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# On a Grouping Method for Constructing Mixed Orthogonal Arrays

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## ABSTRACT

Mixed orthogonal arrays of strength two and size $s^{mn}$ are constructed by grouping points in the finite projective geometry $PG(mn-1,s)$. $PG(mn-1,s)$ can be partitioned into $\left[(s^{mn}-1)/(s^n-1)\right]$ $(n-1)$-flats such that each $(n-1)$-flat is associated with a point in $PG(m-1,s^n)$. An orthogonal array $L_{s^{mn}}\left((s^n)^{(s^{mn}-1)/(s^n-1)}\right)$ can be constructed by using $(s^{mn}-1)/(s^n-1)$ points in $PG(m-1,s^n)$. A set of $(s^t-1)/(s-1)$ points in $PG(m-1,s^n)$ is called a $(t-1)$-flat over *GF*(*s*) if it is isomorphic to $PG(t-1,s)$. If there exists a $(t-1)$-flat over *GF*(*s*) in $PG(m-1,s^n)$, then we can replace the corresponding $\left[(s^t-1)/(s-1)\right]$ $s^n$-level columns in $L_{s^{mn}}\left((s^n)^{(s^{mn}-1)/(s^n-1)}\right)$ by $\left[(s^n-1)/(s-1)\right]$ $s^t$-level columns and obtain a mixed orthogonal array. Many new mixed orthogonal arrays can be obtained by this procedure. In this paper, we study methods for finding disjoint $(t-1)$-flats over *GF*(*s*) in $PG(m-1,s^n)$ in order to construct more mixed orthogonal arrays of strength two. In particular, if *m* and *n* are relatively prime then we can construct an $L_{s^{mn}}\left((s^m)^{\frac{s^{mn}-1}{s^m-1}-\frac{i(s^n-1)}{s-1}}(s^n)^{\frac{i(s^m-1)}{s-1}}\right)$ for any $0\le i\le\frac{(s^{mn}-1)(s-1)}{(s^m-1)(s^n-1)}$. New orthogonal arrays of sizes 256, 512, and 1024 are obtained by using *PG*(7, 2), *PG*(8, 2) and *PG*(9, 2) respectively.

**Keywords:** Finite Field; Finite Projective Geometry; $(t-1)$-Flat over *GF*(s) in $PG(m-1,s^n)$; Geometric Orthogonal Array; Matrix Representation; Minimal Polynomial; Orthogonal Main-Effect Plan; Primitive Element; Tight

## 1. Introduction

Orthogonal arrays of strength two are used as orthogonal main-effect plans in fractional factorial experiments. In an orthogonal main-effect plan, the main effects of each factor can be optimally estimated assuming the interactions of all factors are negligible.

Let $L_N(s_1\cdots s_k)$ denote an orthogonal arrays of strength two with *N* rows, *k* columns, and $s_i$ levels in the *i*th column for $i=1,\cdots,k$. In every *N* × 2 subarray of $L_N(s_1\cdots s_k)$, all possible combinations of levels occur equally often as rows. It is known that $N-1\ge\sum(s_i-1)$ in an $L_N(s_1\cdots s_k)$ and the orthogonal array is called *tight* if the equality holds. Orthogonal array $L_N(s_1\cdots s_k)$ is called *symmetric* if $s_1=\cdots=s_k$, otherwise it is called *asymmetric* or *mixed*. Symmetric orthogonal arrays have been constructed in [1-3]. Mixed orthogonal arrays were introduced in [4], and they have drawn the attentions of many researchers in recent years. Methods for constructing mixed orthogonal arrays of strength two have been developed in [5-9], and many other authors. These methods use Hadamard matrices, difference schemes, generalized Kronecker sums, finite projective geometries, and orthogonal projection matrices. We refer to [10] for more constructions and applications of orthogonal arrays.

The method of grouping was used in [11] to replace three two-level columns in symmetric orthogonal arrays by one four-level column for constructing mixed orthogonal arrays having two-level and four-level columns. A systematic method [12] was developed for identifying disjoint sets of three two-level columns for constructing $L_N(2^m4^n)$. The method was generalized in [6] for constructing $L_{s^k}\left(s^m\left(s^{r_1}\right)^{n_1}\cdots\left(s^{r_t}\right)^{n_t}\right)$, where $s$ is a prime power. Mixed orthogonal arrays of strength $t$ were constructed by using mixed spreads of strength $t$ in finite geometries in [13]. This method was also independently discovered in [14] for constructing mixed orthogonal arrays of strength three and four. Orthogonal arrays constructed by this method are called *geometric*. Geometric orthogonal arrays $L_{64}(8^64^7)$, $L_{64}(8^34^{14})$, $L_{64}(8^44^{10}2^5)$ and $L_{64}(8^14^{17}2^5)$ were constructed in [13]. However, the method is restricted to constructing mixed orthogonal arrays with the number of levels in each column a power of 2. In this paper, we shall use finite projective geometries to develop a general procedure for constructing more mixed orthogonal arrays. Moreover, the procedure allows us to construct mixed orthogonal arrays with the number of levels in each column a power of any given prime number. We start with a symmetric orthogonal array $L_{s^{mn}}\left(\left(s^n\right)^{\left(s^{mn}-1\right)/\left(s^n-1\right)}\right)$, and then construct mixed orthogonal arrays by replacing a group of columns with another group of columns. Our grouping method uses properties of finite projective geometries, which is different from the grouping method in [6]. Hence we are able to obtain some new series of mixed orthogonal arrays.

## 2. Geometric Orthogonal Arrays

For $r \ge 1$ and $s$ a prime power, let $PG(r-1,s)$ denote the $(r-1)$-dimensional finite projective geometry over the Galois field $GF(s)$. A point in $PG(r-1,s)$ is denoted by an $r$-tuple $(x_1,\cdots,x_r)$, where $x_i$'s are elements of $GF(s)$ and at least one $x_i$ is not 0. Two $r$-tuples represent the same point in $PG(r-1,s)$ if one is a multiple of the other. Hence there are $\left(s^r-1\right)/(s-1)$ points in $PG(r-1,s)$. A $(t-1)$-flat in $PG(r-1,s)$ is a set of $\left(s^t-1\right)/(s-1)$ points which are linear combinations of $t$ independent points. A spread $\mathcal{F}$ of $(t-1)$-flats of $PG(r-1,s)$ is a set of $(t-1)$-flats which partition $PG(r-1,s)$. It is known [15] that there exists a spread $\mathcal{F}$ of $(t-1)$-flats of $PG(r-1,s)$ if and only if $t$ divides $r$.

We call a set of flats $\mathcal{F}=\{F_1,\cdots,F_k\}$ a *mixed spread* of $PG(r-1,s)$ if it partitions $PG(r-1,s)$ and at least two flats in $\mathcal{F}$ have different dimensions. Mixed spreads are useful for constructing mixed orthogonal arrays of strength two. Specifically, we give the following theorem for constructing an orthogonal array from a (mixed) spread. The theorem is the special case of strength two of Theorem 2.1 [14] in finite projective geometry's language.

Theorem 1. *Let* $\mathcal{F}=\{F_1,\cdots,F_k\}$ *be a* (*mixed*) *spread of* $PG(r-1,s)$, *where* $F_i$ *is a* $(t_i-1)$*-flat for* $i=1,\cdots,k$. *Then we can construct an orthogonal array* $L_{s^r}\left(\left(s^{t_1}\right)\cdots\left(s^{t_k}\right)\right)$.

We now describe the procedure to construct the orthogonal array in Theorem 1. For $i=1,\cdots,k$, let $G_i$ be an $r\times t_i$ matrix such that the $t_i$ columns are any choice of $t_i$ independent points of the $(t_i-1)$-flat $F_i$. Let $G$ be the $r\times\sum t_i$ matrix $[G_1,\cdots,G_k]$. The $L_{s^r}\left(\left(s^{t_1}\right)\cdots\left(s^{t_k}\right)\right)$ consists of $s^r$ rows which are the elements of the row space of $G$, where the $t_i$ $s$-level columns of $G_i$ is replaced by an $s^{t_i}$-level column for each $i=1,\cdots,k$. We call orthogonal arrays *geometric* if they can be obtained by Theorem 1. Geometric orthogonal arrays have been constructed in [1, 9,13,16]. Examples of geometric orthogonal arrays are:

1) $L_{s^r}\left(s^{\left(s^r-1\right)/(s-1)}\right)$;

2) $L_{s^r}\left(\left(s^t\right)^{\left(s^r-1\right)/\left(s^t-1\right)}\right)$ if $t$ divides $r$;

3) $L_{s^r}\left(\left(s^{r-t}\right)^1\left(s^t\right)^{s^{r-t}}\right)$ if $r\ge 2t$; and

4) $L_{s^r}\left(\left(s^t\right)^k s^l\right)$, where,

$$k=s^j\left(s^{it}-1\right)/\left(s^t-1\right)-s^j+1,$$

$$l=s^t\left(s^j-1\right)/(s-1),\quad r=it+j,\quad 0\le j<t.$$

## 3. Main Results

It is proved in Lemma 12 [13] that if $V_1$, $V_2$, $V_3$ are three disjoint $(n-1)$-flats of $PG(2n-1,2)$ then their union can be regrouped into $2^n-1$ disjoint 1-flats. Hence three $2^n$-level columns in an $L_{2^{2n}}\left(\left(2^n\right)^{2^n+1}\right)$ can be replaced by $\left(2^n-1\right)$ 4-level columns. By applying this result to a spread of 2-flats of $PG(5, 2)$, $L_{64}(8^64^7)$ and $L_{64}(8^34^{14})$ were constructed. Generalizing the idea, we would like to find a sufficient condition that a set of $\left[\left(s^t-1\right)/(s-1)\right]$ $(n-1)$-flats in $PG(mn-1,s)$ can be regrouped into a set of $\left[\left(s^n-1\right)/(s-1)\right]$ $(t-1)$-flats.

Since there exists a spread of $(n-1)$-flats of $PG(mn-1,s)$, we can, by Theorem 1, construct an $L_{s^{mn}}\left(\left(s^n\right)^{\left(s^{mn}-1\right)/\left(s^n-1\right)}\right)$. If there exist $\left[\left(s^t-1\right)/(s-1)\right]$ $(n-1)$-flats in the spread such that their union can be regrouped into $\left[\left(s^n-1\right)/(s-1)\right]$ $(t-1)$-flats, then we can replace the corresponding $\left[\left(s^t-1\right)/(s-1)\right]$ $s^n$-level columns in the $L_{s^{mn}}\left(\left(s^n\right)^{\left(s^{mn}-1\right)/\left(s^n-1\right)}\right)$ by $\left[\left(s^n-1\right)/\left(s-1\right)\right]$ $s^t$-level columns and obtain an $L_{s^{mn}}\left(\left(s^n\right)^{\frac{s^{mn}-1}{s^n-1}-\frac{s^t-1}{s-1}}\left(s^t\right)^{\frac{s^n-1}{s-1}}\right)$. By repeating this process, many orthogonal arrays can be obtained.

First we would like to establish a one-to-one correspondence between the $\left(s^{mn}-1\right)/\left(s^n-1\right)$ disjoint $(n-1)$-flats in $PG(mn-1,s)$ and the $\left(s^{mn}-1\right)/\left(s^n-1\right)$ points in $PG\left(mn-1,s^n\right)$. Let $\omega$ be a primitive element of $GF(s^n)$ and let the minimum polynomial of $GF\left(s^n\right)$ be $\omega^n+\alpha_{n-1}\omega^{n-1}+\cdots+\alpha_1\omega+\alpha_0$, where $\alpha_0,\cdots,\alpha_{n-1}$ are elements of $GF(s)$. The companion matrix of the minimum polynomial is an $n\times n$ matrix

$$\boldsymbol{W}=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} \end{bmatrix}.$$

If $\omega$ is a primitive element of $GF\left(s^n\right)$, then $0,1,\omega,\cdots,\omega^{s^n-2}$ are the $s^n$ elements of $GF\left(s^n\right)$. The elements of $GF\left(s^n\right)$ can be represented by $n\times n$ matrices with entries from $GF(s)$. The element $\omega^i$ is represented by $W^i$, and the elements 0 and 1 are represented by the zero matrix and the identity matrix respectively. Denote the matrix representation of an element $x$ in $GF\left(s^n\right)$ by $W(x)$. Let each point $\left(x_1,\cdots,x_m\right)$ in $PG\left(m-1,s^n\right)$ correspond to the $(n-1)$-flat in $PG(mn-1,s)$ which consists of points that are linear combinations of row vectors of the $n\times mn$ matrix $\left[W\left(x_1\right),\cdots,W\left(x_m\right)\right]$ over $GF(s)$. It can be shown that the $\left[\left(s^{mn}-1\right)/\left(s^n-1\right)\right]$ $(n-1)$-flats corresponding to the $\left[\left(s^{mn}-1\right)/\left(s^n-1\right)\right]$ points of $PG\left(m-1,s^n\right)$ partition $PG(mn-1,s)$. This establishes a one-to-one correspondence between the $\left(s^{mn}-1\right)/\left(s^n-1\right)$ disjoint $(n-1)$-flats in $PG(mn-1,s)$ and the $\left(s^{mn}-1\right)/\left(s^n-1\right)$ points in $PG\left(m-1,s^n\right)$.

**Definition 1.** A set of $\left(s^t-1\right)/\left(s-1\right)$ points in $PG\left(m-1,s^n\right)$ is said to be a $(t-1)$-flat over $GF(s)$ if it is possible to find coordinates for this set of $\left(s^t-1\right)/\left(s-1\right)$ points such that it is isomorphic to $PG(t-1,s)$ over $GF(s)$.

Note that whether a set of $\left(s^t-1\right)/\left(s-1\right)$ points in $PG\left(m-1,s^n\right)$ is isomorphic to $PG(t-1,s)$ over $GF(s)$ depends not only on the choice of the points but also on the choice of the coordinates for these points. For example, the set $S_1=\left\{(1,\omega),\left(\omega,\omega^3\right),\left(\omega^3,1\right)\right\}$ in Example 1 (given after Theorem 2) is an 1-flat over $GF(2)$ in $PG(1,8)$ since it is isomorphic to $PG(1,2)$ over $GF(2)$. But if we choose different coordinates for $S_1=\left\{(1,\omega),\left(1,\omega^2\right),\left(1,\omega^4\right)\right\}$, then it is not isomorphic to $PG(1,2)$ over $GF(2)$. Hence it is important to specify the correct coordinates when a $(t-1)$-flat over $GF(s)$ in $PG\left(m-1,s^n\right)$ is mentioned. Also we note that it is possible to have $t>m$ for a $(t-1)$-flat over $GF(s)$ in $PG\left(m-1,s^n\right)$. For example, $S_1$ and $S_2$ in Example 2 (given after Theorem 2) are 2-flats over $GF(2)$ in $PG(1,16)$.

We now give a sufficient condition that a set of $\left(s^t-1\right)/\left(s-1\right)$ disjoint $(n-1)$-flats in $PG(mn-1,s)$ can be regrouped into a set of $\left(s^n-1\right)/\left(s-1\right)$ disjoint $(t-1)$-flats.

**Theorem 2.** *A set of* $\left(s^t-1\right)/\left(s-1\right)$ *disjoint* $(n-1)$-*flats in* $PG(mn-1,s)$ *can be regrouped into a set of* $\left(s^n-1\right)/\left(s-1\right)$ *disjoints* $(t-1)$-*flats, if the set of* $\left(s^t-1\right)/\left(s-1\right)$ *corresponding points in* $PG\left(m-1,s^n\right)$ *is a* $(t-1)$-*flat over GF*(*s*).

**Proof.** Let the coordinates of the $\left(s^t-1\right)/\left(s-1\right)$ corresponding points of the $(t-1)$-flat over $GF(s)$ in $PG\left(m-1,s^n\right)$ be $\left(x_{1j},\cdots,x_{mj}\right)$ for $j=1,\cdots,\left(s^t-1\right)/\left(s-1\right)$. Also let $L$ be an $\left[\left(s^n-1\right)/\left(s-1\right)\right]\times n$ matrix such that the rows are the points of $PG(n-1,s)$. Then the $(n-1)$-flat in $PG(mn-1,s)$ corresponding to the point $\left(x_{1j},\cdots,x_{mj}\right)$ consists of points which are the rows of the $\left[\left(s^n-1\right)/\left(s-1\right)\right]\times mn$ matrix $M_j=L\left[W\left(x_{1j}\right),\cdots,W\left(x_{mj}\right)\right]$, where $W(x)$ is the $n\times n$ matrix representation of $x$. We can verify that for each $i=1,\cdots,\left(s^n-1\right)/\left(s-1\right)$, the set of $\left(s^t-1\right)/\left(s-1\right)$ points which consists of the $i$th rows of $M_1,\cdots,M_{\left(s^t-1\right)/\left(s-1\right)}$ is a $(t-1)$-flat in $PG(mn-1,s)$. □

Note that in general there are more ways of regrouping a set of $\left(s^t-1\right)/\left(s-1\right)$ disjoint $(n-1)$-flats in

$PG(mn-1,s)$ into disjoint flats if the $(s^t-1)/(s-1)$ corresponding points in $PG(m-1,s^n)$ is a $(t-1)$-flat over $GF(s)$. Let $P_{ij}$ be the point in $PG(mn-1,s)$ with the $i$th row of $M_j$ as its coordinates. The $\left[(s^n-1)/(s-1)\right]\times\left[(s^t-1)/(s-1)\right]$ array of points $\mathbf{P}=\left[P_{ij}\right]$ has the following properties:

1) Each row (column) of **P** is a $(t-1)$-flat ($(n-1)$-flat).

2) If $(s^u-1)/(s-1)$ points in a given row (column) form a $(u-1)$-flat, then the $(s^u-1)/(s-1)$ points at the same positions in any other row (column) also form a $(u-1)$-flat.

For example, if there exists a 2-flat over $GF(2)$ in $PG(1, 16)$, then each of the 7 points in the 2-flat over $GF(2)$ corresponds to 15 points in $PG(7, 2)$. The 105 points in $PG(7, 2)$ corresponding to the 2-flat over $GF(2)$ in $PG(1, 16)$ can be arranged into a 15 × 7 array such that each row is a 2-flat and each column is a 3-flat. Since a 3-flat can be partitioned into five 1-flats, the 15 × 7 array of points can be partitioned into five 3 × 7 subarrays such that each column is a 1-flat and each row is a 2-flat. Also, consider a 15 × 3 subarray of the 15 × 7 array such that each row is a 1-flat. We can select a 7 × 3 subarray such that each column is a 2-flat. Each of the remaining eight rows is a 1-flat. Hence the 15 × 3 subarray can be partitioned into three 2-flat and eight 1-flats. Therefore, these 105 points can be grouped into: 1) $(7-i)$ 3-flats and $5i$ 1-flats for $i=0,\cdots,7$; 2) $(15-3i)$ 2-flats and $7i$ 1-flats for $i=0,\cdots,5$; or 3) four 3-flats, three 2-flats, and eight 1-flats.

**Example 1.** Let $0,1,\omega,\cdots,\omega^6$ be the 8 elements of $GF(8)$ with $\omega^3=\omega+1$. Consider $PG(1, 8)$ with nine points (0, 1), (1, 0), (1, 1), $(1, \omega)$, $(1, \omega^2)$, $(1, \omega^3)$, $(1, \omega^4)$, $(1, \omega^5)$ and $(1, \omega^6)$. Each point of $PG(1, 8)$ corresponds to a 2-flat in $PG(5, 2)$, and the nine 2-flats partition $PG(5, 2)$. We can construct an $L_{64}(8^9)$ by Theorem 1. Let

$$S_1=\left\{(1,\omega),(\omega,\omega^3),(\omega^3,1)\right\},$$
$$S_2=\left\{(1,\omega^3),(\omega^3,\omega),(\omega,1)\right\}, \text{ and}$$
$$S_3=\left\{(0,1),(1,0),(1,1)\right\}.$$

We can verify that $S_1$, $S_2$, and $S_3$ are disjoint 1-flats over $GF(2)$ in $PG(1, 8)$. The 3 × 3 matrix representation $W$ of $\omega$ and the 7 × 3 matrix $\boldsymbol{L}$ given in the proof of Theorem 2 are

$$\boldsymbol{W}=\begin{bmatrix}0&1&0\\0&0&1\\1&1&0\end{bmatrix};\quad \boldsymbol{L}=\begin{bmatrix}1&0&0&1&0&1&1\\0&1&0&1&1&1&0\\0&0&1&0&0&1&1\end{bmatrix}^{\mathrm{T}}.$$

The three points of $S_1$ correspond to the three 2-flats in $PG(5, 2)$ which are rows of the following three matrices $\boldsymbol{M_1}$, $\boldsymbol{M_2}$, and $\boldsymbol{M_3}$ respectively.

$$\boldsymbol{M_1}=\boldsymbol{L}\left[W(1),W(\omega)\right]=\begin{bmatrix}1&0&0&0&1&0\\0&1&0&0&0&1\\0&0&1&1&1&0\\1&1&0&0&1&1\\0&1&1&1&1&1\\1&1&1&1&0&1\\1&0&1&1&0&0\end{bmatrix},$$

$$\boldsymbol{M_2}=\boldsymbol{L}\left[W(\omega),W(\omega^3)\right]=\begin{bmatrix}0&1&0&1&1&0\\0&0&1&0&1&1\\1&1&0&1&1&1\\0&1&1&1&0&1\\1&1&1&1&0&0\\1&0&1&0&1&0\\1&0&0&0&0&1\end{bmatrix},$$

$$\boldsymbol{M_3}=\boldsymbol{L}\left[W(\omega^3),W(1)\right]=\begin{bmatrix}1&1&0&1&0&0\\0&1&1&0&1&0\\1&1&1&0&0&1\\1&0&1&1&1&0\\1&0&0&0&1&1\\0&1&0&1&1&1\\0&0&1&1&0&1\end{bmatrix}.$$

We observe that for each $i=1,\cdots,7$, the $i$th rows of $\boldsymbol{M_1}$, $\boldsymbol{M_2}$, and $\boldsymbol{M_3}$ are three points of a 1-flat in $PG(5, 2)$. Hence we can replace the three 8-level columns corresponding to $S_1$ in $L_{64}(8^9)$ by seven 4-level columns to obtain an $L(8^6 4^7)$. Continuing this procedure, we can replace the three 8-level columns corresponding to $S_2$ in $L_{64}(8^6 4^7)$ by seven 4-level columns to obtain an $L_{64}(8^3 4^{14})$. □

Note that $L_{64}(8^6 4^7)$ and $L_{64}(8^3 4^{14})$ were also construct in [13] using a different method. However, Theorem 2 is more versatile as shown in following example.

**Example 2.** Let $0,1,\omega,\cdots,\omega^{14}$ be the 16 elements of $GF(16)$ with $\omega^4=\omega+1$. Consider $PG(1, 16)$ with 17 points (0, 1), (1, 0), (1, 1), $(1, \omega)$, $\cdots$, $(1, \omega^{14})$. Each point of $PG(1, 16)$ corresponds to a 3-flat in $PG(7, 2)$, and the seventeen 3-flats partition $PG(7, 2)$. We can construct an $L_{256}(16^{17})$ by Theorem 1. Let

$$S_1=\left\{(1,\omega^7),(\omega,\omega^9),(\omega^2,\omega^{12}),(\omega^4,1),(\omega^5,\omega^8),(\omega^8,\omega^2),(\omega^{10},\omega^{11})\right\},$$
$$S_2=\left\{(1,\omega^{12}),(\omega,\omega^3),(\omega^2,\omega),(\omega^4,\omega^{10}),(\omega^5,\omega^9),(\omega^8,\omega^{13}),(\omega^{10},\omega^8)\right\},$$

 

$$T_1=\{(0,1),(1,0),(1,1)\},\ T_2=\{(1,\omega),(\omega,\omega^4),(\omega^4,1)\},\ T_3=\{(1,\omega^2),(\omega^2,\omega^8),(\omega^8,1)\},$$
$$T_4=\{(1,\omega^4),(\omega^4,\omega),(\omega,1)\},\text{ and } T_5=\{(1,\omega^8),(\omega^8,\omega^2),(\omega^2,1)\}.$$

We can verify that $S_1$ and $S_2$ are disjoint 2-flats and $T_1,\cdots,T_5$ are disjoint 1-flats over $GF(2)$ in $PG(1, 16)$. Moreover, $S_1$, $S_2$, and $T_1$ partition $PG(1, 16)$. By the discussion following Theorem 2, we can replace the subarray $L_{256}(16^7)$ corresponding to $S_1$ or $S_2$ in $L_{256}(16^{17})$ by $L_{256}(16^4 8^3 4^8)$ or $L_{256}(8^{15-3i}4^{7i})$, $0 \le i \le 5$. Similarly, we can replace the subarray $L_{256}(16^3)$ corresponding to $T_1,\cdots,T_5$ in $L_{256}(16^{17})$ by $L_{256}(8^3 4^8)$. Many mixed orthogonal arrays such as $L_{256}(16^{10}8^{15})$, $L_{256}(16^3 8^{30})$, $L_{256}(16^{10}8^{12}4^7)$, $L_{256}(16^{10}8^9 4^{14})$, $L_{256}(16^{14}8^3 4^8)$, $L_{256}(16^7 8^{18} 4^8)$, $L_{256}(16^{11}8^6 4^{16})$, $\cdots$, can be obtained by this procedure. □

## 4. Construction of More Orthogonal Arrays

In this section, methods for finding disjoint flats over $GF(s)$ in $PG(m-1,s^n)$ are developed to construct more orthogonal arrays. Let $\alpha$ be a primitive element of $GF(s^m)$, and let the $m \times m$ matrix representation of $\alpha$ in $GF(s)$ be $W$. Since $\alpha^{(s^m-1)/(s-1)}$ is an element of $GF(s)$ and $W^{(s^m-1)/(s-1)}$ is the matrix representation of $\alpha^{(s^m-1)/(s-1)}$, we have $W^{(s^m-1)/(s-1)}=\alpha^{(s^m-1)/(s-1)}I_m$, where $I_m$ is the $m \times m$ identity matrix. Then for any fixed point $x=(x_1,\cdots,x_m)$ in $PG(m-1,s^n)$, the set $S_x=\{xW^i : i\ge 0\}$ contains at most $(s^m-1)/(s-1)$ points in $PG(m-1,s^n)$ since $xW^{(s^m-1)/(s-1)}$ ($=x\alpha^{(s^m-1)/(s-1)}I_m=\alpha^{(s^m-1)/(s-1)}x$) and $x$ represent the same point. Moreover, if $\beta$ and $\gamma$ are any elements of $GF(s)$ and $xW^i$ and $xW^j$ are elements of $S_x$, then $\beta xW^i+\gamma xW^j = x(\beta W^i+\gamma W^j)=xW^l$ for some $l$, since $\beta W^i+\gamma W^j$ is the matrix representation of the element $\beta\alpha^i+\gamma\alpha^j$ of $GF(s^m)$. $S_x$ has the structure of a flat over $GF(s)$ in $PG(m-1,s^n)$ since linear combinations of points in $S_x$ are also points in $S_x$. In fact, $S_x$ is a $(t-1)$-flat over $GF(s)$ in $PG(m-1,s^n)$ if and only if the number of points in $S_x$ is $(s^t-1)/(s-1)$ for some integer $t$. Now if $x$ and $y$ are two points in $PG(m-1,s^n)$ and $S_x\cap S_y\ne\phi$, then there exist $i$ and $j$ such that $xW^i = yW^j$. We have $y=xW^{i-j}\in S_x$, hence $S_x = S_y$.

**Theorem 3.** *Let $x$ be a point in $PG(m-1,s^n)$, and let $S_x=\{xW^i : i\ge 0\}$. Then $S_x$ is a $(t-1)$-flat over $GF(s)$ in $PG(m-1,s^n)$ if and only if the number of points in $S_x$ is $(s^t-1)/(s-1)$ for some integer $t$. Moreover, for any two points $x$ and $y$ in $PG(m-1,s^n)$ either $S_x = S_y$ or $S_x\cap S_y=\phi$. Hence $PG(m-1,s^n)$ can be partitioned into disjoint sets of $S_x$'s.*

**Example 3.** We illustrate how we obtain the three disjoint 1-flats over $GF(2)$ in $PG(1, 8)$ in Example 1. Let $\omega$ be a primitive element of $GF(8)$ with $\omega^3 = \omega + 1$, and let $\alpha$ be a primitive element of $GF(4)$ with $\alpha^2 = \alpha + 1$ and matrix representation

$$W=\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}.$$

Then

$$S_{(0,1)}=\{(0,1),(0,1)W,(0,1)W^2\}=\{(0,1),(1,1),(1,0)\},$$
$$S_{(1,\omega)}=\{(1,\omega),(1,\omega)W,(1,\omega)W^2\}=\{(1,\omega),(\omega,\omega^3),(\omega^3,1)\},\text{ and}$$
$$S_{(1,\omega^3)}=\{(1,\omega^3),(1,\omega^3)W,(1,\omega^3)W^2\}=\{(1,\omega^3),(\omega^3,\omega),(\omega,1)\}$$

are three disjoint 1-flats over $GF(2)$ in $PG(1, 8)$. □

**Example 4.** Let $\omega$ be a primitive element of $GF(16)$ with $\omega^4 = \omega + 1$, and let $\alpha$ be a primitive element of $GF(4)$ with $\alpha^2 = \alpha + 1$ and matrix representation $W$ given in Example 3. The 17 points of $PG(1, 16)$ can be partitioned into the following flats over $GF(2)$:

$$S_{(0,1)}=\{(0,1),(1,1),(1,0)\},$$
$$S_{(1,\omega)}=\{(1,\omega),(\omega,\omega^4),(\omega^4,1)\},$$
$$S_{(1,\omega^2)}=\{(1,\omega^2),(\omega^2,\omega^8),(\omega^8,1)\},$$
$$S_{(1,\omega^4)}=\{(1,\omega^4),(\omega^4,\omega),(\omega,1)\},$$
$$S_{(1,\omega^8)}=\{(1,\omega^8),(\omega^8,\omega^2),(\omega^2,1)\},$$
$$S_{(1,\omega^5)}=\{(1,\omega^5)\},\text{ and } S_{(1,\omega^{10})}=\{(1,\omega^{10})\}.$$

The five disjoint 1-flats over $GF(2)$ are $T_1,\cdots,T_5$ in Example 2. □

**Theorem 4.** *If $s$ is a prime power and $m$ and $n$ are relatively prime, then we can construct mixed orthogonal arrays*

$$L_{s^{mn}}\left(\left(s^n\right)^{\left(s^{mn}-1\right)/\left(s^n-1\right)-i\left(s^m-1\right)/\left(s-1\right)}\left(s^m\right)^{i\left(s^n-1\right)/\left(s-1\right)}\right)$$

*for* $i = 0,\cdots,\left[\left(s^{mn}-1\right)\left(s-1\right)\right]/\left[\left(s^m-1\right)\left(s^n-1\right)\right]$.

**Proof.** We can construct an $L_{s^{mn}}\left(\left(s^n\right)^{\left(s^{mn}-1\right)/\left(s^n-1\right)}\right)$ from $PG\left(m-1,s^n\right)$. From the proof of Theorem 4.3.6 [15], if $m$ and $n$ are relatively prime then $S_x$ is an $(m-1)$-flat over $GF(s)$ in $PG\left(m-1,s^n\right)$ for every point $x$ in $PG\left(m-1,s^n\right)$. Hence $PG\left(m-1,s^n\right)$ can be partitioned into $\left[\left(s^{mn}-1\right)\left(s-1\right)\right]/\left[\left(s^m-1\right)\left(s^n-1\right)\right]$ $(m-1)$-flats over $GF(s)$. Each $S_x$ represents $\left[\left(s^m-1\right)/\left(s-1\right)\right]$ $s^n$-level columns in $L_{s^{mn}}\left(\left(s^n\right)^{\left(s^{mn}-1\right)/\left(s^n-1\right)}\right)$, and by Theorem 2 it can be replaced by $\left(s^n-1\right)/\left(s-1\right)$ $s^m$-level columns. □

The following result which follows from Theorem 4 is a generalization of Theorem 4.

**Corollary 1.** *If* s *is a prime power and d is the greatest common divisor of integers m and n, then we can construct mixed orthogonal arrays*

$$L_{s^{mn/d}}\left(\left(s^n\right)^{\left(s^{mn}-1\right)/\left(s^n-1\right)-i\left(s^m-1\right)/\left(s^d-1\right)}\left(s^m\right)^{i\left(s^n-1\right)/\left(s^d-1\right)}\right)$$

for $i = 0,\cdots,\left[\left(s^{mn/d}-1\right)\left(s^d-1\right)\right]/\left[\left(s^m-1\right)\left(s^n-1\right)\right]$.

**Proof.** If $d$ is the greatest common divisor of $m$ and $n$, then $m/d$ and $n/d$ are relatively prime. By substituting $m$, $n$, and $s$ with $m/d$, $n/d$, and $s^d$ respectively in Theorem 4, we obtain the mixed orthogonal arrays. □

By using Theorem 4 and Corollary 1, we obtain the following *new* series of tight orthogonal arrays for any prime power $s$.

1) $L_{s^6}\left(\left(s^3\right)^{\frac{s^6-1}{s^3-1}-\frac{i\left(s^2-1\right)}{s-1}}\left(s^2\right)^{\frac{i\left(s^3-1\right)}{s-1}}\right)$,

$0 \le i \le s^2 - s + 1$;

2) $L_{s^{10}}\left(\left(s^5\right)^{\frac{s^{10}-1}{s^5-1}-\frac{i\left(s^2-1\right)}{s-1}}\left(s^2\right)^{\frac{i\left(s^5-1\right)}{s-1}}\right)$,

$0 \le i \le \left(s^5+1\right)/\left(s+1\right)$;

3) $L_{s^{12}}\left(\left(s^4\right)^{\frac{s^{12}-1}{s^4-1}-\frac{i\left(s^3-1\right)}{s-1}}\left(s^3\right)^{\frac{i\left(s^4-1\right)}{s-1}}\right)$,

$0 \le i \le \left(s^4-s^2+1\right)\left(s^2-s+1\right)$;

4) $L_{s^{12}}\left(\left(s^6\right)^{\frac{s^{12}-1}{s^6-1}-\frac{i\left(s^4-1\right)}{s^2-1}}\left(s^4\right)^{\frac{i\left(s^6-1\right)}{s^2-1}}\right)$,

$0 \le i \le s^4 - s^2 + 1$;

5) $L_{s^{14}}\left(\left(s^7\right)^{\frac{s^{14}-1}{s^7-1}-\frac{i\left(s^2-1\right)}{s-1}}\left(s^2\right)^{\frac{i\left(s^7-1\right)}{s-1}}\right)$,

$0 \le i \le \left(s^7+1\right)/\left(s+1\right)$; and

6) $L_{s^{15}}\left(\left(s^5\right)^{\frac{s^{15}-1}{s^5-1}-\frac{i\left(s^3-1\right)}{s-1}}\left(s^3\right)^{\frac{i\left(s^5-1\right)}{s-1}}\right)$,

$0 \le i \le \left(s^{10}+s^5+1\right)/\left(s^2+s+1\right)$.

The following theorem gives a set of $s-1$ disjoint $(n-1)$-flats over $GF(s)$ in $PG(1, s^n)$.

**Theorem 5.** *For* $i = 0,\cdots,s-2$, *let* $T_i = \left\{\left(\gamma,\omega^i\gamma^s\right):\gamma\in GF\left(s^n\right)\backslash\{0\}\right\}$, *where* ω *is a primitive element of* $GF\left(s^n\right)$. *Then* $T_0,\cdots,T_{s-2}$, *are* $s-1$ *disjoint* $(n-1)$ *-flats over* $GF(s)$ *in* $PG(1, s^n)$.

**Proof.** $T_i$ is a set of $\left(s^n-1\right)/\left(s-1\right)$ points in $PG(1, s^n)$, since $\left(\alpha\gamma,\omega^i\left(\alpha\gamma\right)^s\right)$ $\left(=\alpha\left(\gamma,\omega^i\gamma^s\right)\right)$ represents the same point for each nonzero element $\alpha$ of $GF(s)$. To show that $T_i$ is an $(n-1)$-flat over $GF(s)$, we prove that any linear combination of elements in $T_i$ is again in $T_i$. If $\left(\gamma_1,\omega^i\gamma_1{}^s\right),\left(\gamma_2,\omega^i\gamma_2{}^s\right)\in T_i$ and $\alpha_1,\alpha_2\in GF\left(s^n\right)$, then

$$\begin{aligned}&\alpha_1\left(\gamma_1,\omega^i\gamma_1{}^s\right)+\alpha_2\left(\gamma_2,\omega^i\gamma_2{}^s\right)\\&=\left(\alpha_1\gamma_1+\alpha_2\gamma_2,\omega^i\left(\alpha_1\gamma_1+\alpha_2\gamma_2\right)^s\right)\in T_i.\end{aligned}$$

For $0 \le i < j \le s-2$, if $\left(\gamma_1,\omega^i\gamma_1{}^s\right)\in T_i$ and $\left(\gamma_2,\omega^j\gamma_2{}^s\right)\in T_j$ represent the same points in $PG(1, s^n)$, then $\omega^i\gamma_1^{s-1}=\omega^j\gamma_2^{s-1}$. Hence $\omega^{j-i}=\left(\gamma_1/\gamma_2\right)^{s-1}$. But $\left(\gamma_1/\gamma_2\right)^{s-1}=\omega^{k(s-1)}$ for some $0\le k\le\left(s^n-1\right)/\left(s-1\right)-1$, which contradicts $0 \le i < j \le s-2$. Hence $T_i$ and $T_j$ are disjoint for all $0 \le i < j \le s-2$. □

**Corollary 2.** $L_{s^{2n}}\left(\left(s^n\right)^{s^n+1-i\left(s^{n-1}-1\right)/\left(s-1\right)}\left(s^{n-1}\right)^{i\left(s^n-1\right)/\left(s-1\right)}\right)$ *can be constructed for any integer n, prime power s, and* $i=1,\cdots,s-1$.

**Proof.** We can construct an $L_{s^{2n}}\left(\left(s^n\right)^{s^n+1}\right)$ from $PG(1, s^n)$. For each $i=0,\cdots,s-2$, let $T_i'\subset T_i$ be an $(n-2)$-flat over $GF(s)$ in $PG(1, s^n)$. $\left\{T_i':i=0,\cdots,s-2\right\}$ is a set of $s-1$ disjoint $(n-2)$-flats over $GF(s)$ in $PG(1, s^n)$. Then for each $T_i'$ we replace the corresponding

$\left[\left(s^{n-1}-1\right)/(s-1)\right]$ $s^n$-level columns in $L_{s^{2n}}\left(\left(s^n\right)^{s^n+1}\right)$ by $\left[\left(s^n-1\right)/(s-1)\right]$ $s^{n-1}$-level columns to obtain the orthogonal array. □

**Example 5.** Let $\omega$ be the primitive element of $GF(16)$ with $\omega^4 = \omega + 1$.

$$T_0=\left\{\left(\gamma,\gamma^2\right):\gamma\in GF(16)\setminus\{0\}\right\}=\left\{(1,1),\left(\omega,\omega^2\right),\left(\omega^2,\omega^4\right),\left(\omega^3,\omega^6\right),\left(\omega^4,\omega^8\right),\left(\omega^5,\omega^{10}\right),\left(\omega^6,\omega^{12}\right),\left(\omega^7,\omega^{14}\right),\left(\omega^8,\omega\right),\left(\omega^9,\omega^3\right),\left(\omega^{10},\omega^5\right),\left(\omega^{11},\omega^7\right),\left(\omega^{12},\omega^9\right),\left(\omega^{13},\omega^{11}\right),\left(\omega^{14},\omega^{13}\right)\right\}$$

is a 3-flat over $GF(2)$ in $PG(1, 16)$ and

$$T_0'=\left\{(1,1),\left(\omega,\omega^2\right),\left(\omega^2,\omega^4\right),\left(\omega^4,\omega^8\right),\left(\omega^5,\omega^{10}\right),\left(\omega^8,\omega\right),\left(\omega^{10},\omega^5\right)\right\}\subset T_0$$

is a 2-flat over $GF(2)$ in $PG(1, 16)$. □

Note that we are able to find two disjoint 2-flats over $GF(2)$ in $PG(1, 16)$ in Example 2 by trial and error. However, we do not have a method to find more than $s$-1 disjoint ($n$-2)-flats over $GF(s)$ in $PG(1, s^n)$. With $n = 4, 5, 6$ and 7 in Corollary 2, we obtain the following new series of tight orthogonal arrays for any prime power $s$ and $i=1,\cdots,s-1$.

1) $L_{s^8}\left(\left(s^4\right)^{s^4+1-i\left(s^3-1\right)/(s-1)}\left(s^3\right)^{i\left(s^4-1\right)}\right)$;

2) $L_{s^{10}}\left(\left(s^5\right)^{s^5+1-i\left(s^4-1\right)/(s-1)}\left(s^4\right)^{i\left(s^5-1\right)}\right)$;

3) $L_{s^{12}}\left(\left(s^6\right)^{s^6+1-i\left(s^5-1\right)/(s-1)}\left(s^5\right)^{i\left(s^6-1\right)}\right)$; and

4) $L_{s^{14}}\left(\left(s^7\right)^{s^7+1-i\left(s^6-1\right)/(s-1)}\left(s^6\right)^{i\left(s^7-1\right)}\right)$.

The following theorem gives an $n$-flat over $GF(s)$ in $PG(2, s^n)$. The proof is omitted since it is similar to that of Theorem 5.

**Theorem 6.** *For any integer* $n\geq 2$ *and* $\beta\in GF\left(s^n\right)\setminus\{0\}$,

$$T_\beta=\left\{\left(\gamma,\gamma^s,\alpha\beta\right):\gamma\in GF\left(s^n\right),\alpha\in GF(s),(\alpha,\gamma)\neq(0,0)\right\}$$

*is an n-flat over GF(s) in PG(2, $s^n$).*

However, for $\beta_1 \neq \beta_2$ the $n$-flats over $GF(s)$ $T_{\beta_1}$ and $T_{\beta_2}$ are not disjoint in $PG(2, s^n)$. But if $s = 2$, we can find more disjoint $n$-flats over $GF(2)$ in $PG(2, 2^n)$.

**Theorem 7.** *Let ω be a primitive element of GF($2^n$), and let*

$$S=\left\{\left(\gamma,\alpha\omega,\gamma^2\right):\gamma\in GF\left(2^n\right),\alpha\in GF(2),(\alpha,\gamma)\neq(0,0)\right\},$$

$$T=\left\{\left(\gamma,\gamma^2,\alpha\right):\gamma\in GF\left(2^n\right),\alpha\in GF(2),(\alpha,\gamma)\neq(0,0)\right\},$$

$$U=\left\{\left(\gamma^2,\alpha\omega,\gamma\right):\gamma\in GF\left(2^n\right),\alpha\in GF(2),(\alpha,\gamma)\neq(0,0)\right\}, \text{ and}$$

$$V=\left\{\left(\alpha\omega^2,\gamma^2,\gamma\right):\gamma\in GF\left(2^n\right),\alpha\in GF(2),(\alpha,\gamma)\neq(0,0)\right\}.$$

*Then we have*

1) *S and T are disjoint n-flats over GF(2) in PG(2, $2^n$) for $n\geq 2$.*

2) *T, U and V are three disjoint n-flats over GF(2) in PG(2, $2^n$) if n is even.*

**Proof.** By Theorem 6, $S$, $T$, $U$ and $V$ are $n$-flats over $GF(2)$ in $PG(2, 2^n)$. We now prove that $S$ and $T$ are disjoint. Assume that $\left(\gamma_1,\alpha_1\omega,\gamma_1^2\right)\in S$ and $\left(\gamma_2,\gamma_2^2,\alpha_2\right)\in T$ represent the same point in $PG(2, 2^n)$, where $\alpha_1,\alpha_2\in GF(2)$ and $\gamma_1,\gamma_2\in GF\left(2^n\right)$. Clearly, $\alpha_1, \alpha_2, \gamma_1, \gamma_2 \neq 0$, hence $\alpha_1 = \alpha_2 = 1$ and $\left(\gamma_1,\omega,\gamma_1^2\right)$ and $\left(\gamma_2,\gamma_2^2,1\right)$ represent the same point. We have $\omega/\gamma_1=\gamma_2$ and $\gamma_1=1/\gamma_2$, which imply $\omega = 1$, a contradiction. Hence $S$ and $T$ are disjoint. Now we show that $T$ and $U$ are disjoint if $n$ is even. If $n$ is even then 3 divides $2^n - 1$. For any $\gamma\in GF\left(2^n\right)\setminus\{0\}$, $\gamma^3 = \omega^{3k}$ for some $0\leq k\leq\left(2^n-1\right)/3-1$. Assume that $\left(\gamma_2,\gamma_2^2,\alpha_2\right)\in T$ and $\left(\gamma_3^2,\alpha_3\omega,\gamma_3\right)\in U$ represent the same point in $PG(2, 2^n)$, where $\alpha_2,\alpha_3\in GF(2)$ and $\gamma_2,\gamma_3\in GF\left(2^n\right)$. Clearly, $\alpha_2, \alpha_3, \gamma_2, \gamma_3 \neq 0$, hence $\alpha_2 = \alpha_3 = 1$ and $\left(\gamma_2,\gamma_2^2,1\right)$ and $\left(\gamma_3^2,\omega,\gamma_3\right)$ represent the same point. We have $\gamma_2 = \gamma_3$ and $\gamma_2^2=\omega/\gamma_3$, which imply $\omega=\gamma_2^3=\omega^{3k}$ for some $0\leq k\leq\left(2^n-1\right)/3-1$, a contradiction. Hence $T$ and $U$ are disjoint. We can similarly prove that $T$ and $V$ are disjoint and that $U$ and $V$ are disjoint if $n$ is even. □

An $L_{s^{3n}}\left(\left(s^n\right)^{s^{2n}+s^n+1}\right)$ can be constructed from $PG(2, s^n)$. By applying Theorems 2, 6, and 7, we obtain the following orthogonal arrays.

**Corollary 3.** *For any prime power s, we can construct*

1) $L_{s^{3n}}\left(\left(s^{n+1}\right)^{\left(s^n-1\right)/(s-1)}\left(s^n\right)^{s^{2n}-\left(s^n-s\right)/(s-1)}\right),\ n\geq 2$;

2) $L_{2^{3n}}\left(\left(2^{n+1}\right)^{2^{n+1}-2}\left(2^n\right)^{2^{2n}-3\cdot 2^n+3}\right),\ n\geq 2$; *and*

3) $L_{2^{6n}}\left(\left(2^{2n+1}\right)^{3\cdot 2^{2n}-3}\left(2^{2n}\right)^{2^{4n}-5\cdot 2^n+4}\right),\ n\geq 1$

**Example 6.** Let $\omega$ be the primitive element of $GF(8)$ with $\omega^3=\omega+1$. Let

$$S=\left\{(1,1,0),\left(\omega,\omega^2,0\right),\left(\omega^2,\omega^4,0\right),\left(\omega^3,\omega^6,0\right),\left(\omega^4,\omega,0\right),\left(\omega^5,\omega^3,0\right),\left(\omega^6,\omega^5,0\right),(0,0,1),\right.$$
$$\left.(1,1,1),\left(\omega,\omega^2,1\right),\left(\omega^2,\omega^4,1\right),\left(\omega^3,\omega^6,1\right),\left(\omega^4,\omega,1\right),\left(\omega^5,\omega^3,1\right),\left(\omega^6,\omega^5,1\right)\right\}$$

and

$$T=\left\{(1,0,1),\left(\omega,0,\omega^2\right),\left(\omega^2,0,\omega^4\right),\left(\omega^3,0,\omega^6\right),\left(\omega^4,0,\omega\right),\left(\omega^5,0,\omega^3\right),\left(\omega^6,0,\omega^5\right),(0,\omega,0),\right.$$
$$\left.(1,\omega,1),\left(\omega,\omega,\omega^2\right),\left(\omega^2,\omega,\omega^4\right),\left(\omega^3,\omega,\omega^6\right),\left(\omega^4,\omega,\omega\right),\left(\omega^5,\omega,\omega^3\right),\left(\omega^6,\omega,\omega^5\right)\right\}$$

be two disjoint 3-flats over $GF(2)$ in $PG(2,8)$. An $L_{512}(8^{73})$ can be constructed from $PG(2,8)$. We can replace the subarray $L_{512}(8^{15})$ corresponding to $S$ or $T$ by an $L_{512}(16^7)$ to obtain $L_{512}(16^7 8^{58})$ and $L_{512}(16^{14}8^{43})$. □

The following two examples are obtained by applying Theorems 3 and 5 and by trial and error.

**Example 7.** Let $\omega$ be the primitive element of $GF(8)$ with $\omega^3=\omega+1$. Let

$$A_1=\left\{\left(1,\omega,\omega^2\right),\left(0,1,\omega^2\right),\left(1,\omega^3,0\right)\right\},$$
$$B_1=\left\{\left(1,\omega^2,\omega^4\right),\left(0,1,\omega^4\right),\left(1,\omega^6,0\right)\right\},$$
$$C_1=\left\{\left(1,\omega^4,\omega\right),\left(0,1,\omega\right),\left(1,\omega^5,0\right)\right\},$$

and $W$ be the 3 × 3 matrix defined in Example 1. For $i=2,\cdots,7$, let $A_i$ (or $B_i$, $C_i$) be the set obtained by multiplying each element in $A_i$ (or $B_i$, $C_i$) by $W$. For Example,

$$A_2=\left\{\left(1,\omega,\omega^2\right)W,\left(0,1,\omega^2\right)W,\left(1,\omega^3,0\right)W\right\}=\left\{\left(\omega^2,\omega^6,\omega\right),\left(\omega^2,\omega^2,1\right),\left(0,1,\omega^3\right)\right\}.$$

It can be verified that $A_1,\cdots,A_7$, $B_1,\cdots,B_7$, $C_1,\cdots,C_7$, and $\left\{(1,0,0),(0,1,0),(1,1,0)\right\}$ are 22 disjoint 1-flats over $GF(2)$ in $PG(2, 8)$. An $L_{512}(8^{73})$ can be constructed from $PG(2, 8)$. We can replace the subarray $L_{512}(8^3)$ corresponding to each 1-flat over $GF(2)$ in $PG(2, 8)$ by an $L_{512}(4^7)$ to obtain $L_{512}(8^{73-3i}4^{7i})$ for $i=1,\cdots,22$. □

**Example 8.** Let $\omega$ be the primitive element of $GF(32)$ with $\omega^5=\omega^2+1$. An $L_{1024}(32^{33})$ can be constructed from $PG(1, 32)$.

1)

$$A_1=\left\{(1,0),(0,1),(1,1)\right\},$$
$$A_2=\left\{(1,\omega),\left(\omega,\omega^{18}\right),\left(\omega^{18},1\right)\right\},$$
$$A_3=\left\{\left(1,\omega^2\right),\left(\omega^2,\omega^5\right),\left(\omega^5,1\right)\right\},$$
$$A_4=\left\{\left(1,\omega^4\right),\left(\omega^4,\omega^{10}\right),\left(\omega^{10},1\right)\right\},$$
$$A_5=\left\{\left(1,\omega^5\right),\left(\omega^5,\omega^2\right),\left(\omega^2,1\right)\right\},$$
$$A_6=\left\{\left(1,\omega^8\right),\left(\omega^8,\omega^{20}\right),\left(\omega^{20},1\right)\right\},$$
$$A_7=\left\{\left(1,\omega^9\right),\left(\omega^9,\omega^{16}\right),\left(\omega^{16},1\right)\right\},$$
$$A_8=\left\{\left(1,\omega^{10}\right),\left(\omega^{10},\omega^4\right),\left(\omega^4,1\right)\right\},$$
$$A_9=\left\{\left(1,\omega^{14}\right),\left(\omega^{14},\omega^{13}\right),\left(\omega^{13},1\right)\right\},$$
$$A_{10}=\left\{\left(1,\omega^{16}\right),\left(\omega^{16},\omega^9\right),\left(\omega^9,1\right)\right\},\text{ and}$$
$$A_{11}=\left\{\left(1,\omega^{19}\right),\left(\omega^{19},\omega^{11}\right),\left(\omega^{11},1\right)\right\}$$

are eleven disjoint 1-flats over $GF(2)$ in $PG(1, 32)$. We can replace the subarray $L_{1024}(32^3)$ corresponding to each 1-flat over $GF(2)$ in the $L_{1024}(32^{33})$ by an $L_{1024}(16^3 4^{16})$ to obtain $L_{1024}(32^{33-3i}16^{3i}4^{16i})$ for $i=1,\cdots,11$.

2)

$$B_1=\left\{\left(1,\omega^{21}\right),\left(\omega,\omega^{23}\right),\left(\omega^{18},\omega^{26}\right),\left(\omega^2,\omega^{25}\right),\left(\omega^5,1\right),\left(\omega^{19},\omega^{28}\right),\left(\omega^{11},\omega^{12}\right)\right\},$$
$$B_2=\left\{\left(1,\omega^{25}\right),\left(\omega,\omega^8\right),\left(\omega^{18},\omega^7\right),\left(\omega^2,\omega^{14}\right),\left(\omega^5,\omega^2\right),\left(\omega^{19},\omega^4\right),\left(\omega^{11},\omega^{29}\right)\right\},$$
$$B_3=\left\{\left(1,\omega^2\right),\left(\omega,\omega^5\right),\left(\omega^{18},1\right),\left(\omega^2,\omega^{16}\right),\left(\omega^5,\omega^{15}\right),\left(\omega^{19},\omega^{24}\right),\left(\omega^{11},\omega^9\right)\right\},\text{ and}$$
$$B_4=\left\{\left(1,\omega^6\right),\left(\omega,\omega^{25}\right),\left(\omega^{18},\omega^{17}\right),\left(\omega^2,\omega^{19}\right),\left(\omega^5,\omega^{20}\right),\left(\omega^{19},\omega^{15}\right),\left(\omega^{11},\omega^{22}\right)\right\}$$

are four disjoint 2-flats over $GF(2)$ in $PG(1,32)$. We can replace the subarray $L_{1024}(32^7)$ corresponding to each 2-flat over $GF(2)$ in the $L_{1024}(32^{33})$ by an $L_{1024}(16^7 8^{16})$ or an $L_{1024}(8^{31})$ to obtain $L_{1024}(32^{33-7i-7j}16^{7i}8^{16i+31j})$ for $1\leq i+j\leq 4$.

3)

$$C_1 = B_1 \cup \{(\omega^3, \omega^{27}), (\omega^{29}, \omega^{17}), (\omega^6, \omega^2), (\omega^{27}, \omega^{13}), (\omega^{20}, \omega^{30}), (\omega^8, \omega^6), (\omega^{12}, \omega^{14}), (\omega^{23}, \omega^5)\} \text{ and}$$
$$C_2 = B_2 \cup \{(\omega^3, \omega^{18}), (\omega^{29}, \omega^9), (\omega^6, \omega^{12}), (\omega^{27}, \omega^{26}), (\omega^{20}, \omega^{24}), (\omega^8, \omega^{11}), (\omega^{12}, \omega^{17}), (\omega^{23}, \omega^6)\}$$

are two disjoint 3-flats over $GF(2)$ in $PG(1,32)$, where $B_1$ and $B_2$ are 2-flats over $GF(2)$ in 2). Moreover, $C_1$, $C_2$, and $A_1$ in 1) partition $PG(1, 32)$. We can replace the subarray $L_{1024}(32^{15})$ corresponding to $C_1$ or $C_2$ in the $L_{1024}(32^{33})$ by an $L_{1024}(16^{24}8^{15})$ or an $L_{1024}(16^{31})$ to obtain $L_{1024}(32^{18}16^{24}8^{15})$, $L_{1024}(32^3 16^{48} 8^{30})$, $L_{1024}(32^{18}16^{31})$, $L_{1024}(32^3 16^{62})$, and $L_{1024}(32^3 16^{55} 8^{15})$. □

## 5. Discussion

We use $t$-flats over $GF(s)$ in $PG(m-1, s^n)$ to find different ways to regroup a set of $(n-1)$-flats in $PG(m-1, s^n)$ into disjoint flats. However, many problems remain unsolved. For example, we do not know how many disjoint $(n-2)$-flats over $GF(s)$ exist in $PG(1, s^n)$. Since there are $s^n + 1$ $(= (s^2 - s)(s^{n-1} - 1)/(s-1) + s + 1)$ points in $PG(1, s^n)$, the upper bound for the number of disjoint $(n-2)$-flats over $GF(s)$ equals $s^2$-$s$ if $n \geq 4$ and equals $s^2 - s + 1$ if $n = 3$. An obvious conjecture is that $PG(1, s^n)$ can be partitioned into $(s^2 - s)$ $(n-2)$-flats and one 1-flat over $GF(s)$. This conjecture is true for $n = 3$, since $PG(1, s^3)$ can be partitioned into $(s^2 - s + 1)$ 1-flats over $GF(s)$ by Theorem 4. It is also true for $s = 2$ and $n = 4, 5$, which are shown in Example 2 for $n = 4$ and shown in Example 8(3) for $n = 5$. If the conjecture is true, we can construct an $L_{s^{2n}}\left((s^n)^{s^n + 1 - i(s^{n-1}-1)/(s-1)} (s^{n-1})^{i(s^n - 1)/(s-1)}\right)$ for $n \geq 3$ and $i = 1, \cdots, s^2 - s$, which would be a significant improvement of Corollary 2.

Also we do not know how many disjoint $n$-flats over $GF(s)$ exist in $PG(2, s^n)$. The number of points in $PG(2, s^n)$ is

$$s^{2n} + s^n + 1 = (s^n - s^{n-1})(s^{n+1} - 1)/(s-1) + s^n + s^{n-1} + 1.$$

Hence an upper bound for the number of disjoint $n$-flats over $GF(s)$ in $PG(2, s^n)$ is $s^n - s^{n-1}$ if $n \geq 3$ and is $s^2 - s + 1$ if $n = 2$. The upper bound is attained for $n = 2$, since $PG(2, s^n)$ can be partitioned into $(s^2 - s + 1)$ 2-flats over $GF(s)$ by Theorem 4. In general, the difference between the upper bound and what can be obtained in Theorems 6 and 7 is considerably large for $n \geq 3$. There may be better ways to find disjoint $n$-flats over $GF(s)$ in $PG(2, s^n)$ than the approach used in Theorems 6 and 7. So far, we do not know any example having $s^n - s^{n-1}$ disjoint $n$-flats over $GF(s)$ in $PG(2, s^n)$ for $n \geq 3$.

Another problem which cannot be solved by the approach of this paper is the construction of orthogonal arrays having $s^n$ rows, where $n$ is a prime number. For example, it is known that $L_{128}(16^1 8^{16})$ can be constructed by a mixed spread of $PG(6, 2)$, which consists of a 3-flat and 16 2-flats . But it is not known that if it is possible to find, among those 16 2-flats, disjoint sets of three 2-flats such that each set can be regrouped into seven 1-flats. We could construct an $L_{128}(16^1 8^{16-3i} 4^{7i})$ if there exist $i$ such disjoint sets of three 2-flats.

## REFERENCES


[1] R. C. Bose and K. A. Bush, "Orthogonal Arrays of Strength Two and Three," *The Annals of Mathematical Statistics*, Vol. 23, No. 4, 1952, pp. 508-524. doi:10.1214/aoms/1177729331

[2] R. L. Plackett and J. P. Burma, "The Design of Optimum Multifactorial Experiments," *Biometrika*, Vol. 33, No. 4, 1946, pp. 305-325.

[3] C. R. Rao, "Factorial Experiments Derivable from Combinatorial Arrangements of Arrays," *Journal of Royal Statistical Society* (*Supplement*), Vol. 9, No. 1, 1947, pp. 128-139. doi:10.2307/2983576

[4] C. R. Rao, "Some Combinatorial Problems of Arrays and Applications to Design of Experiments," In: J. N. Srivastava, Ed., *A Survey of Combinatorial Theory*, North-Holland, Amsterdam, 1973, pp. 349-359.

[5] J. C. Wang and C. F. J. Wu, "An Approach to the Construction of Asymmetrical Orthogonal Arrays," *Journal of American Statistical Association*, Vol. 86, No. 414, 1991, pp. 450-456. doi:10.2307/2290593

[6] C. F. J. Wu, R. C. Zhang and R. Wang, "Construction of Asymmetrical Orthogonal Array of Type $OA(s^k, s^m(s^{r_1})^{n_1} \cdots (s^{r_t})^{n_t})$," *Statistica Sinica*, Vol. 2, No. 1, 1992, pp. 203-219.

[7] A. Dey and C. K. Midha, "Construction of Some Asymmetrical Orthogonal Arrays," *Statistics & Probability Letters*, Vol. 28, No. 3, 1996, pp. 211-217. doi:10.1016/0167-7152(95)00126-3

[8] Y. S. Zhang, Y. Q. Lu and S. Q. Pang, "Orthogonal Arrays Obtained by Orthogonal Decomposition of Projection Matrices," *Statistica Sinica*, Vol. 9, No. 2, 1999, pp. 595-604.

[9] C. Suen and W. F. Kuhfeld, "On the Construction of Mixed Orthogonal Arrays of Strength Two," *Journal of Statistical Planning and Inference*, Vol. 133, No. 2, 2005, pp. 555-560. doi:10.1016/j.jspi.2004.03.018

[10] A. S. Hedayat, N. J. A. Sloane and J. Stufken, "Orthogonal Arrays," Springer, New York, 1999. doi:10.1007/978-1-4612-1478-6

[11] S. Addelman, "Orthogonal Main Effect Plans for Asym-

metrical Factorial Experiments," *Technometrics*, Vol. 4, No. 1, 1962, pp. 21-46. doi:10.2307/1266170

[12] C. F. J. Wu, "Construction of $2^m4^n$ Deigns via a Grouping Scheme," *Annals of Statistics*, Vol. 17, No. 4, 1989, pp. 1880-1885. doi:10.1214/aos/1176347399

[13] E. M. Rains, N. J. A. Sloane and J. Stufken, "The Lattice of *N*-Run Orthogonal Arrays," *Journal of Statistical Planning and Inference*, Vol. 102, No. 2, 2002, pp. 477-500. doi:10.1016/S0378-3758(01)00119-7

[14] C. Suen, A. Das and A. Dey, "On the Construction of Asymmetric Orthogonal Arrays," *Statistica Sinica*, Vol. 11, No. 1, 2001, pp. 241-260.

[15] J. W. P. Hirschefld, "Projective Geometries over Finite Fields," Oxford University Press, Oxford, 1979.

[16] C. Suen and A. Dey, "Construction of Asymmetric Orthogonal Arrays through Finite Geometries," *Journal of Statistical Planning and Inference*, Vol. 115, No. 2, 2003, pp. 623-635. doi:10.1016/S0378-3758(02)00165-9

 