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PC-VAR Estimation of Vector Autoregressive Models*

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ABSTRACT

In this paper PC-VAR estimation of vector autoregressive models (VAR) is proposed. The estimation strategy successfully lessens the curse of dimensionality affecting VAR models, when estimated using sample sizes typically available in quarterly studies. The procedure involves a dynamic regression using a subset of principal components extracted from a vector time series, and the recovery of the implied unrestricted VAR parameter estimates by solving a set of linear constraints. PC-VAR and OLS estimation of unrestricted VAR models show the same asymptotic properties. Monte Carlo results strongly support PC-VAR estimation, yielding gains, in terms of both lower bias and higher efficiency, relatively to OLS estimation of high dimensional unrestricted VAR models in small samples. Guidance for the selection of the number of components to be used in empirical studies is provided.

Keywords: Vector Autoregressive Model; Principal Components Analysis; Statistical Reduction Techniques

1. Introduction

In this paper principal components vector autoregressive estimation (PC-VAR) for large scale dynamic econometric models is proposed. Vector autoregressive models (VAR), when estimated using economic time series of sample sizes typically available in empirical quarterly studies, are subject to the curse of dimensionality. Recent contributions, dealing with this issue in the framework of factor augmented VAR (FVAR) models ([1-4]; see also [5] for a survey and new results), have mostly relied on principal components (PC) estimation (in the frequency or time domain) of the underlying unobserved common factor structure. Results of [1,6,7] have in fact proved consistency and asymptotic normality of PC estimation under various scenarios, including the exact and approximate factor model case, weakly stationary (short memory) and $I(1)$ integrated processes, also showing conditional heteroskedasticity; see also [8] for additional implications of temporal dependence for PC estimation.

The proposed approach is different from FVAR modeling, as no reference to an underlying factor structure is made, and PC estimation is performed to yield accurate estimation of the parameters of an unrestricted VAR model, rather than of a dynamic factor model. The procedure involves a dynamic regression using a subset of principal components extracted from a vector time series, and the recovery of the implied unrestricted VAR parameter estimates by solving a set of linear constraints. PC-VAR and OLS estimation of unrestricted VAR models show the same asymptotic properties.

Monte Carlo results strongly support PC-VAR estimation, yielding gains, in terms of both lower bias and higher efficiency, relatively to OLS estimation of high dimensional unrestricted VAR models in small samples. Guidance for the selection of the number of components to be used in empirical studies is provided.

After this introduction, the paper is organized as follows. In section two PC-VAR estimation is presented, while in section three Monte Carlo analysis is performed; see [9] for an empirical application of the procedure.

2. PC-VAR Estimation of VAR Models

Consider the vector autoregressive (VAR) model

$$\begin{aligned} \left(\boldsymbol{I}-\boldsymbol{P}(\boldsymbol{L})\right)\boldsymbol{x}_t &= \boldsymbol{\eta}_t \\ \boldsymbol{\eta}_t &\sim i.i.d.\left(\boldsymbol{0},\boldsymbol{\Sigma}_\eta\right), \end{aligned} \tag{1}$$

where $\boldsymbol{x}_t$ is an $r\times 1$ zero mean $I(0)$ vector process, $t=1,\cdots,T$, and $\boldsymbol{P}(L)\equiv \boldsymbol{P}_1 L+\boldsymbol{P}_2 L^2+\cdots+\boldsymbol{P}_p L^p$ has all the roots outside the unit circle, with $\boldsymbol{P}_j$, $j=1,\cdots,p$, being a square matrix of coefficients of order r.

PC-VAR estimation relies on the following identity

$$\boldsymbol{x}_t \equiv \hat{\boldsymbol{\Xi}}\hat{\mathbf{f}}_t, \tag{2}$$

where $\hat{\mathbf{f}}_t=\hat{\boldsymbol{\Xi}}'\boldsymbol{x}_t$ is the $r\times 1$ vector of estimated principal components of x_t, $\hat{\Xi}$ is the $r \times r$ matrix of orthogonal eigenvectors associated with the r (ordered) eigenvalues of $\hat{\Sigma}$ ($\Sigma = E[x_t x_t']$). This follows from the eigenvalue-eigenvector decomposition of Σ, *i.e.*, $\hat{\Xi}^{-1}\hat{\Sigma}\hat{\Xi} = \hat{\Gamma}$, where $\hat{\Gamma} = \text{diag}(\hat{\gamma}_1, \cdots, \hat{\gamma}_r)$ is the $r \times r$ diagonal matrix containing the (ordered) eigenvalues of $\hat{\Sigma}$.

The validity of PC estimation for weakly dependent processes follows from results in [7] and [8]. In particular, in [7], under some general conditions, $\sqrt{r}$ consistency and asymptotic normality of PC estimation of the unobserved common factors has been established, at each point in time, for $r, T \to \infty$ and $\sqrt{r}/T \to 0$, when both the unobserved factors and the idiosyncratic components show limited serial correlation, and the latter also display limited heteroskedasticity in both their time-series and cross-sectional dimensions (see Theorem 1, p. 145); moreover, the invariance of the singular value decomposition to row ordering is discussed in [8] (see the Lemma on the eigenvalues matrix in [8], p. 175).

PC-VAR estimation of $\mathbf{P}(L)$ is then be implemented as follows:

1) apply PCA to x_t and compute $\hat{\mathbf{f}}_t = \hat{\Xi}' x_t$;

2) obtain $D(L)$ by means of OLS estimation of the stationary dynamic vector regression model

$$x_t = D(L)\hat{\mathbf{f}}_t + \varepsilon_t \qquad \varepsilon_t \sim i.i.d.(\mathbf{0}, \Sigma_\varepsilon) \tag{3}$$

where $D(L) \equiv D_1 L + D_2 L^2 + \cdots + D_p L^p$ has all the roots outside the unit circle;

3) recover the (implied OLS) estimate of the actual parameters yield by the unrestricted VAR model in (1) by solving the linear constraints

$$\hat{\mathbf{P}}(L)_{PCVAR} = \hat{\mathbf{D}}(L)\hat{\Xi}'. \tag{4}$$

Note that, by construction, the PC-VAR estimator and the OLS estimator of the unrestricted VAR model in (1) are the same estimator, *i.e.*, $\hat{\mathbf{P}}(L)_{OLS} = \hat{\mathbf{P}}(L)_{PCVAR}$.

In fact, substituting (2) in (1) yields

$$x_t = P(L)\hat{\Xi}\hat{\mathbf{f}}_t + \eta_t, \tag{5}$$

i.e., the dynamic vector regression in (3), with $D(L) = P(L)\hat{\Xi}$ and $\eta_t = \varepsilon_t$.

The implied $\mathbf{P}(L)$ matrix is then estimated by computing

$$\hat{\mathbf{D}}(L)\hat{\Xi}' = \hat{\mathbf{P}}(L)\hat{\Xi}\hat{\Xi}' = \hat{\mathbf{P}}(L),$$

as $\hat{\Xi}\hat{\Xi}' = I_r$ due to the orthonormality of the eigenvectors. The PC-VAR estimator would therefore show the same asymptotic properties of the OLS estimator.

The case considered is however of no interest for empirical implementations, as it does not allow for any dimensionality reduction, relatively to the estimation of the unrestricted VAR model.

2.1. The Unfeasible Case

Consider the case in which only the first s, $s < r$, principal components associated with the s largest ordered eigenvalues of $\hat{\Sigma}$ are considered, with $\hat{\gamma}_j = 0$, $j = s + 1, \cdots, r$. The same results as obtained above ($s = r$, implicitly) would hold.

Rewrite the identity in (2) as

$$x_t = \hat{\Xi}_s \hat{\mathbf{f}}_{s,t} + \hat{\Xi}_{r-s}\hat{\mathbf{f}}_{r-s,t} = x_{*,t} + \tau_t = x_{*,t} \tag{6}$$

where $x_{*,t} \equiv \hat{\Xi}_s \hat{\mathbf{f}}_{s,t}$, $\tau_t \equiv \hat{\Xi}_{r-s}\hat{\mathbf{f}}_{r-s,t} = \mathbf{0}$ as $\hat{\mathbf{f}}_{r-s,t} = \mathbf{0}$,

$$\underset{(r\times 1)}{\hat{\mathbf{f}}_t} = \begin{bmatrix} \underset{(1\times s)}{\hat{\mathbf{f}}'_{s,t}} & \underset{(1\times r-s)}{\hat{\mathbf{f}}'_{r-s,t}} \end{bmatrix}', \quad \underset{(r\times r)}{\hat{\Xi}} = \begin{bmatrix} \underset{(r\times s)}{\hat{\Xi}_s} & \underset{(r\times r-s)}{\hat{\Xi}_{r-s}} \end{bmatrix}'.$$

Then, substituting (6) in (1) yields

$$x_t = P(L)(x_{*,t} + \tau_t) + \eta_t = P(L)x_{*,t} + \eta_t. \tag{7}$$

PC-VAR would then entail OLS estimation of

$$x_t = D(L)\hat{\mathbf{f}}_{s,t} + \varepsilon_t \qquad \varepsilon_t \sim i.i.d.(\mathbf{0}, \Sigma_\varepsilon). \tag{8}$$

It then follows

$$\hat{\mathbf{D}}_*(L)\hat{\Xi}' = \hat{\mathbf{P}}_*(L)\hat{\Xi}' = \hat{\mathbf{P}}(L) \odot \hat{\Xi}\hat{\Xi}' = \hat{\mathbf{P}}(L),$$

i.e.,

$$\hat{\mathbf{P}}(L)_{PCVAR} = \hat{\mathbf{D}}_*(L)\hat{\Xi}' = \hat{\mathbf{D}}(L)\hat{\Xi}_s', \tag{9}$$

where $D_*(L) \equiv D_{*1}L + D_{*2}L^2 + \cdots + D_{*p}L^p$,

$$D_{*j}(L) = \begin{bmatrix} \underset{(r\times s)}{D_j(L)} & \underset{(r\times r-s)}{\mathbf{0}} \end{bmatrix}, \; j = 1, \cdots, p,$$

$$P_*(L) \equiv P_{*1}L + P_{*2}L^2 + \cdots + P_{*p}L^p$$

$$P_{*j}(L) = \begin{bmatrix} \underset{(r\times s)}{P_j\hat{\Xi}_s} & \underset{(r\times r-s)}{\mathbf{0}} \end{bmatrix} = \begin{bmatrix} \underset{(r\times r)}{P_j} & \underset{(r\times r-s)}{\mathbf{0}} \end{bmatrix} \odot \hat{\Xi}, \; j = 1, \cdots, p,$$

and $\odot$ is the Hadamart product.

2.2. The Feasible Case

Consider the case in which only the first s, $s < r$, principal components associated with the s largest ordered eigenvalues of $\hat{\Sigma}$ are considered, with $\hat{\gamma}_j \simeq 0$, $j = s + 1, \cdots, r$.

By rewriting (7) as

$$\begin{aligned} \mathbf{x}_t &= P(L)x_{*,t} + P(L)\tau_t + \eta_t \\ &= P(L)\hat{\Xi}_s\hat{\mathbf{f}}_{s,t} + P(L)\hat{\Xi}_{r-s}\hat{\mathbf{f}}_{r-s,t} + \eta_t \\ &= P(L)\hat{\Xi}_s\hat{\mathbf{f}}_{s,t} + \lambda_t, \end{aligned}$$

where $\lambda_t = \eta_t + P(L)\tau_t \simeq \eta_t$, as, for $\gamma_j = 0$, $\hat{\mathbf{f}}_{r-s,t} = \mathbf{0}$, consistency of the PC-VAR estimator in (9), obtained from OLS estimation of (8), would require the limiting uncorrelation condition $\text{plim}\left(\hat{\mathbf{f}}_s'\boldsymbol{\lambda}/T\right)=\mathbf{0}$ to hold, where $\hat{\mathbf{f}}_s=\left[\hat{\mathbf{f}}_{s,-1}\ \cdots\ \hat{\mathbf{f}}_{s,-p}\right]$ is the $T\times(s\times p)$ design matrix containing the temporal information on the lagged principal components and $\boldsymbol{\lambda}$ is the $T\times 1$ vector containing the temporal information on the error process. The latter condition would necessarily hold for the $p=1$ case, as $\text{plim}\left(\boldsymbol{x}_{*,t-1}\boldsymbol{\tau}_{t-1}/T\right)=\mathbf{0}$ by construction, due to the orthogonality of $\hat{\mathbf{f}}_{s,t}$ and $\hat{\mathbf{f}}_{r-s,t}$, and therefore of $\boldsymbol{x}_{*,t}$ and $\boldsymbol{\tau}_t$. The condition $\text{plim}\left(\boldsymbol{x}_{*,t-i}\boldsymbol{\tau}_{t-j}/T\right)=\mathbf{0}$, $i,j=1,\cdots,p$, $i\neq j$, would on the other hand appear to be required for the $p>1$ case. As under the weak stationarity assumption, for any generic element in the $\boldsymbol{x}_{*,t}$ and $\boldsymbol{\tau}_t$ vectors, the Wold decomposition would yield

$$x_{*m,t}=\gamma(L)\varepsilon_{x_{*m},t}\qquad m=1,\cdots,s$$

$$\varepsilon_{x_{*m},t}\sim w.n.\left(0,\sigma^2_{\varepsilon_{x*m}}\right)$$

$$\tau_{n,t}=\theta(L)\varepsilon_{\tau_n,t}\qquad n=1,\cdots,r-s$$

$$\varepsilon_{\tau_n,t}\sim w.n.\left(0,\sigma^2_{\varepsilon_{\tau_n}}\right),$$

with $\gamma(L)$ and $\theta(L)$ stationary infinite order polynomials in the lag operator, provided $E\left[\varepsilon_{x_{*m},t},\varepsilon_{\tau_n,t}\right]=0$, the necessary conditions for consistency would then be satisfied. Asymptotic normality would also follow under the same conditions of validity of OLS estimation of unrestricted VAR models.

3. Monte Carlo Results

Consider the following data generation process (DGP) for the $n\times 1$ vector process $\boldsymbol{x}_t$

$$\boldsymbol{\Phi}(L)\boldsymbol{x}_t=\boldsymbol{v}_t$$

$$\boldsymbol{v}_t\sim \text{n.i.d.}\left(\mathbf{0},\boldsymbol{\Sigma}_v\right),$$

where $n=25$, $\boldsymbol{\Phi}(L)=\mathbf{I}-\sum_{j=1}^{p}\boldsymbol{\Phi}_jL^j$, $p=\{1,\cdots,4\}$, is a polynomial matrix in the lag operator, where the $\boldsymbol{\Phi}_j$ coefficient matrices contain randomly extracted values from the interval (−0.4,0.4), constrained to yield a weakly stationary vector autoregressive process;

$$\boldsymbol{\Sigma}_v=\begin{bmatrix}1 & \rho_{1,2} & \cdots & \rho_{1,n}\\ \rho_{1,2} & 1 & \ddots & \vdots\\ \vdots & \vdots & \ddots & \rho_{n-1,n}\\ \rho_{1,n} & \cdots & \rho_{n-1,n} & 1\end{bmatrix},$$

with $\rho_{i,j}$ coefficients randomly extracted from the interval (−1,1) and constrained to yield an average absolute off-diagonal element (correlation coefficient)

$$\frac{1}{\left(n(n-1)/2\right)}\sum_{i=1}^{n}\sum_{j>i}^{n}\left|\rho_{i,j}\right|=k,$$

$k=\{0,0.05,0.10,0.15,0.20,0.25,0.30\}$, covering the cases of main interest.

The estimated models are the PC-VAR(*p*,*r*) models, considering r principal components, $r=2,4,\cdots,24$ and $p=1,\cdots,4$ lags, and the unrestricted VAR(*p*) model, equivalent to the PC-VAR(*p*,*r*) model with $r=n$ (25). The temporal (usable) sample size is $T=100$ and the number of replications is 10,000.

System level simulation results, *i.e.*, the mean absolute bias and root mean square error, across parameters and equations, are reported in **Tables 1** and **2**.

As shown in **Tables 1** and **2**, PC-VAR estimation improves upon unrestricted OLS VAR estimation in terms

Table 1. Monte Carlo results.[a]

ρ = 0.00	***# PC*** ***(explained total variance)***												
	2 (0.17)	**4** (0.31)	**6** (0.43)	**8** (0.53)	**10** (0.62)	**12** (0.70)	**14** (0.77)	**16** (0.83)	**18** (0.88)	**20** (0.93)	**22** (0.96)	**24** (0.99)	**25** (1.0)
							***p* = 1**						
Bias	0.025	0.023	0.021	0.019	0.017	0.016	0.014	0.013	0.012	0.012	0.012	0.012	0.012
RMSE	0.054	0.060	0.064	0.067	0.071	0.074	0.078	0.083	0.088	0.095	0.102	0.112	0.118
							***p* = 2**						
Bias	0.024	0.022	0.020	0.019	0.017	0.015	0.014	0.013	0.012	0.012	0.012	0.013	0.014
RMSE	0.051	0.057	0.062	0.067	0.072	0.077	0.083	0.090	0.098	0.108	0.119	0.134	0.142
							***p* = 3**						
Bias	0.023	0.022	0.020	0.018	0.017	0.016	0.014	0.014	0.013	0.014	0.015	0.017	0.019
RMSE	0.049	0.056	0.062	0.068	0.074	0.082	0.090	0.101	0.114	0.130	0.150	0.178	0.197
							***p* = 4**						
Bias	0.022	0.021	0.019	0.018	0.017	0.016	0.015	0.015	0.015	0.017	0.021	0.030	0.044
RMSE	0.047	0.054	0.061	0.069	0.077	0.087	0.100	0.116	0.138	0.169	0.221	0.326	0.499

Continued

ρ = 0.05	# *PC* (*explained total variance*)												
	2 (0.19)	**4** (0.33)	**6** (0.45)	**8** (0.55)	**10** (0.64)	**12** (0.72)	**14** (0.78)	**16** (0.84)	**18** (0.89)	**20** (0.93)	**22** (0.96)	**24** (0.99)	**25** (1.0)
							$p = 1$						
Bias	0.028	0.026	0.024	0.022	0.020	0.017	0.016	0.014	0.013	0.012	0.012	0.012	0.013
RMSE	0.053	0.060	0.064	0.067	0.071	0.075	0.079	0.084	0.090	0.096	0.105	0.115	0.121
							$p = 2$						
Bias	0.026	0.024	0.022	0.020	0.019	0.017	0.015	0.014	0.013	0.012	0.012	0.013	0.014
RMSE	0.050	0.056	0.062	0.067	0.072	0.078	0.084	0.091	0.100	0.110	0.122	0.137	0.146
							$p = 3$						
Bias	0.025	0.023	0.022	0.020	0.018	0.017	0.016	0.014	0.014	0.014	0.015	0.017	0.019
RMSE	0.048	0.055	0.061	0.067	0.074	0.082	0.091	0.102	0.115	0.132	0.153	0.182	0.202
							$p = 4$						
Bias	0.024	0.022	0.021	0.019	0.018	0.017	0.016	0.015	0.016	0.017	0.021	0.030	0.044
RMSE	0.046	0.053	0.061	0.068	0.077	0.087	0.100	0.117	0.140	0.173	0.226	0.334	0.511

ρ = 0.10	# *PC* (*explained total variance*)												
	2 (0.23)	**4** (0.39)	**6** (0.51)	**8** (0.60)	**10** (0.68)	**12** (0.75)	**14** (0.81)	**16** (0.86)	**18** (0.91)	**20** (0.94)	**22** (0.97)	**24** (0.99)	**25** (1.0)
							$p = 1$						
Bias	0.030	0.029	0.027	0.025	0.022	0.020	0.017	0.015	0.014	0.013	0.012	0.013	0.013
RMSE	0.052	0.058	0.063	0.068	0.072	0.076	0.081	0.087	0.094	0.101	0.111	0.122	0.129
							$p = 2$						
Bias	0.027	0.027	0.025	0.023	0.021	0.019	0.017	0.015	0.014	0.013	0.013	0.014	0.015
RMSE	0.048	0.055	0.061	0.067	0.073	0.079	0.086	0.095	0.104	0.115	0.129	0.145	0.155
							$p = 3$						
Bias	0.026	0.026	0.024	0.022	0.020	0.018	0.017	0.016	0.015	0.015	0.016	0.018	0.020
RMSE	0.046	0.053	0.060	0.067	0.075	0.084	0.094	0.106	0.120	0.139	0.162	0.193	0.214
							$p = 4$						
Bias	0.025	0.024	0.023	0.021	0.019	0.018	0.017	0.016	0.016	0.018	0.022	0.032	0.047
RMSE	0.044	0.052	0.059	0.068	0.077	0.089	0.103	0.121	0.146	0.181	0.238	0.355	0.542

ρ = 0.15	# *PC* (*explained total variance*)												
	2 (0.29)	**4** (0.46)	**6** (0.58)	**8** (0.67)	**10** (0.74)	**12**(0.80)	**14** (0.85)	**16** (0.89)	**18** (0.92)	**20** (0.95)	**22** (0.98)	**24** (0.99)	**25** (1.0)
							$p = 1$						
Bias	0.031	0.031	0.030	0.027	0.025	0.022	0.019	0.017	0.015	0.014	0.013	0.014	0.014
RMSE	0.051	0.057	0.063	0.068	0.073	0.079	0.085	0.092	0.101	0.110	0.121	0.135	0.143
							$p = 2$						
Bias	0.028	0.028	0.027	0.025	0.022	0.020	0.018	0.016	0.015	0.014	0.014	0.015	0.016
RMSE	0.047	0.053	0.060	0.067	0.074	0.082	0.090	0.100	0.112	0.125	0.140	0.159	0.171
							$p = 3$						
Bias	0.027	0.027	0.025	0.024	0.022	0.020	0.018	0.017	0.016	0.016	0.017	0.020	0.022
RMSE	0.045	0.052	0.059	0.067	0.076	0.086	0.098	0.112	0.129	0.150	0.177	0.212	0.236
							$p = 4$						
Bias	0.025	0.025	0.024	0.022	0.021	0.019	0.018	0.017	0.018	0.020	0.024	0.034	0.051
RMSE	0.043	0.050	0.058	0.068	0.079	0.092	0.109	0.130	0.158	0.197	0.260	0.390	0.600

Continued

ρ = 0.20	# PC (explained total variance)												
	2(0.35)	**4** (0.54)	**6** (0.66)	**8** (0.74)	**10** (0.80)	**12** (0.85)	**14** (0.89)	**16** (0.92)	**18** (0.94)	**20** (0.97)	**22** (0.98)	**24** (1.0)	**25** (1.0)
							***p* = 1**						
Bias	0.031	0.032	0.031	0.029	0.027	0.024	0.021	0.019	0.017	0.015	0.015	0.016	0.016
RMSE	0.050	0.056	0.062	0.068	0.075	0.083	0.092	0.101	0.112	0.124	0.139	0.156	0.166
							***p* = 2**						
Bias	0.028	0.028	0.028	0.026	0.024	0.022	0.020	0.018	0.017	0.016	0.016	0.017	0.018
RMSE	0.046	0.052	0.059	0.067	0.076	0.086	0.097	0.110	0.124	0.141	0.160	0.183	0.198
							***p* = 3**						
Bias	0.027	0.027	0.026	0.025	0.023	0.021	0.020	0.018	0.018	0.018	0.019	0.022	0.025
RMSE	0.044	0.050	0.058	0.067	0.078	0.091	0.106	0.124	0.144	0.170	0.202	0.244	0.273
							***p* = 4**						
Bias	0.025	0.026	0.025	0.024	0.022	0.021	0.019	0.019	0.019	0.022	0.027	0.039	0.053
RMSE	0.042	0.049	0.057	0.067	0.081	0.097	0.117	0.142	0.175	0.222	0.296	0.447	0.692

ρ = 0.25	# PC (explained total variance)												
	2 (0.41)	**4** (0.63)	**6** (0.75)	**8** (0.83)	**10** (0.88)	**12** (0.91)	**14** (0.94)	**16** (0.96)	**18** (0.97)	**20** (0.98)	**22** (0.99)	**24** (1.0)	**25** (1.0)
							***p* = 1**						
Bias	0.031	0.032	0.032	0.031	0.030	0.028	0.025	0.023	0.021	0.020	0.019	0.021	0.022
RMSE	0.049	0.055	0.061	0.068	0.078	0.091	0.106	0.124	0.144	0.167	0.193	0.222	0.240
							***p* = 2**						
Bias	0.028	0.029	0.029	0.028	0.026	0.025	0.023	0.022	0.021	0.020	0.021	0.023	0.025
RMSE	0.045	0.051	0.058	0.067	0.078	0.093	0.111	0.133	0.157	0.185	0.217	0.255	0.278
							***p* = 3**						
Bias	0.027	0.027	0.027	0.026	0.025	0.024	0.023	0.022	0.022	0.023	0.025	0.030	0.033
RMSE	0.043	0.049	0.057	0.067	0.081	0.100	0.123	0.151	0.184	0.225	0.274	0.339	0.383
							***p* = 4**						
Bias	0.026	0.026	0.025	0.025	0.024	0.023	0.022	0.022	0.024	0.028	0.036	0.053	0.081
RMSE	0.042	0.048	0.055	0.067	0.084	0.107	0.138	0.176	0.227	0.297	0.406	0.623	0.977

ρ = 0.30	# PC (explained total variance)												
	2 (0.44)	**4** (0.67)	**6** (0.79)	**8** (0.86)	**10** (0.88)	**12** (0.91)	**14** (0.94)	**16** (0.96)	**18** (0.97)	**20** (0.98)	**22** (0.99)	**24** (1.0)	**25** (1.0)
							***p* = 1**						
Bias	0.031	0.032	0.032	0.031	0.030	0.029	0.028	0.026	0.025	0.025	0.026	0.029	0.031
RMSE	0.049	0.055	0.061	0.069	0.079	0.095	0.115	0.141	0.174	0.213	0.260	0.313	0.346
							***p* = 2**						
Bias	0.028	0.029	0.029	0.028	0.027	0.026	0.025	0.024	0.024	0.025	0.027	0.031	0.034
RMSE	0.045	0.051	0.057	0.067	0.080	0.098	0.121	0.151	0.189	0.233	0.286	0.348	0.386
							***p* = 3**						
Bias	0.027	0.028	0.027	0.027	0.026	0.025	0.025	0.025	0.026	0.028	0.033	0.040	0.045
RMSE	0.043	0.049	0.056	0.067	0.083	0.106	0.137	0.176	0.226	0.289	0.367	0.468	0.535
							***p* = 4**						
Bias	0.026	0.026	0.026	0.025	0.024	0.024	0.024	0.025	0.029	0.035	0.047	0.071	0.111
RMSE	0.041	0.047	0.055	0.067	0.086	0.115	0.154	0.208	0.282	0.386	0.546	0.862	1.365

[a]The Table reports Monte Carlo (absolute) bias and RMSE statistics (average across parameters and equations) from PC-VAR and unrestricted OLS VAR estimation of first, second, third, and fourth order systems. The average absolute residuals correlation coefficient is $\boldsymbol{\rho}$ = (0.00, 0.05, 0.10, 0.15, 0.20, 0.25, 0.30), the temporal sample size is $\boldsymbol{T}$ = 100, the cross-sectional sample size is $\boldsymbol{n}$ = 25, and the number of replications is 10,000. The estimated models are the PC-VAR model, considering r principal components, r = 2, 4, ···, 24, and the unrestricted VAR model, equivalent to the PC-VAR model with $r = n$ (25) principal components.

Table 2. Monte Carlo results (ratio of PC-VAR to OLS figures).[a]

***ρ* = 0.00**	***# PC* (*explained total variance*)**												
	2 (0.17)	**4** (0.31)	**6** (0.43)	**8** (0.53)	**10** (0.62)	**12** (0.70)	**14** (0.77)	**16** (0.83)	**18** (0.88)	**20** (0.93)	**22** (0.96)	**24** (0.99)	**25** (1.0)
							***p* = 1**						
Bias	2.044	1.854	1.683	1.529	1.386	1.259	1.144	1.047	0.974	0.933	0.928	0.962	1.000
RMSE	0.461	0.509	0.542	0.571	0.599	0.629	0.663	0.703	0.749	0.804	0.869	0.950	1.000
							***p* = 2**						
Bias	1.722	1.590	1.463	1.342	1.227	1.119	1.022	0.943	0.886	0.861	0.878	0.945	1.000
RMSE	0.355	0.400	0.435	0.469	0.504	0.542	0.585	0.633	0.691	0.758	0.839	0.938	1.000
							***p* = 3**						
Bias	1.249	1.160	1.073	0.989	0.910	0.838	0.777	0.735	0.718	0.735	0.795	0.911	1.000
RMSE	0.247	0.282	0.313	0.344	0.377	0.415	0.459	0.512	0.577	0.659	0.763	0.903	1.000
							***p* = 4**						
Bias	0.512	0.478	0.444	0.411	0.382	0.357	0.338	0.333	0.347	0.389	0.479	0.677	1.000
RMSE	0.094	0.109	0.122	0.137	0.154	0.174	0.199	0.232	0.276	0.339	0.442	0.653	1.000

***ρ* = 0.05**	***# PC* (*explained total variance*)**												
	2 (0.19)	**4** (0.33)	**6** (0.45)	**8** (0.55)	**10** (0.64)	**12** (0.72)	**14** (0.78)	**16** (0.84)	**18** (0.89)	**20** (0.93)	**22** (0.96)	**24** (0.99)	**25** (1.0)
							***p* = 1**						
Bias	2.199	2.061	1.889	1.711	1.539	1.378	1.232	1.108	1.013	0.952	0.934	0.965	1.000
RMSE	0.441	0.492	0.527	0.557	0.586	0.617	0.652	0.693	0.741	0.797	0.865	0.948	1.000
							***p* = 2**						
Bias	1.815	1.712	1.587	1.454	1.323	1.198	1.085	0.988	0.915	0.876	0.884	0.946	1.000
RMSE	0.339	0.385	0.422	0.456	0.492	0.531	0.574	0.625	0.683	0.752	0.834	0.937	1.000
							***p* = 3**						
Bias	1.299	1.231	1.146	1.057	0.969	0.887	0.814	0.760	0.733	0.740	0.796	0.910	1.000
RMSE	0.236	0.271	0.302	0.334	0.368	0.406	0.451	0.505	0.571	0.654	0.760	0.902	1.000
							***p* = 4**						
Bias	0.533	0.504	0.470	0.435	0.402	0.372	0.350	0.340	0.351	0.391	0.478	0.678	1.000
RMSE	0.090	0.105	0.118	0.133	0.151	0.171	0.197	0.229	0.274	0.338	0.442	0.654	1.000

***ρ* = 0.10**	***# PC* (*explained total variance*)**												
	2 (0.23)	**4** (0.39)	**6** (0.51)	**8** (0.60)	**10** (0.68)	**12** (0.75)	**14** (0.81)	**16** (0.86)	**18** (0.91)	**20** (0.94)	**22** (0.97)	**24** (0.99)	**25** (1.0)
							***p* = 1**						
Bias	2.262	2.217	2.059	1.865	1.668	1.477	1.306	1.157	1.040	0.962	0.934	0.961	1.000
RMSE	0.402	0.453	0.492	0.525	0.557	0.591	0.629	0.674	0.726	0.786	0.858	0.946	1.000
							***p* = 2**						
Bias	1.836	1.788	1.670	1.530	1.387	1.249	1.122	1.010	0.928	0.881	0.883	0.945	1.000
RMSE	0.308	0.352	0.392	0.429	0.467	0.509	0.555	0.608	0.670	0.742	0.828	0.934	1.000
							***p* = 3**						
Bias	1.308	1.276	1.198	1.105	1.010	0.919	0.839	0.776	0.738	0.740	0.794	0.909	1.000
RMSE	0.215	0.248	0.280	0.313	0.349	0.389	0.436	0.492	0.561	0.646	0.755	0.900	1.000
							***p* = 4**						
Bias	0.529	0.515	0.485	0.449	0.413	0.381	0.356	0.342	0.349	0.388	0.477	0.678	1.000
RMSE	0.082	0.095	0.109	0.125	0.143	0.164	0.190	0.224	0.269	0.334	0.439	0.654	1.000

Continued

$\rho = 0.15$	# PC (explained total variance)												
	2 (0.29)	**4** (0.46)	**6** (0.58)	**8** (0.67)	**10** (0.74)	**12** (0.80)	**14** (0.85)	**16** (0.89)	**18** (0.92)	**20** (0.95)	**22** (0.98)	**24** (0.99)	**25** (1.0)
							$p = 1$						
Bias	2.149	2.165	2.073	1.911	1.717	1.518	1.334	1.173	1.045	0.959	0.932	0.962	1.000
RMSE	0.354	0.399	0.439	0.476	0.513	0.553	0.597	0.647	0.704	0.770	0.848	0.942	1.000
							$p = 2$						
Bias	1.736	1.737	1.657	1.539	1.404	1.266	1.135	1.020	0.934	0.880	0.880	0.942	1.000
RMSE	0.273	0.311	0.350	0.390	0.432	0.478	0.529	0.586	0.652	0.730	0.820	0.931	1.000
							$p = 3$						
Bias	1.229	1.227	1.174	1.094	1.005	0.916	0.835	0.772	0.735	0.737	0.792	0.906	1.000
RMSE	0.190	0.219	0.250	0.284	0.322	0.366	0.416	0.475	0.547	0.635	0.748	0.897	1.000
							$p = 4$						
Bias	0.493	0.490	0.468	0.437	0.403	0.372	0.347	0.335	0.343	0.382	0.471	0.672	1.000
RMSE	0.072	0.083	0.097	0.113	0.132	0.154	0.181	0.216	0.263	0.329	0.434	0.650	1.000

$\rho = 0.20$	# PC (explained total variance)												
	2 (0.35)	**4** (0.54)	**6** (0.66)	**8** (0.74)	**10** (0.80)	**12** (0.85)	**14** (0.89)	**16** (0.92)	**18** (0.94)	**20** (0.97)	**22** (0.98)	**24** (1.0)	**25** (1.0)
							$p = 1$						
Bias	1.903	1.944	1.901	1.801	1.655	1.475	1.297	1.137	1.012	0.931	0.913	0.955	1.000
RMSE	0.301	0.337	0.372	0.410	0.452	0.499	0.551	0.609	0.674	0.749	0.835	0.938	1.000
							$p = 2$						
Bias	1.537	1.565	1.521	1.437	1.330	1.214	1.097	0.991	0.910	0.864	0.870	0.940	1.000
RMSE	0.232	0.263	0.298	0.338	0.384	0.435	0.492	0.556	0.629	0.712	0.810	0.927	1.000
							$p = 3$						
Bias	1.086	1.096	1.066	1.013	0.945	0.870	0.798	0.742	0.712	0.721	0.781	0.902	1.000
RMSE	0.161	0.185	0.212	0.245	0.286	0.333	0.388	0.452	0.529	0.622	0.739	0.894	1.000
							$p = 4$						
Bias	0.436	0.438	0.425	0.403	0.378	0.352	0.331	0.321	0.330	0.370	0.460	0.663	1.000
RMSE	0.061	0.070	0.082	0.097	0.117	0.140	0.169	0.206	0.254	0.321	0.429	0.647	1.000

$\rho = 0.25$	# PC (explained total variance)												
	2 (0.41)	**4** (0.63)	**6** (0.75)	**8** (0.83)	**10** (0.88)	**12** (0.91)	**14** (0.94)	**16** (0.96)	**18** (0.97)	**20** (0.98)	**22** (0.99)	**24** (1.0)	**25** (1.0)
							$p = 1$						
Bias	1.417	1.461	1.453	1.416	1.347	1.262	1.158	1.044	0.946	0.890	0.885	0.945	1.000
RMSE	0.205	0.229	0.254	0.285	0.326	0.379	0.442	0.517	0.601	0.696	0.802	0.925	1.000
							$p = 2$						
Bias	1.134	1.160	1.144	1.106	1.055	0.998	0.937	0.877	0.831	0.818	0.845	0.928	1.000
RMSE	0.163	0.184	0.208	0.240	0.282	0.336	0.402	0.478	0.566	0.667	0.781	0.917	1.000
							$p = 3$						
Bias	0.812	0.825	0.813	0.789	0.755	0.718	0.684	0.659	0.656	0.687	0.761	0.896	1.000
RMSE	0.114	0.129	0.148	0.174	0.211	0.260	0.321	0.394	0.481	0.587	0.717	0.885	1.000
							$p = 4$						
Bias	0.316	0.319	0.314	0.304	0.293	0.281	0.274	0.277	0.296	0.346	0.440	0.649	1.000
RMSE	0.043	0.049	0.057	0.069	0.086	0.110	0.141	0.181	0.232	0.304	0.416	0.638	1.000

Continued

ρ = 0.30	# PC (explained total variance)												
	2 (0.44)	**4** (0.67)	**6** (0.79)	**8** (0.86)	**10** (0.88)	**12** (0.91)	**14** (0.94)	**16** (0.96)	**18** (0.97)	**20** (0.98)	**22** (0.99)	**24** (1.0)	**25** (1.0)
							***p* = 1**						
Bias	1.004	1.036	1.031	1.009	0.979	0.934	0.891	0.849	0.815	0.800	0.827	0.925	1.000
RMSE	0.142	0.158	0.176	0.198	0.230	0.274	0.333	0.409	0.504	0.617	0.751	0.906	1.000
							***p* = 2**						
Bias	0.824	0.844	0.834	0.814	0.785	0.754	0.725	0.709	0.709	0.732	0.798	0.916	1.000
RMSE	0.117	0.131	0.149	0.173	0.206	0.253	0.314	0.392	0.488	0.603	0.740	0.903	1.000
							***p* = 3**						
Bias	0.599	0.609	0.602	0.586	0.569	0.552	0.541	0.543	0.569	0.629	0.734	0.887	1.000
RMSE	0.081	0.092	0.105	0.126	0.155	0.198	0.255	0.329	0.423	0.540	0.686	0.874	1.000
							***p* = 4**						
Bias	0.232	0.235	0.231	0.225	0.219	0.215	0.216	0.229	0.259	0.319	0.423	0.644	1.000
RMSE	0.030	0.035	0.040	0.049	0.063	0.084	0.113	0.153	0.207	0.283	0.400	0.631	1.000

[a]The Table reports the *ratio* Monte Carlo (absolute) bias and RMSE statistics (average across parameters and equations) from PC-VAR estimation of first, second, third, and fourth order VAR systems, relative to unrestricted OLS estimation figures. The average absolute residuals correlation coefficient is ρ = (0.00, 0.05, 0.10, 0.15, 0.20, 0.25, 0.30), the temporal sample size is T = 100, the cross-sectional sample size is n = 25, and the number of replications is 10,000. The estimated models are the PC-VAR model, considering r principal components, r = 2, 4, ···, 24, and the unrestricted VAR model, equivalent to the PC-VAR model with $r = n$ (25) principal components.

of both lower bias and higher efficiency, independently of the order of the system and the strength of the contemporaneous cross-sectional correlation relating the error terms. By following a bias minimization criterion, two broad cases may however be distinguished, *i.e.*, the (contemporaneously) non-correlated errors ($\rho = 0$) and correlated errors ($\rho > 0$) cases. For the former one ($\rho = 0$), the optimal proportion of total variance to be explained by the selected PCs ranges between 77% and 88%, depending on the order of the system, falling as the order of the VAR increases. While the bias improvement is small for the VAR(1) and VAR(2) cases, for which the degrees of freedom are not smaller than 50% of the sample size, PC-VAR estimation yields a much more dramatic bias reduction for the VAR(3) and VAR(4) cases (−30% and −350%, respectively), as the degrees of freedom fall to 25% of the sample size and 0, respectively. The improvement in relative efficiency is also large, between 20% and 70%, increasing with the order of the VAR, *i.e.*, as the number of degrees of freedom decreases.

On the other hand, for the latter case ($\rho > 0$) a higher optimal level of explained variance would appear to be determined, increasing with ρ, and decreasing with the order of the VAR model, *i.e.*, 93% to 99% for the VAR(1) and VAR(2) models, 89% to 97% for the VAR(3) model and 84% to 94% for the VAR (4) model. Large improvements in bias reduction (−5% to −70%) and relative efficiency (20% to 80%), increasing as the number of degrees of freedom falls, would then appear to be attained also for the contemporaneously correlated case.

Overall, Monte Carlo results point to important gains, in terms of bias and efficiency, of PC-VAR estimation over OLS estimation of high dimensional unrestricted VAR models, in small samples. Concerning the cases of main empirical interest for VAR analysis, *i.e.*, showing a low ($\rho \le 0.1$) or null average degree of contemporaneous correlation of the error terms, and a number of degrees of freedom about or below 25% of the sample size, a target proportion of total variance, to be explained by the selected PCs, in the range 80% to 90%, may then be expected to yield highly satisfactory results, and therefore advisable for empirical applications.

4. Conclusion

In this paper principal components vector autoregressive estimation (PC-VAR) for large scale dynamic econometric models is proposed. The procedure involves a dynamic regression using a subset of principal components extracted from a vector time series, and the recovery of the implied unrestricted VAR parameter estimates by solving a set of linear constraints. PC-VAR and OLS estimation of unrestricted VAR models show the same asymptotic properties. Monte Carlo results strongly support PC-VAR estimation, yielding gains, in terms of both lower bias and higher efficiency, relatively to OLS estimation of high dimensional unrestricted VAR models in small samples.

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