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Stochastic Orders Comparisons of Negative Binomial Distribution with Negative Binomial—Lindley Distribution

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ABSTRACT

The purpose of this study is to compare a negative binomial distribution with a negative binomial—Lindley by using stochastic orders. We characterize the comparisons in usual stochastic order, likelihood ratio order, convex order, expectation order and uniformly more variable order based on theorem and some numerical example of comparisons between negative binomial random variable and negative binomial—Lindley random variable.

Keywords: Stochastic Orders; Negative Binomial Distribution; Negative Binomial—Lindley Distribution

1. Introduction

The negative binomial (NB) distribution is a mixture of Poisson distribution by mixing the Poisson distribution and gamma distribution. The NB distribution is employed as a functional form that relaxes the overdispersion (variance is greater than the mean) restriction of the Poisson distribution (see [1]). If X denote a random variable of NB distributed with parameter r and p then its probability mass function is in form

$$f(x)=\binom{r+x-1}{x}p^r(1-p)^x,\quad x=0,1,2,\cdots,$$

for $r>0$ and $0<p<1$, with $E(X)=\frac{r(1-p)}{p}$ and $\mathrm{Var}(X)=\frac{r(1-p)}{p^2}$.

The negative binomial—Lindley (NB-L) distribution which is a mixed negative binomial distribution obtained by mixing the negative binomial distribution with a Lindley distribution. The NB-L distribution was introduced by Zamani and Ismail in [2] and it provides a model for count data of insurance claims. If Y is a NB-L random variable with parameter r and θ then its probability mass function is in form

$$g(y)=\binom{r+y-1}{y}\sum_{j=0}^{y}\binom{y}{j}(-1)^j\frac{\theta^2(\theta+r+j+1)}{(\theta+r+j)^2},$$

$y=0,1,2,\cdots$, for r and $\theta>0$, with $E(Y)=\frac{r\theta^3}{(\theta+1)(\theta-1)^2}-r$, when $\theta>1$, and

$$\mathrm{Var}(Y)=\frac{(r^2+r)\theta^2(\theta-1)}{(\theta+1)(\theta-2)^2}-\frac{(2r^2+r)\theta^3}{(\theta+1)(\theta-1)^2}-\frac{r^2\left(\theta^3-(\theta+1)(\theta-1)^2\right)^2}{(\theta+1)^2(\theta-1)^4}+r^2,$$

when $\theta>2$.

In this respect, the aim of this work is to compare a negative binomial distribution with negative binomial—Lindley distribution base on stochastic orders such as usual stochastic order, likelihood ratio order, convex order, expectation order and uniformly more variable order.

2. Stochastic Orders

Stochastic orders are useful in comparing random variables measuring certain characteristics in many areas. Such areas include insurance, operations research, queuing theory, survival analysis and reliability theory (see [3]). The simplest comparison is through comparing the expected value of the two comparable random variables. The following, we will define some notions of the stochastic orders which will be used in the context of the paper. For more details, we refer to Ross [4], Misra [5], Shaked [6,7] and Singh [8].

Definition 1. *Let* X *and* Y *be random variables with densities* f *and* g, *respectively, such that* $g(k)/f(k)$ *is non-decreasing function in* k *over the union of the supports of* X *and* Y, *or, equivalently,* $f(u)g(v)\ge f(v)g(u)$, *for all* $u\le v$. *Then* X *is smaller than* Y *in the likelihood ratio order which is denoted by* $X\le_{lr}Y$.

Definition 2. *Let* X *and* Y *be two random variables such that* $\Pr(Y \le k) \le \Pr(X \le k)$, *for all* $k \in \mathbb{R}$. *Then* X *is smaller than* Y *in the usual stochastic order which is denoted by* $X \le_{st} Y$.

Definition 3. *Let* X *and* Y *be two random variables such that* $E(\varphi(X)) \le E(\varphi(Y))$, *for every real valued convex function* φ *where expectations are assumed to be existed. Then* X *is smaller than* Y *in convex order which is denoted by* $X \le_{cx} Y$.

Definition 4. *Let* X *and* Y *be two random variables such that* $E(X) \le E(Y)$, *where expectations are assumed to be existed. Then* X *is smaller than Y in the expectation order which is denoted by* $X \le_E Y$.

Definition 5. *Let* X *and* Y *be two random variables with densities* f *and* g, *respectively. Recall that* supp(X) *and* supp(Y) *denote the respective support of* X *and respective support of* Y, *such that* supp(X) $\subseteq$ supp(Y) *and the ratio* $f(k)/g(k)$ *is a unimodal function over* supp(Y). *Then* X *is smaller than* Y *in uniformly more variable order which is denoted by* $X \le_{uv} Y$.

3. Comparison

We make comparisons between the negative binomial random variable and negative binomial—Lindley random variable with respect to the likelihood ratio order, stochastic order, convex order, expectation order and uniform more variable order. The following lemma will be useful in proving the main results.

Lemma 1. Define,

$$a(k) = 1 - \frac{\sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^j \frac{(\theta+r+j+1)}{(\theta+r+j)^2}}{\sum_{j=0}^{k} \binom{k}{j} (-1)^j \frac{(\theta+r+j+1)}{(\theta+r+j)^2}} \quad \text{and}$$

$$\varphi_k(m) = \sum_{j=0}^{k} \binom{j+r-1}{j} m \left(1 - m^{\frac{1}{r}}\right)^j,$$

$\forall m$, $0 < m < 1$, $k = 0,1,2,\cdots$ Then,

1) $a(k)$ is a non- increasing function of $k \in \{0,1,2,\cdots\}$,

2) For each fixed $k \in \{0,1,2,\cdots\}$, $\varphi_k(m)$ is concave function of $m \in (0,1)$.

Proof.

1) We may write for $a(k)$, $k = 0,1,2,\cdots$ that

$$a(k) = 1 - \frac{\int_0^\infty e^{-\lambda r}\left(1-e^{-\lambda}\right)^{k+1} h(\lambda;\theta)\,d\lambda}{\int_0^\infty e^{-\lambda r}\left(1-e^{-\lambda}\right)^{k} h(\lambda;\theta)\,d\lambda} = E(W_k),$$

where $h(\cdot)$ is the Lindley distribution defined by

$$h(z;\theta) = \frac{\theta^2}{\theta+1}(1+z)e^{-\theta z}, \quad z > 0 \quad \text{and} \quad \theta > 0$$

and W_k is a random variable having the probability density function:

$$\psi_k(z) = \frac{(\theta+1)e^{-rz}\left(1-e^{-z}\right)^k h(z;\theta)}{\sum_{j=0}^{k}\binom{k}{j}(-1)^j \frac{\theta^2(\theta+r+j+1)}{(\theta+r+j)^2}}, \quad z > 0.$$

For fixed $k \in \{0,1,2,\cdots\}$, the ratio $\psi_{k+1}(x)/\psi_k(x)$ is obviously a non-increasing function of $x > 0$. Then, by Definitions 1 and 2, we have $W_k \ge_{lr} W_{k+1}$, which yields $W_k \ge_{st} W_{k+1}$ and therefore $E(W_k) \ge E(W_{k+1})$ or, equivalently, $a(k) \ge a(k+1)$. This proves $a(k)$ is a non-increasing function of $k \in \{0,1,2,\cdots\}$.

2) For $k = 0$, note that $\varphi_0(m) = m$ is both convex and concave. For $k = 1,2,\cdots$, we can write

$$\varphi_k(m) = 1 - \sum_{j=k+1}^{\infty}\binom{j+r-1}{j} m\left(1-m^{\frac{1}{r}}\right)^j, \ \forall m, \ 0 < m < 1. \tag{1}$$

The relationship between negative binomial and beta probabilities is of the form

$$\sum_{j=k}^{\infty}\binom{r+r-1}{j} p^r (1-p)^j = \frac{(k+r-1)!}{(k-1)!(r-1)!}\int_0^{1-p} t^{k-1}(1-t)^{r-1}dt,$$

$k = 0,1,2,\cdots$.

Therefore, $\varphi_k(m)$ in Equation (1) can be written as

$$\varphi_k(m) = 1 - \frac{(k+r)!}{k!(r-1)!}\int_0^{1-m^{\frac{1}{r}}} t^k (1-t)^{r-1}dt,$$

$\forall m$, $0 < m < 1$.

Thus,

$$\frac{\partial^2}{\partial m^2}\varphi_k(m) = -\frac{1}{r}\frac{(k+r)!}{(r-1)!r!} m^{\frac{1}{r}-1}\left(1-m^{\frac{1}{r}}\right)^{k-1} < 0,$$

$\forall m$, $0 < m < 1$.

which proves concavity. □

Theorem 1. Let $X \sim NB(r,p)$, $Y \sim NB\text{-}L(r,\theta)$ and

$$p_0 = \frac{(\theta+r+2)(\theta+r)^2}{(\theta+r+1)^3}, \quad p_1 = \left[\frac{\theta^2(\theta+r+1)}{(\theta+r)^2}\right]^{\frac{1}{r}},$$

$$p_2 = \frac{(\theta+1)(\theta-1)^2}{\theta^3}. \text{ Then } 0 < p_2 < p_1 < p_0 < 1.$$

Furthermore,

1) $X \le_{lr} Y$ if and only if $p \ge p_0$,

2) $X \le_{st} Y$ if and only if $p \ge p_1$,

3) $X \le_E (\ge_E) Y$ if and only if $p \ge (\le) p_2$,

Proof.

1) The likelihood ratio order between Y and X can be written as

$$\mathrm{l}(\mathrm{k})=\frac{\Pr(\mathrm{Y}=\mathrm{k})}{\Pr(\mathrm{X}=\mathrm{k})}=\frac{\sum_{j=0}^{k}\binom{k}{j}(-1)^{j}\frac{\theta^{2}(\theta+r+j+1)}{(\theta+r+j)^{2}}}{p^{r}(1-p)^{k}(\theta+1)},\quad (2)$$

$k=0,1,2,\cdots$.

By Definition 1, we have

$$\mathrm{X}\le_{\mathrm{lr}}\mathrm{Y}\Leftrightarrow \mathrm{l}(\mathrm{k})\le \mathrm{l}(\mathrm{k}+1),\ \forall k,\ k\in\{0,1,2,\cdots\},$$

$$\Leftrightarrow p\ge 1-\frac{\sum_{j=0}^{k+1}\binom{k+1}{j}(-1)^{j}\frac{(\theta+r+j+1)}{(\theta+r+j)^{2}}}{\sum_{j=0}^{k}\binom{k}{j}(-1)^{j}\frac{(\theta+r+j+1)}{(\theta+r+j)^{2}}},$$

$$\Leftrightarrow p\ge a(k),\ \forall k,\ k\in\{0,1,2,\cdots\},$$

$$\Leftrightarrow p\ge p_0=a(0).$$

Since $a(k)$ is non-increasing in k (by part 1) in Lemma 1), then $p\ge p_0$ which provides a necessary and sufficient condition for the $l(k)$ in Equation (2) to be non-decreasing. This completes the proof of the result.

2) Let $X\le_{st}Y$ by Definition 2, we have

$$\Pr(Y\le 0)\le \Pr(X\le 0)\Rightarrow p^{r}\ge \frac{\theta^{2}(\theta+r+1)}{(\theta+r)^{2}},$$

$$\Rightarrow p\ge\left[\frac{\theta^{2}(\theta+r+1)}{(\theta+r)^{2}}\right]^{\frac{1}{r}}=p_1.$$

Conversely, suppose that $0<p_1<p<1$.

For $k=0,1,2,\cdots$, consider

$$\Delta_p(k)=\Pr(X\le k)-\Pr(Y\le k)=\sum_{i=0}^{k}\binom{i+r-1}{i}p^{r}(1-p)^{i}-\sum_{i=0}^{k}\left[\binom{i+r-1}{i}\sum_{j=0}^{i}\binom{i}{j}(-1)^{j}\frac{\theta^{2}(\theta+r+j+1)}{(\theta+1)(\theta+r+j)^{2}}\right],$$

and if $X\sim NB(r,p)$ and $X\sim NB(r,p_1)$, then $X\le_{lr}X_1$. Hence, we get $X\le_{st}X_1$.

Consequently,

$$\sum_{i=0}^{k}\binom{i+r-1}{i}p^{r}(1-p)^{i}\ge\sum_{i=0}^{k}\binom{i+r-1}{i}p_1^{r}(1-p_1)^{i},$$

and therefore $\Delta_p(k)\ge\Delta_{p_1}(k)$.

For fixed $k\in\{0,1,2,\cdots\}$, $p_1=\left[\frac{\theta^{2}(\theta+r+1)}{(\theta+r)^{2}}\right]^{\frac{1}{r}}$, $p=\exp(-\lambda)$ and $\lambda\sim \mathrm{Lindley}(\theta)$. We get

$$\Delta_{p_1}(k)=\sum_{i=0}^{k}\binom{i+r-1}{i}p_1^{r}(1-p_1)^{i}-\sum_{i=0}^{k}\left[\binom{i+r-1}{i}\sum_{j=0}^{i}\binom{i}{j}(-1)^{j}\frac{\theta^{2}(\theta+r+j+1)}{(\theta+1)(\theta+r+j)^{2}}\right],$$

$$\Delta_{p_1}(k)=\sum_{i=0}^{k}\binom{i+r-1}{i}E(P^{r})\left[1-\left(E(P^{r})\right)^{\frac{1}{r}}\right]^{i}-\sum_{i=0}^{k}\binom{i+r-1}{i}E\left(P^{r}(1-P)^{i}\right),$$

Using concave function (by part 2) in Lemma 1), $\Delta_{p_1}(k)$ can be written as

$$\Delta_{p_1}(k)=\varphi_k\left(E(P^{r})\right)-E\left(\varphi_k(P^{r})\right).$$

Applying Jensen's inequality to concave function, we have $\varphi_k\left(E(P^{r})\right)\ge E\left(\varphi_k(P^{r})\right)$ and $\Delta_{p_1}(k)\ge 0$ for $\forall k$, $k\in\{0,1,2,\cdots\}$. Conversely, where $X\le_{st}Y$ implies that $\Pr(Y\le 0)\le\Pr(X\le 0)$. This proves $p\ge p_1$.

3) The proofs of the results are obvious. □

Theorem 2. Suppose that, for every $0<t<1$, $\Pr(t<p<1)>0$, then

1) No value of $0<p<1$ can ensure that $X\le_{st}Y$,

2) $X\le_{uv}Y$ if and only if $0<p<p_1<1$.

Proof.

1) We find $R(k)=\frac{\Pr(X\ge k)}{\Pr(Y\ge k)}$, $k=0,1,2,\cdots$, by refrying the numerator and denominator as following:

$$\Pr(X\ge k)=\sum_{j=k}^{\infty}\binom{j+r-1}{j}p^{r}(1-p)^{j},$$

$$=\frac{(k+r-1)!}{(k-1)!(r-1)!}\int_0^{1-p}t^{k-1}(1-t)^{r-1}dt,$$

$$\le(1-p)^{k-1}\frac{(k+r-1)!}{(k-1)!(r-1)!}\int_0^{1-p}(1-t)^{r-1}dt,$$

$$=\frac{(k+r-1)!}{(k-1)!(r-1)!}\frac{(1-p^{r})(1-p)^{k}}{r(1-p)}.$$

For any $0<p<p_1<1$, $k=0,1,2,\cdots$,

$$\Pr(Y\ge k)=\sum_{i=k}^{\infty}\left[\binom{r+i-1}{i}\sum_{j=0}^{i}\binom{i}{j}(-1)^{j}\frac{\theta^{2}(\theta+r+j+1)}{(\theta+r+j)^{2}}\right],$$

$$=\sum_{i=k}^{\infty}\left[\binom{r+i-1}{i}\int_0^{\infty}e^{-\lambda r}\left(1-e^{-\lambda}\right)^{i}h(\lambda;\theta)d\lambda\right],$$

$$=\int_0^{\infty}\left(\sum_{i=k}^{\infty}\binom{r+i-1}{i}e^{-\lambda r}\left(1-e^{-\lambda}\right)^{i}\right)h(\lambda;\theta)d\lambda,$$

$$=\int_0^{\infty}\left(\frac{(k+r-1)!}{(k-1)!(r-1)!}\int_0^{1-e^{-\lambda}}t^{k-1}(1-t)^{r-1}dt\right)h(\lambda;\theta)d\lambda,$$

$$=\int_0^{1}\frac{(k+r-1)!}{(k-1)!(r-1)!}t^{k-1}(1-t)^{r-1}\left(\int_0^{\log\left(\frac{1}{1-t}\right)}h(\lambda;\theta)d\lambda\right)dt,$$

$$=\int_0^{1}\frac{(k+r-1)!}{(k-1)!(r-1)!}t^{k-1}(1-t)^{r-1}H\left(\log\left(\frac{1}{1-t}\right);\theta\right)dt,$$

Table 1. Stochastic orders comparisons of NB random variables with NB-L random variables.

Random Variables		Order Comparisons of NB Random Variables with NB-L Random Variables				
$NB(r,p)$	$NB\text{-}L(r,\theta)$	Usual stochastic order	Likelihood ratio order	Convex order	Expectation order	Uniformly more variable order
	$Y_1 \sim NB\text{-}L(3,1.5)$	$X \le_{st} Y_1$	$X \le_{lr} Y_5$	-	$X \le_E Y_1$	-
	$Y_2 \sim NB\text{-}L(3,2.0)$	$X \le_{st} Y_2$	-	-	$X \le_E Y_2$	-
$X \sim NB(3,0.8)$	$Y_3 \sim NB\text{-}L(3,3.5)$	-	-	-	$X \le_E Y_3$	$X \le_{uv} Y_3$
	$Y_4 \sim NB\text{-}L(3,4.5)$	-	-	-	$X \le_E Y_4$	$X \le_{uv} Y_4$
	$Y_5 \sim NB\text{-}L(3,5.7)$	-	-	$X \le_{cx} Y_5$	$X =_E Y_5$	$X \le_{uv} Y_5$

where $H(\cdot)$ is cumulative distribution function of Lindley distribution:

$$H(z;\theta) = 1 - \frac{\theta+1+\theta z}{\theta+1} e^{-\theta z}, \quad z > 0 \quad \text{and} \quad \theta > 0 .$$

$$\Pr(Y \ge k) \ge H(p_1;\theta) p_1^{r-1} \int_0^{1-p_1} \frac{(k+r-1)!}{(k-1)!(r-1)!} t^{k-1} dt,$$

$$= \frac{(k+r-1)!}{(k-1)!(r-1)!} \frac{H(p_1;\theta) p_1^{r-1} (1-p_1)^k}{k}.$$

So, $R(k) \le \dfrac{k(1-p^r)}{r(1-p) p_1^{r-1} H(p_1;\theta)} \left(\dfrac{1-p}{1-p_1}\right)^k .$

Since $\lim\limits_{k\to\infty} \dfrac{k(1-p^r)}{r(1-p) p_1^{r-1} H(p_1;\theta)} \left(\dfrac{1-p}{1-p_1}\right)^k = 0$, we have then $\lim\limits_{k\to\infty} R(k) = 0$, $\forall p_1 < p < 1$.

Therefore, it follows that, for any $0 < p < 1$, there exists a sufficiently large k such that $\Pr(X \ge k) < \Pr(Y \ge k)$. This validates the result.

2) Suppose that $0 < p_1 < p < 1$. Then, from part 2) in Theorem 1 and part 1) in Theorem 2, it is clear that random variables X and Y are not ordered by the usual stochastic order. Also, from the arguments used in the proof of part 1) in Theorem 1, since $p < p_0$ it follows that $\Pr(X = k)/\Pr(Y = k)$ is non-increasing and unimodal, implying that $X \le_{uv} Y$. The converse part follows by using the similar arguments. □

Theorem 3. Suppose that $p = p_2$. Then,

1) $X \le_{uv} Y$

2) $X \le_{cx} Y$.

Proof.

1) Follows from part 2) in Theorem 2, we have $X \le_{uv} Y$, where $p_2 < p_1$.

2) Since $X \le_{uv} Y$ and

$$E(Y) = \frac{r\theta^3}{(\theta+1)(\theta-1)^2} - r = \frac{r(1-p)}{p} = E(X),$$

by the result of Shaked in [4], $X \le_{cx} Y$.

Next, We shows some numerical examples of the comparisons between negative binomial random variable with negative binomial—Lindley random variable in usual stochastic order, likelihood ratio order, convex order, expectation order and uniformly more variable order and the results are provided in **Table 1**.

Then, we explain that negative binomial random variable (X) is smaller than negative binomial—Lindley random variable (Y) in the usual stochastic order implies that $X \le_E Y$. In addition, if X and Y have respective supports supp(X) and supp(Y), such that supp(X) $\subseteq$ supp(Y) and the ratio $\Pr(X = k)/\Pr(Y = k)$ is a unimodal function over supp(Y) but X and Y are not ordered in the usual stochastic order. Furthermore, if X and Y have a same mean. Then $X \le_{uv} Y$ implies that $X \le_{cx} Y$.

4. Conclusion

This paper shows stochastic orders comparison of negative binomial random variable with a negative binomial—Lindley random variable by usual stochastic order, likelihood ratio order, convex order, expectation order and uniformly more variable order. Some advantages of stochastic orders comparison between negative binomial random variable and negative binomial—Lindley random variable are as follows: If negative binomial random variable (X) is smaller than negative binomial—Lindley random variable (Y) in the usual stochastic order. Its usefulness is that it gives a simple sufficient condition for X is smaller than Y in the expectation order. Next, if supp(X) $\subseteq$ supp(Y) is that it implies that the ratio $\Pr(X = k)/\Pr(Y = k)$ is a unimodal function over supp(Y) but X and Y are not ordered in the usual stochastic order. Finally, If X and Y have a same mean, it is known that X is smaller than Y in uniformly more variable order implies that X is smaller than Y in convex order. This conclusion is supported by numerical examples.

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