E\left(\sum_{k=1}^{n}\left(I_{A_k} - EI_{A_k}\right)\right) I_{\bigcup_{j=1}^{n} A_j} + \sum_{k=1}^{n} P(A_k) P\left(\bigcup_{j=1}^{n} A_j\right) \\
&\le \left(E\left(\sum_{k=1}^{n}\left(I_{A_k} - EI_{A_k}\right)\right)^2 EI_{\bigcup_{j=1}^{n} A_j}\right)^{1/2} + (1-\alpha_n)\sum_{k=1}^{n} P(A_k) \\
&\le \left(c\left(\sum_{k=1}^{n} E\left(I_{A_k} - EI_{A_k}\right)^2\right)^2 P\left(\bigcup_{j=1}^{n} A_j\right)\right)^{1/2} \\
&\quad + (1-\alpha_n)\sum_{k=1}^{n} P(A_k) \\
&\le \left(\frac{c(1-\alpha_n)}{\alpha_n}\alpha_n \sum_{k=1}^{n} P(A_k)\right)^{1/2} + (1-\alpha_n)\sum_{k=1}^{n} P(A_k) \\
&\le \frac{1}{2}\left(\frac{c(1-\alpha_n)}{\alpha_n} + \alpha_n \sum_{k=1}^{n} P(A_k)\right) + (1-\alpha_n)\sum_{k=1}^{n} P(A_k)
\end{aligned}$$

Thus

$$\alpha_n^2 \sum_{k=1}^{n} P(A_k) \le c(1-\alpha_n),$$

i.e.,

$$\left(1 - P\left(\max_{1\le k\le n}|X_k| > x\right)\right)^2 \sum_{k=1}^{n} P\left(|X_k| > x\right) \le cP\left(\max_{1\le k\le n}|X_k| > x\right).$$

2. Complete Convergence

In the following, let $a(x) \sim b(x)$ denote $a(x)/b(x) \to 1,\ x \to \infty$, and $a_n \ll b_n$ $(a_n \gg b_n)$ denote that there exists a constant $c > 0$ such that $a_n \le cb_n$ $(a_n \ge cb_n)$ for sufficiently large *n*, logx mean $\ln(\max(x, \mathrm{e}))$, and $S_n = \sum_{j=1}^{n} X_i$.

Definition 2.1. A measurable function $l(x) > 0\,(x > 0)$ is said to be a slowly varying function at ∞ if for any $c > 0$, $\lim_{x\to\infty}\frac{l(cx)}{l(x)} = 1$.

Lemma 2.1 ([13], Lemma 1). *Let* $l(x)$ *be a slowly varying function at* ∞. *Then*

i) $\lim_{k\to\infty}\sup_{2^k \le x < 2^{k+1}} \frac{l(x)}{l(2^k)} = 1$.

ii) $\lim_{x\to\infty} x^{\delta} l(x) = \infty, \quad \lim_{x\to\infty} x^{-\delta} l(x) = 0,$ *for any* $\delta > 0$.

iii) *For any* $r > 0$ *and* $\eta > 0$, *there exist positive constants* c_1 *and* c_2 (*depending only on* r, η, *and the function* $l(\cdot)$) *such that for any positive number k,*

$$c_1 2^{kr} l\left(2^k \eta\right) \le \sum_{j=1}^{k} 2^{jr} l\left(2^j \eta\right) \le c_2 2^{kr} l\left(2^k \eta\right).$$

iv) *For any* $r < 0$ *and* $\eta > 0$, *there exist positive constants* d_1 *and* d_2 (*depending only on* r, η, *and the function* $l(\cdot)$) *such that for any positive number k,*

$$d_1 2^{kr} l\left(2^k \eta\right) \le \sum_{j=k}^{\infty} 2^{jr} l\left(2^j \eta\right) \le d_2 2^{kr} l\left(2^k \eta\right).$$

Theorem 2.1. *Let* $\{X_n; n \ge 1\}$ *be a* $\tilde{\rho}$*-mixing sequence of identically distributed random variables. Suppose that* $l(x) > 0$ *is a slowly varying function at* ∞, *and also assume that for each* $a > 0$, *the function* $l(x)$ *is bounded on the interval* $(0, a)$. *Suppose* $0 < p < 2$ *and* $\alpha p > 1$; *and if* $\alpha \le 1$ *then suppose also that* $EX_1 = 0$. *Then*

$$E\left(|X_1|^p\, l\left(|X_1|^{1/\alpha}\right)\right) < \infty \qquad (2.1)$$

and

$$\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\left(\max_{1\le j\le n}\left|S_j\right| > \varepsilon n^{\alpha}\right) < \infty, \quad \forall \varepsilon > 0 \tag{2.2}$$

are equivalent.

For $\alpha p = 1$, we also have the following theorem under adding the condition that $l(x)$ is a monotone non-decreasing function.

Theorem 2.2. *Let* $\{X_n; n \ge 1\}$ *be a* $\tilde{\rho}$*-mixing sequence of identically distributed random variables. Let* $l(x) > 0$ *is a slowly varying function at* ∞ *and monotone non-decreasing function. Suppose* $\alpha > 1/2$; *and if* $\alpha \le 1$ *then suppose also that* $EX_1 = 0$. *Then*

$$E\left(\left|X_1\right|^{1/\alpha} l\left(\left|X_1\right|^{1/\alpha}\right)\right) < \infty \tag{2.3}$$

and

$$\sum_{n=1}^{\infty} n^{-1} l(n) P\left(\max_{1\le j\le n}\left|S_j\right| > \varepsilon n^{\alpha}\right) < \infty, \quad \forall \varepsilon > 0 \tag{2.4}$$

are equivalent.

Taking $l(x) = 1$ and $l(x) = \log x$ respectively in Theorems 2.1 and 2.2 we can immediately obtain the following corollaries.

Corollary 2.1. *Let* $\{X_n; n \ge 1\}$ *be a* $\tilde{\rho}$*-mixing sequence of identically distributed random variables. Suppose* $0 < p < 2$ *and* $\alpha p > 1$; *and if* $\alpha \le 1$ *then suppose also that* $EX_1 = 0$. *Then*

$$E\left|X_1\right|^p < \infty$$

and

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le j\le n}\left|S_j\right| > \varepsilon n^{\alpha}\right) < \infty, \quad \forall \varepsilon > 0$$

are equivalent.

Corollary 2.2. *Let* $\{X_n; n \ge 1\}$ *be a* $\tilde{\rho}$*-mixing sequence of identically distributed random variables. Suppose* $0 < p < 2$ *and* $\alpha p > 1$; *and if* $\alpha \le 1$ *then suppose also that* $EX_1 = 0$. *Then*

$$E\left(\left|X_1\right|^p \log\left|X_1\right|\right) < \infty$$

and

$$\sum_{n=1}^{\infty} n^{\alpha p-2} \log n P\left(\max_{1\le j\le n}\left|S_j\right| > \varepsilon n^{\alpha}\right) < \infty, \quad \forall \varepsilon > 0$$

are equivalent.

Remark 2.1. When $\{X_n; n \ge 1\}$ i.i.d., Corollary 2.5 becomes the Baum and Katz [12] complete convergence theorem. So Theorems 2.1 and 2.2 extend and improve the Baum and Katz complete convergence theorem from the i.i.d. case to $\tilde{\rho}$-mixing sequences.

Remark 2.2. Letting $l(x)$ take various forms in Theorems 2.1 and 2.2, we can get a variety of pairs of equivalent statements, one involving a moment condition and the other involving a complete convergence condition.

Proof of Theorem 2.1. $(2.1) \Rightarrow (2.2)$. Let

$$Y_i = Y_i^{(n)} = X_i I_{\left(\left|X_i\right| \le n^{\alpha}\right)},$$

$Y_i = Y_i^{(n)} = X_i I_{\left(\left|X_i\right| \le n^{\alpha}\right)}, i = 1, 2, \cdots, n$. Firstly, we prove that

$$n^{-\alpha} \max_{1\le j\le n}\left|\sum_{i=1}^{j} EY_i\right| \to 0, \quad n \to \infty. \tag{2.5}$$

By Lemma 2.1 and (2.1), it is easy to show that

$$E\left|X_1\right|^{p-\delta} < \infty, \text{ for any } \delta > 0. \tag{2.6}$$

i) For $\alpha \le 1$, we have $p > 1/\alpha \ge 1$, and $EX_1 = 0$. Let $0 < \delta < \min\left(\frac{\alpha p - 1}{\alpha}, p - 1\right)$ in (2.6), by

$$E\left|X_1\right|^{p-\delta} < \infty, 1 - \alpha p + \alpha\delta < 0,$$

$$\begin{aligned}
n^{-\alpha} \max_{1\le j\le n}\left|\sum_{i=1}^{j} EY_i\right| &\le n^{-\alpha} \sum_{i=1}^{n}\left|EY_i\right| \\
&= n^{1-\alpha}\left|EX_1 I_{\left(\left|X_1\right| > n^{\alpha}\right)}\right| \\
&\le n^{1-\alpha} E\left|X_1\right| \frac{\left|X_1\right|^{p-1-\delta}}{n^{\alpha(p-1-\delta)}} I_{\left(\left|X_1\right| > n^{\alpha}\right)} \\
&\ll n^{1-\alpha p+\alpha\delta} E\left|X_1\right|^{p-\delta} \to 0.
\end{aligned}$$

ii) For $\alpha > 1, p \ge 1$, let $0 < \delta < \frac{\alpha - 1}{\alpha}$ in (2.6), then $E\left|X_1\right|^{1-\delta} < \infty$ and $1 - \alpha + \alpha\delta < 0$. Hence

$$\begin{aligned}
n^{-\alpha} \max_{1\le j\le n}\left|\sum_{i=1}^{j} EY_i\right| &\le n^{1-\alpha} E\left|X_1\right| I_{\left(\left|X_1\right| \le n^{\alpha}\right)} \\
&\le n^{1-\alpha+\alpha\delta} E\left|X_1\right|^{1-\delta} \to 0.
\end{aligned}$$

iii) For $\alpha > 1, p < 1$,

$$\begin{aligned}
n^{-\alpha} \max_{1\le j\le n}\left|\sum_{i=1}^{j} EY_i\right| &\le n^{-\alpha} \sum_{i=1}^{n}\left|EY_i\right| \\
&= n^{1-\alpha}\left|EY_1\right| \le n^{1-\alpha} E\left|X_1\right| I_{\left(\left|X_1\right| \le n^{\alpha}\right)} \\
&= n^{1-\alpha} \sum_{i=1}^{n} E\left|X_1\right| I_{\left((i-1)^{\alpha} < \left|X_1\right| \le i^{\alpha}\right)}.
\end{aligned}$$

Noting $p < 1, \alpha p > 1$, let $0 < \delta < \frac{\alpha p - 1}{\alpha}$ in (2.6). By $1 - \alpha p + \alpha\delta < 0$ and $E\left|X_1\right|^{p-\delta} < \infty$, we get

$$\sum_{i=1}^{\infty} i^{1-\alpha} E|X_1| I_{\left((i-1)^\alpha < |X_1| \le i^\alpha\right)}$$

$$\le \sum_{i=1}^{\infty} i^{1-\alpha p+\alpha\delta} E|X_1|^{p-\delta} I_{\left((i-1)^\alpha < |X_1| \le i^\alpha\right)}$$

$$\le \sum_{i=1}^{\infty} E|X_1|^{p-\delta} I_{\left((i-1)^\alpha < |X_1| \le i^\alpha\right)} < \infty.$$

By $i^{\alpha-1} \uparrow \infty$ and the Kronecker lemma,

$$n^{1-\alpha} \sum_{i=1}^{n} E|X_1| I_{\left((i-1)^\alpha < |X_1| \le i^\alpha\right)} \to 0,\ n \to \infty.$$

Hence (2.5) holds. So to prove (2.2) it suffices to prove that

$$\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\left(\bigcup_{i=1}^{n}\left(|X_i| > n^\alpha\right)\right) < \infty, \qquad (2.7)$$

and $\forall \varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}\left(Y_i - EY\right)_i\right| > \varepsilon n^\alpha\right) < \infty. \quad (2.8)$$

By Lemmas 2.1 (i), (iii), (2.1), and for each $a > 0$, the function $l(x)$ is bounded on the interval $(0,a)$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\left(\bigcup_{i=1}^{n}\left(|X_i| > n^\alpha\right)\right)$$

$$\le \sum_{n=1}^{\infty} n^{\alpha p-1} l(n) P\left(|X_1| > n^\alpha\right)$$

$$= \sum_{j=0}^{\infty} \sum_{2^j \le n < 2^{j+1}} n^{\alpha p-1} l(n) P\left(|X_1| > n^\alpha\right)$$

$$\ll \sum_{j=1}^{\infty} 2^{j(\alpha p-1)} 2^j l\left(2^j\right) P\left(|X_1| > 2^{\alpha j}\right)$$

$$= \sum_{j=1}^{\infty} 2^{\alpha pj} l\left(2^j\right) \sum_{k=j}^{\infty} P\left(2^{\alpha k} < |X_1| \le 2^{\alpha(k+1)}\right)$$

$$= \sum_{k=1}^{\infty} \sum_{j=1}^{k} 2^{\alpha pj} l\left(2^j\right) P\left(2^{\alpha k} < |X_1| \le 2^{\alpha(k+1)}\right)$$

$$\ll \sum_{k=1}^{\infty} 2^{\alpha pk} l\left(2^k\right) P\left(2^{\alpha k} < |X_1| \le 2^{\alpha(k+1)}\right)$$

$$\ll E\left(|X_1|^p l\left(|X_1|^{1/\alpha}\right)\right) < \infty.$$

i.e., (2.7) holds.

By the Markov inequality, Lemma 1.2, Lemmas 2.1 (i), (iv), (2.1), and for each $a > 0$, the function $l(x)$ is bounded on the interval $(0,a)$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} Y_i - EY_i\right| \ge \varepsilon n^\alpha\right)$$

$$\ll \sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha} l(n) \sum_{i=1}^{n} E\left(Y_i - EY_i\right)^2$$

$$\le \sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha} l(n) EX_1^2 I_{\left(|X_1| \le n^\alpha\right)}$$

$$= \sum_{j=1}^{\infty} \sum_{2^{j-1} \le n < 2^j} n^{\alpha p-1-2\alpha} l(n) EX_1^2 I_{(|X_1| \le n^\alpha)}$$

$$\ll \sum_{j=1}^{\infty} 2^{j\alpha(p-2)} l(2^j) EX_1^2 I_{(|X_1| \le 2^{\alpha j})}$$

$$\ll \sum_{j=1}^{\infty} 2^{\alpha(p-2)j} l(2^j) \sum_{k=1}^{j} EX_1^2 I_{(2^{\alpha(k-1)} < |X_1| \le 2^{\alpha k})}$$

$$= \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} 2^{\alpha(p-2)j} l(2^j) EX_1^2 I_{(2^{\alpha(k-1)} < |X_1| \le 2^{\alpha k})}$$

$$\ll \sum_{k=1}^{\infty} 2^{\alpha(p-2)k} l\left(2^k\right) E|X_1|^p 2^{\alpha(2-p)k} I_{\left(2^{\alpha(k-1)} < |X_1| \le 2^{\alpha k}\right)}$$

$$\ll E\left(|X_1|^p l\left(|X_1|^{1/\alpha}\right)\right) < \infty.$$

Hence, (2.8) holds.

Now we prove that (2.2) $\Rightarrow$ (2.1). Obviously (2.2) implies

$$\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\left(\max_{1\le k\le n}|X_k| \ge \varepsilon n^\alpha\right) < \infty, \quad \forall \varepsilon > 0. \qquad (2.9)$$

Noting $\alpha p - 2 > -1$, by Lemma 2.1 (ii), we have

$$\sum_{m=1}^{\infty} P\left(\max_{1\le j\le 2^m}|X_j| \ge \varepsilon 2^{\alpha(m+1)}\right)$$

$$\ll \sum_{m=0}^{\infty} \sum_{2^m \le n < 2^{m+1}} \frac{1}{n} P\left(\max_{1\le j\le n}|X_j| > \varepsilon n^\alpha\right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1\le j\le n}|X_j| \ge \varepsilon n^\alpha\right)$$

$$\le \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\left(\max_{1\le j\le n}|X_j| \ge \varepsilon n^\alpha\right) < \infty.$$

Thus,

$$\max_{2^{m-1} \le n < 2^m} P\left(\max_{1\le j\le n}|X_j| \ge \varepsilon 2^{2\alpha} n^\alpha\right)$$

$$\le P\left(\max_{1\le j<2^m}|X_j| \ge \varepsilon 2^{\alpha(m+1)}\right) \to 0.$$

Therefore, for sufficiently large *n*,

$$P\left(\max_{1\le j\le n}|X_j| \ge 2^{2\alpha} \varepsilon n^\alpha\right) < \frac{1}{2},$$

which, in conjunction with Lemma 1.2, gives

$$\sum_{k=1}^{n} P\left(|X_k| \ge \varepsilon 2^{2\alpha} n^\alpha\right) \le 4cP\left(\max_{1\le j\le n}|X_j| \ge 2^{2\alpha} \varepsilon n^\alpha\right).$$

Putting this one into (2.9), we get furthermore

$$\sum_{n=1}^{\infty} n^{\alpha p-1} l(n) P\left(|X_1| \geq 2^{2\alpha} \varepsilon n^{\alpha}\right) < \infty, \quad \forall \varepsilon > 0.$$

Thus, by Lemmas 2.1 (i), (iii),

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{\alpha p-1} l(n) P\left(|X_1| \geq 2^{2\alpha} \varepsilon n^{\alpha}\right) \\
&\geq \sum_{j=1}^{\infty} \sum_{2^j \leq n < 2^{j+1}} n^{\alpha p-1} l(n) P\left(|X_1| \geq 2^{2\alpha} \varepsilon n^{\alpha}\right) \\
&\gg \sum_{j=1}^{\infty} 2^{j\alpha p} l\left(2^j\right) P\left(|X_1| \geq \varepsilon 2^{\alpha} 2^{(j+1)\alpha} \hat{=} \varepsilon_0 2^{\alpha j}\right) \\
&= \sum_{j=1}^{\infty} 2^{\alpha pj} l\left(2^j\right) \sum_{k=j}^{\infty} P\left(\varepsilon_0 2^{\alpha k} \leq |X_1| < \varepsilon_0 2^{\alpha(k+1)}\right) \\
&= \sum_{k=1}^{\infty} \sum_{j=1}^{k} 2^{\alpha pj} l\left(2^j\right) P\left(\varepsilon_0 2^{\alpha k} \leq |X_1| < \varepsilon_0 2^{\alpha(k+1)}\right) \\
&\gg \sum_{k=1}^{\infty} 2^{\alpha pk} l\left(2^k\right) P\left(\varepsilon_0 2^{\alpha k} \leq |X_1| < \varepsilon_0 2^{\alpha(k+1)}\right) \\
&\gg E\left(|X_1|^p l\left(|X_1|^{1/\alpha}\right)\right).
\end{aligned}$$

This completes the proof of Theorem 2.1.

Proof of Theorem 2.2. (2.3) $\Rightarrow$ (2.4). Let $Y_i = Y_i^{(n)} = X_i I_{\left(|X_i| \leq n^{\alpha}\right)}$, $i = 1, 2, \cdots, n$, the method of proof of Theorem 2.2 is similar to method used to prove the above Theorem 2.1. Only the method of prove of (2.5) is not the same. In what follows, we prove that (2.5) holds. Since $l(x) > 0$ is a monotone non-decreasing function, we have

$$\begin{aligned}
|X_1|^{1/\alpha} &= |X_1|^{1/\alpha} I_{(|X_1| \leq 1)} \\
&\quad + |X_1|^{1/\alpha} l\left(|X_1|^{1/\alpha}\right) \frac{1}{l\left(|X_1|^{1/\alpha}\right)} I_{(|X_1| > 1)} \\
&\leq 1 + |X_1|^{1/\alpha} l\left(|X_1|^{1/\alpha}\right) \frac{1}{l(1)}.
\end{aligned}$$

Hence, by (2.3),

$$E|X_1|^{1/\alpha} < \infty. \tag{2.10}$$

i) For $\alpha \leq 1$, by $EX_1 = 0$ and (2.10),

$$\begin{aligned}
n^{-\alpha} \max_{1 \leq j \leq n} \left|\sum_{i=1}^{j} EY_i\right| &\leq n^{1-\alpha} \left|EX_1 I_{(|X_1| > n^{\alpha})}\right| \\
&\leq n^{1-\alpha} E|X_1| \left(\frac{|X_1|}{n^{\alpha}}\right)^{1/\alpha - 1} I_{(|X_1| > n^{\alpha})} \\
&= E|X_1|^{1/\alpha} I_{(|X_1| > n^{\alpha})} \to 0,
\end{aligned}$$

ii) For $\alpha > 1$, *i.e.*, $1/\alpha < 1$,

$$\begin{aligned}
n^{-\alpha} \max_{1 \leq j \leq n} \left|\sum_{i=1}^{j} EY_i\right| &\leq n^{1-\alpha} |EY_1| \\
&\leq n^{1-\alpha} E|X_1| I_{\left(|X_1| \leq n^{\alpha}\right)} \\
&= n^{1-\alpha} \sum_{i=1}^{n} E|X_1| I_{\left((i-1)^{\alpha} < |X_1| \leq i^{\alpha}\right)} \to 0,
\end{aligned}$$

from the Kronecker lemma and

$$\begin{aligned}
&\sum_{i=1}^{\infty} i^{1-\alpha} E|X_1| I_{\left((i-1)^{\alpha} < |X_1| \leq i^{\alpha}\right)} \\
&\leq \sum_{i=1}^{\infty} E|X_1|^{1/\alpha} I_{\left((i-1)^{\alpha} < |X_1| \leq i^{\alpha}\right)} \\
&= E|X_1|^{1/\alpha} < \infty.
\end{aligned}$$

Hence (2.5) holds. The rest of the proof is similar to the corresponding part of the proof of Theorem 2.1, so we omit it.

3. Weak Law of Large Numbers

Theorem 3.1. *Suppose* $p > 1/2$. *Let* $\{X_n; n \geq 1\}$ *be a* $\tilde{\rho}$ *-mixing sequence of identically distributed random variables satisfying*

$$\lim_{n \to \infty} nP\left(|X_1| > n^p\right) = 0. \tag{3.1}$$

Then

$$\frac{S_n}{n^p} - n^{1-p} EX_1 I_{\left(|X_1| \leq n^p\right)} \xrightarrow{P} 0. \tag{3.2}$$

Remark 3.1. When $p = 1$ and $\{X_n; n \geq 1\}$ i.i.d., then Theorem 3.1 is the weak law of large numbers (WLLN) due to Feller [11]. So, Theorem 3.1 extends the sufficient part of the Feller's WLLN from the i.i.d. case to a $\tilde{\rho}$ -mixing setting.

Proof of Theorem 3.1. Let $X_j' = X_j I_{\left(|X_j| \leq n^p\right)}$ for $1 \leq j \leq n$ and $S_n' = \sum_{j=1}^{n} X_j'$. Then, for each $n \geq 2$, $\{X_j'; 1 \leq j \leq n\}$ are $\tilde{\rho}$ -mixing identically distributed random variables and for every $\varepsilon > 0$,

$$\begin{aligned}
P\left(\left|\frac{S_n}{n^p} - \frac{S_n'}{n^p}\right| > \varepsilon\right) &\leq P\left(\frac{S_n}{n^p} \neq \frac{S_n'}{n^p}\right) \\
&= P\left(\bigcup_{j=1}^{n} \left(X_j \neq X_j'\right)\right) \\
&\leq \sum_{j=1}^{n} P\left(|X_j| > n^p\right) = nP\left(|X_1| > n^p\right) \to 0,
\end{aligned}$$

via (3.1). So that (3.1) entails

$$\frac{S_n'}{n^p} - \frac{S_n}{n^p} \xrightarrow{P} 0.$$

Thus, to prove (3.2) it suffices to verify that

$$\frac{S_n'}{n^p} - n^{1-p} EX_1 I_{(|X_1| \le n^p)} \xrightarrow{P} 0. \qquad (3.3)$$

By (3.1) and the Toeplitz lemma,

$$\frac{\sum_{k=1}^{n} k^{2p-2} \cdot kP(|X_1| > k^p)}{\sum_{k=1}^{n} k^{2p-2}} \to 0, \; n \to \infty.$$

Thus, together with $\sum_{k=1}^{n} k^{2p-2} \ll n^{2p-1}$ for $p > 1/2$, we have

$$n^{-2p+1} \sum_{k=1}^{n} k^{2p-1} P(|X_1| > k^p) \to 0, \; n \to \infty,$$

which, in conjunction with Lemma 1.1, yields for every $\varepsilon > 0$,

$$\begin{aligned}
P(|S_n' - ES_n'| > \varepsilon n^p) &\ll n^{-2p} E(S_n' - ES_n')^2 \\
&= n^{-2p} E\left(\sum_{j=1}^{n} (X_j' - EX_j')\right)^2 \\
&\ll n^{-2p} \sum_{j=1}^{n} E(X_j' - EX_j')^2 \le n^{-2p+1} EX_1'^2 \\
&= n^{-2p+1} EX_1^2 I_{(|X_1| \le n^p)} \\
&= n^{-2p+1} \sum_{k=1}^{n} EX_1^2 I_{((k-1)^p < |X_1| \le k^p)} \\
&\le n^{-2p+1} \sum_{k=1}^{n} k^{2p} \left(P(|X_1| > (k-1)^p) - P(|X_1| > k^p) \right) \\
&= n^{-2p+1} \left(\sum_{k=1}^{n-1} (k+1)^{2p} - k^{2p} P(|X_1| > k^p) \right. \\
&\qquad \left. + P(|X_1| > 0) - n^{2p} P(|X_1| > n^p) \right) \\
&\ll n^{-2p+1} \left(\sum_{k=1}^{n} k^{2p-1} P(|X_1| > k^p) + 1 \right) \to 0.
\end{aligned}$$

Thus

$$\frac{S_n' - ES_n'}{n^p} = \frac{S_n'}{n^p} - n^{1-p} EX_1 I_{(|X_1| \le n^p)} \xrightarrow{P} 0.$$

i.e. (3.3) holds.

4. Examples

In this section, we give two examples to show our Theorems.

Example 4.1. Let $\{X_n; n \ge 1\}$ be a $\tilde{\rho}$-mixing sequence of identically distributed random variables. Suppose $0 < p < 2$ and $\alpha p \ge 1$; and if $\alpha \le 1$ then suppose also that $EX_1 = 0$. Assume that $l(x) = \log^r x, r > 0$ and X_1 has a distribution with

$$P(|X_1| > x) \sim \frac{1}{x^{1/\alpha} \log^\beta x}, \; \beta > r + 1.$$

Is easy to verify that $l(x)$ satisfies the conditions of Theorems 2.1 and 2.2, and

$$E\left(X_1^p l\left(|X_1|^{1/\alpha} \right) \right) < \infty.$$

Thus, by Theorems 2.1 and 2.2,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} \log^r n P\left(\max_{1 \le j \le n} |S_j| > \varepsilon n^\alpha \right) < \infty, \; \forall \varepsilon > 0.$$

Example 4.2. Suppose $p > 1/2$. Let $\{X_n; n \ge 1\}$ be a $\tilde{\rho}$-mixing sequence of identically distributed random variables. Assume that X_1 has a distribution with

$$P(|X_1| > x) = o\left(\frac{1}{x^{1/p}} \right),$$

then obviously,

$$\lim_{n \to \infty} nP(|X_1| > n^p) = 0.$$

Thus, by Theorem 3.1,

$$\frac{S_n}{n^p} - n^{1-p} EX_1 I_{(|X_1| \le n^p)} \xrightarrow{P} 0.$$

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REFERENCES

[1] W. Bryc and W. Smolenski, "Moment Conditions for Almost Sure Convergence of Weakly Correlated Random Variables," *Proceedings of the American Mathematical Society*, Vol. 199, No. 2, 1993, pp. 629-635. doi:10.1090/S0002-9939-1993-1149969-7

[2] R. C. Bradley, "On the Spectral Density and Asymptotic Normality of Weakly Dependent Random Fields," *Journal of Theoretical Probability*, Vol. 5, No. 2, 1992, pp. 355-373.

[3] S. C. Yang, "Some Moment Inequalities for Partial Sums of Random Variables and Their Applications," *Chinese Science Bulletin*, Vol. 43, No. 17, 1998, pp. 1823-1827. doi:10.1007/BF02883381

[4] Q. Y. Wu and Y. Y. Jiang, "Some Strong Limit Theorems for $\tilde{\rho}$-Mixing Sequences of Random Variables," *Statistics & Probability Letters*, Vol. 78, No. 8, 2008, pp. 1017-1023.

[5] Q. Y. Wu and Y. Y. Jiang, "Some Strong Limit Theorems for Weighted Product Sums of $\tilde{\rho}$-Mixing Sequences of Random Variables," *Journal of Inequalities and Applications*, 2009. doi:10.1155/2009/174768

[6] Q. Y. Wu and Y. Y. Jiang, "Chover-Type Laws of the K-Iterated Logarithm for $\tilde{\rho}$-Mixing Sequences of Random Variables," *Journal of Mathematical Analysis and Applications*, Vol. 366, No. 2, 2010, pp. 435-443. doi: 10.1016/j.jmaa.2009.

[7] Q. Y. Wu, "Further Study Strong Consistency of Estimator in Linear Model for $\tilde{\rho}$-Mixing Random Samples," *Journal of Systems Science and Complexity*, Vol. 24, No. 5, 2011, pp. 969-980. doi:10.1007/s11424-011-8407-7

[8] M. Peligrad and A. Gut, "Almost-Sure Results for a Class of Dependent Random Variables," *Journal of Theoretical Probability*, Vol. 12, No. 1, 1999, pp. 87-104.

[9] S. X. Gan, "Almost Sure Convergence for $\tilde{\rho}$-Mixing Random Variable Sequences," *Statistics and Probability Letters*, Vol. 67, 2004, pp. 289-298.

[10] S. Utev and M. Peligrad, "Maximal Inequalities and an Invariance Principle for a Class of Weakly Dependent Random Variables," *Journal of Theoretical Probability*, Vol. 16, No. 1, 2003, pp. 101-115. doi:10.1023/A:1022278404634

[11] W. Feller, "A Limit Theorem for Random Variables with Infinite Moments," *American Journal of Mathematics*, Vol. 68, 1946, pp. 257-262.

[12] L. E. Baum and M. Katz, "Convergence Rates in the Law of Large Numbers," *Transactions of the American Mathematical Society*, Vol. 120, No. 1, 1965, pp. 108-123.

[13] Z. D. Bai and C. Su, "The Complete Convergence for Partial Sums of iid Random Variables," *Science in China, Series A*, Vol. 5, 1985, pp. 309-412.

