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Testing for Cross-Sectional Dependence in a Random Effects Model

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ABSTRACT

This paper extends and generalizes the works of [1,2] to allow for cross-sectional dependence in the context of a two-way error components model and consequently develops LM test. The cross-sectional dependence follows the first order spatial autoregressive error (SAE) process and is imposed on the remainder disturbances. It is important to note that this paper does not consider alternative forms of spatial lag dependence other than SAE. It also does not allow for endogeneity of the regressors and requires the normality assumption to derive the LM test.

Keywords: Cross-Sectional Dependence; Error Components Model; Lagrangian Multiplier (LM) Tests

1. Introduction

The standard error components model assumes, among others, spatial independence across cross-sectional units. However, this restrictive assumption may not hold for a lot of panel data applications. When one begins to look at a cross section of regions, states, countries, etc., these aggregate units may exhibit cross-sectional correlation that has to be dealt with (see [3]). Ignoring cross-sectional dependence when in fact it exists, results in biased, inconsistent and inefficient estimates of regression coefficients (see [1,4,5]).

In the literature, several test statistics have been developed for spatial econometrics however in the context of either cross sectional framework or one-way error components model.[1] The specification of cross-sectional dependence in linear regression models by most of these works follows either spatial autoregressive (SAR) process often defined as spatial lag dependence (see [6-10]); spatial moving average processes (SMA) often called spatial error dependence (see [11]); spatial autoregressive error process (SAE) (see [6,12,13]); SARMA (a combination of SAR and SMA) (see [2,4,14]); a combination of SAR and SARE (see [15]); direct representation form of cross-sectional dependence (see [16,17]) or spatial error component process (SEC) suggested by [18]. Consequently, various tests as well as estimators were derived against these different specification forms using either the Maximum Likelihood (ML) approach (see [2,19]; for a survey of the literature) or Instrumental Variables (IV) and Generalized Method of Moments (GMM) (see [9,20,21]).

The present study develops LM test for cross-sectional dependence in the context of panel data framework. The latter is a two-way random effects model where the cross-sectional dependence follows the SAE and is imposed on the remainder disturbances. Prominent papers that have adopted the SAE include [2,4] in the context of cross-sectional framework, and [1,22,23] in the context of one-way error components framework. Thus, the main objective of this work is to extend and generalize the works of [1,2] to allow for cross-sectional dependence in the context of a two-way error components model. The panel data model considered here is the restricted two-way random effects model assuming no cross-sectional dependence in the remainder disturbances. Thus, the LM test will be similar to the one developed by [1] if we further modify the hypothesis to test for cross-sectional dependence assuming the presence of random individual effect only (while ignoring the presence of time effects). In the same vein, the LM test will be similar to [2,4] if the hypothesis is reconstructed to test for cross-sectional dependence ignoring the presence of both the random country and time effects.

In Section 2, the structure of the two-way error random effects model is described in the context of cross-sectional dependence in the remainder disturbance term. Analyses of the LM test are provided in Section 3 and Section 4 concludes the paper.

[1]A review of score test statistics for alternative specifications in spatial econometrics in the context of cross sectional data and one-way error components model can be found in [2,3] respectively.

2. The Model

We consider the following panel data regression model:

$$y_{ti} = x_{ti}\beta + u_{ti}; i = 1,\cdots,N; t = 1,\cdots,T \quad (1)$$

where the index *i* denotes *N* regional units and the index *t* refers to the *T* observations of each region *i*. The *i* subscript, therefore, denotes the cross-sectional dimension whereas *t* denotes time-series dimension. The total number of observations is *NT*. y_{ti} is the observation on the *i*th region over the *t*th time period; x_{ti} is the *ti*th observation on *k* explanatory variables and u_{ti} is the regression disturbance term. The error term u_{ti} follows a two-way random effects with both regional specific and temporal effects; that is,

$$u_{ti} = \varepsilon_{ti} + \mu_i + v_t \quad (2)$$

where μ_i denotes regional specific effects, v_t denotes temporal effects and ε_{ti} represents the remainder disturbance term. Stacking the *N* observations of each time-period *t*, Equation (2) may be written as:

$$u_t = \varepsilon_t + \mu + i_N v_t \quad (3)$$

where $u_t = (u_{t1}, u_{t2}, \cdots, u_{tN})'$, $i_N = (1,\cdots,1)'$ is a vector of ones of *N* dimension, $\varepsilon_t = (\varepsilon_{t1}, \varepsilon_{t2}, \cdots, \varepsilon_{tN})'$ and $\mu = (\mu_1, \mu_2, \cdots, \mu_N)'$.

Assumption 1: Both μ_i and v_t are assumed independent and normally distributed according to,

$$\mu_i \sim N\left(0, \sigma_\mu^2\right); v_t \sim N\left(0, \sigma_v^2\right) \quad (4)$$

The remainder disturbance term (ε_t) is assumed to follow the first order spatial error correlation (see [2,3]), that is:

$$\varepsilon_t = \lambda W \varepsilon_t + e_t \quad (5)$$

where $\varepsilon_t' = (\varepsilon_{t1}, \ldots, \varepsilon_{tN})$ and $e_t' = (e_{t1}, \ldots, e_{tN})$. The term λ is the scalar spatial autoregressive coefficient with $|\lambda| < 1$. The matrix *W* is an $N \times N$ spatial weight matrix which represents the degree of potential interaction between neighboring locations whose diagonal elements are zero and off-diagonal elements are non-zero. Equation (5) can be further simplified as:

$$\varepsilon_t = \left(I_N - \lambda W\right)^{-1} e_t \quad (6)$$

Given Equation (6), the weight matrix *W* also satisfies the condition that $\left(I_N - \lambda W\right)$ is nonsingular for all $|\lambda| < 1$. e_t is also assumed to be independent and normally distributed as:

$$e_t \sim IN\left(0, \sigma_e^2\right) \quad (7)$$

The e_{ti} process is also independent of the μ_i and v_t terms.

The model (1) can be re-written in matrix notation as:

$$y = X\beta + u \quad (8)$$

where *y* is of dimension $NT \times 1$ vector, *X* is an $NT \times k$ matrix, β is $k \times 1$ vector and *u* is $NT \times 1$ vector. The matrix *X* is assumed to be of full column rank and its elements are assumed to be asymptotically bounded in absolute value. Given Equation (6), Equation (3) can be re-written as:

$$u_t = \left(I_N - \lambda W\right)^{-1} e_t + \mu + i_N v_t \quad (9)$$

We can write Equation (8) in vector from as:

$$u = \left(I_T \otimes B^{-1}\right)e + \left(i_T \otimes I_N\right)\mu + \left(I_T \otimes i_N\right)v \quad (10)$$

The variance-covariance (VCV) matrix (Ω) of Equation (10) (that is, the unrestricted model) can be expressed as:

$$\begin{aligned}\Omega = &\left(I_T \otimes \sigma_e^2 \left(B'B\right)^{-1}\right) + \left(J_T \otimes \sigma_\mu^2 I_N\right) \\ &+ \left(I_T \otimes \sigma_v^2 \left(i_N i_N'\right)\right)\end{aligned} \quad (11)[2]$$

where $J_T = i_T i_T'$ and it is a matrix of ones of dimension *T*. To obtain the spectral decomposition of Equation (11), we use the [24] method. Essentially, we replace J_T by $T\bar{J}_T$ and I_T by $E_T + \bar{J}_T$ where $E_T = I_T - \bar{J}_T$ and $\bar{J}_T = J_T/T$ and consequently, we obtain[3]:

$$\begin{aligned}\Omega = &E_T \otimes \left[\sigma_e^2 \left(B'B\right)^{-1} + \sigma_v^2 \left(i_N i_N'\right)\right] \\ &+ \bar{J}_T \otimes \left[\sigma_e^2 \left(B'B\right)^{-1} + T\sigma_\mu^2 I_N + \sigma_v^2 \left(i_N i_N'\right)\right]\end{aligned} \quad (12)$$

Also, using the [25] method of inversion, Equation (12) can be expressed as:

$$\begin{aligned}\Omega^{-1} &= E_T \otimes \left[\sigma_e^2 \left(B'B\right)^{-1} + \sigma_v^2 \left(i_N i_N'\right)\right]^{-1} \\ &+ \bar{J}_T \otimes \left[\sigma_e^2 \left(B'B\right)^{-1} + T\sigma_\mu^2 I_N + \sigma_v^2 \left(i_N i_N'\right)\right]^{-1} \\ &= E_T \otimes A_1 + \bar{J}_T \otimes A_2\end{aligned} \quad (13)$$

where $A_1 = \left[\sigma_e^2 \left(B'B\right)^{-1} + \sigma_v^2 \left(i_N i_N'\right)\right]^{-1}$ and

$A_2 = \left[\sigma_e^2 \left(B'B\right)^{-1} + T\sigma_\mu^2 I_N + \sigma_v^2 \left(i_N i_N'\right)\right]^{-1}$.

3. Derivation of the LM Test

In this section, we derive the LM test for testing for no cross-sectional dependence in a two-way random effects model. We employ the Maximum Likelihood (ML) approach and consequently, the log-likelihood function. The LM test derived is based on the idea that the score of the likelihood function evaluated under the null is equal to zero when the null hypothesis is true, so that a χ^2 test based on the square of the score divided by the appropriate element of the information matrix (since this is the variance of the score) can be constructed. The use of the normal likelihood function requires the assumption of

[2]See the appendix for the derivation.

[3]Note that E_T and $\bar{J}_T$ are symmetric idempotent matrices.

normality of the error term.

Essentially, the derivation of the LM test involves the following steps:

Step 1: Derive the VCV matrix for the unrestricted model;

Step 2: Derive the VCV matrix for the restricted model;

Step 3: Derive the spectral decomposition for the matrices obtained in steps 1 and 2;

Step 4: Derive the inverse of the matrices obtained in steps 1 and 2 using the results from step 3;

Step 5: Derive the general log-likelihood function;

Step 6: Use the information in steps 1 - 5 to derive the score functions of the likelihood evaluated from the restricted ML under H_0^a;

Step 7: Derive the information matrix and its inverse;

Step 8: Use the results obtained in steps 6 and 7 to develop the LM test.

The log likelihood function, L under normality of disturbances is given as:

$$L(\beta,\phi) = c - \frac{1}{2}\log|\Omega| - \frac{1}{2}u'\Omega^{-1}u \tag{14}$$

where $u = y - X\beta$ and the vector of parameters is denoted as $\psi' = (\beta, \sigma_e^2, \sigma_\mu^2, \sigma_v^2, \lambda) = (\beta, \phi)$ where $\phi' = (\sigma_e^2, \sigma_\mu^2, \sigma_v^2, \lambda)$.

Since our test statistic requires information only on the vector of parameters ϕ, consequently, information due to β is ignored. Following [26], the gradient of the log likelihood with respect to ϕ can be expressed as:

$$\begin{aligned}\frac{\partial L}{\partial \phi} &= -\frac{1}{2}tr\left[\Omega^{-1}(\partial\Omega/\partial\phi)\right] \\ &\quad + \frac{1}{2}\left[u'\Omega^{-1}(\partial\Omega/\partial\phi)\Omega^{-1}u\right]\end{aligned} \tag{15}$$

$$I_{\phi\phi} = E\left[-\frac{\partial^2 L}{\partial\phi_i\partial\phi_j}\right] \tag{16}$$

By further simplification, it is easy to show that:

$$I_{\phi\phi} = \frac{1}{2}tr\left[\Omega^{-1}\frac{\partial\Omega}{\partial\phi_i}\Omega^{-1}\frac{\partial\Omega}{\partial\phi_j}\right]$$

For $i, j = 1,2,3,4$. Equations (15) and (16) represent the score function and the information matrix respectively. The information matrix-$I_{\phi\phi}$ is block diagonal. The LM statistic can, therefore, be written generally as:

$$LM_\phi = \tilde{D}_\phi'\left(I_{\tilde{\phi}\tilde{\phi}}\right)^{-1}\tilde{D}_\phi \tag{17}$$

where $\tilde{D}_\phi$ and $I_{\tilde{\phi}\tilde{\phi}}$ are the score function and information matrix respectively evaluated at the null hypothesis. The LM test statistic expressed in (17) is distributed as $\chi^2_{k_\phi}$ (*i.e.* chi-square distributed) with k_ϕ degrees of freedom, k_ϕ being the number of parameters in the vector ϕ. Based on Equation (17), therefore, the following hypotheses can be tested in relation to cross-sectional dependence:

$$H_0^a : \lambda = 0 \left| \sigma_e^2 > 0; \sigma_\mu^2 > 0; \sigma_v^2 > 0 \right. \tag{18}$$

This is a test of no cross-sectional dependence assumeing the presence of random individual and time effects. This is the null hypothesis this study sets out to test.

$$H_0^b : \lambda = 0 \left| \sigma_e^2 > 0; \sigma_\mu^2 > 0; \sigma_v^2 = 0 \right. \tag{19}$$

This hypothesis tests for cross-sectional dependence assuming the presence of random individual effect only (while ignoring the presence of time effects). This test is similar to [1] LM test for spatial error correlation as well as random country effects.

$$H_0^c : \lambda = 0 \left| \sigma_e^2 > 0; \sigma_\mu^2 = 0; \sigma_v^2 = 0 \right. \tag{20}$$

This hypothesis tests for cross-sectional dependence ignoring the presence of both the random country and time effects. This is similar to the LM test by [2,4].

We derive below the score function for the null hypothesis expressed in (18) above which is the focus of this paper; that is:

$$H_0^a : \lambda = 0 \left| \sigma_e^2 > 0; \sigma_\mu^2 > 0; \sigma_v^2 > 0 \right.$$

Under the null hypothesis in (18), the VCV matrix reduces to:[4]

$$\begin{aligned}\Omega &= E_T \otimes \left[\sigma_e^2 I_N + \sigma_v^2 (i_N i_N')\right] \\ &\quad + \bar{J}_T \otimes \left[\sigma_e^2 I_N + T\sigma_\mu^2 I_N + \sigma_v^2 (i_N i_N')\right]\end{aligned} \tag{21}$$

Given that $\lambda = 0$; then $\varepsilon_{ti} = e_{ti}$ and, therefore, $Var(\varepsilon_{ti}) = Var(e_{ti})$. The Equation (21) is the VCV matrix for the restricted model. Using [25] Lemma 2.1, the inverse of Equation (21) can be expressed as:

$$\begin{aligned}\Omega^{-1} &= E_T \otimes \left[\sigma_e^2 I_N + \sigma_v^2 (i_N i_N')\right]^{-1} \\ &\quad + \bar{J}_T \otimes \left[\sigma_e^2 I_N + T\sigma_\mu^2 I_N + \sigma_v^2 (i_N i_N')\right]^{-1} \\ &= E_T \otimes A_1^a + \bar{J}_T \otimes A_2^a\end{aligned} \tag{22}$$

where $A_1 = \left[\sigma_e^2 I_N + \sigma_v^2 (i_N i_N')\right]^{-1}$ and $A_2 = \left[\sigma_e^2 I_N + T\sigma_\mu^2 I_N + \sigma_v^2 (i_N i_N')\right]^{-1}$.

The Equation (22) is the reduced form of Equation (13) and is also the VCV matrix for the familiar two-way random effects error components model. In addition, it is a principal component required in the log-likelihood function to derive the LM test. In particular, both Equations (21) and (22) are required to derive the partial derivatives and information matrix for the LM test.

[4]See the appendix for the derivation.

Using the general formulas on log likelihood differentiation, we derive its gradients evaluated at the restricted ML under H_0^a as follows:

Recall Equation (15):

$$\frac{\partial L}{\partial \phi} = -\frac{1}{2} tr\left[\Omega^{-1}\left(\partial\Omega/\partial\phi\right)\right] + \frac{1}{2}\left[u'\Omega^{-1}\left(\partial\Omega/\partial\phi\right)\Omega^{-1}u\right]$$

Assumption 2: Let $M = \frac{\partial}{\partial\lambda}\left(\left(B'B\right)^{-1}\right)$, then $M = \left(B'B\right)^{-1}\left[W'B + B'W\right]$. Recall, $B = I_N - \lambda W$ and since under H_0^a, $\lambda = 0$; then $B = I_N$ and $M = W' + W$.

Assumption 3: If E_T, I_T and $\bar{J}_T$ are idempotent and symmetric matrices, we can write that $I_T = E_T + \bar{J}_T$ where $E_T = I_T - \bar{J}_T$. Then, E_T and $\bar{J}_T$ are orthogonal (see [3]).

Proposition 1: Based on assumptions 2 and 3, we can write the derivatives $\left.\frac{\partial\Omega}{\partial\phi}\right|_{H_0^a}$ for the parameters, λ, σ_e^2, σ_μ^2 and σ_v^2, respectively, as:

$$\left.\frac{\partial\Omega}{\partial\lambda}\right|_{H_0^a} = \sigma_e^2\left[I_T \otimes M\right]$$

$$\left.\frac{\partial\Omega}{\partial\sigma_e^2}\right|_{H_0^a} = I_T \otimes I_N$$

$$\left.\frac{\partial\Omega}{\partial\sigma_\mu^2}\right|_{H_0^a} = T\bar{J}_T \otimes I_N$$

$$\left.\frac{\partial\Omega}{\partial\sigma_v^2}\right|_{H_0^a} = I_T \otimes i_N i_N'$$

Proof:

$$\begin{aligned}
&(\mathrm{A})\left.\frac{\partial\Omega}{\partial\lambda}\right|_{H_0^a} \\
&= E_T \otimes \frac{\partial}{\partial\lambda}\left[\sigma_e^2\left(B'B\right)^{-1} + \sigma_v^2\left(i_N i_N'\right)\right] \\
&\quad + \bar{J}_T \otimes \frac{\partial}{\partial\lambda}\left[\sigma_e^2\left(B'B\right)^{-1} + T\sigma_\mu^2 I_N + \sigma_v^2\left(i_N i_N'\right)\right] \\
&= E_T \otimes \frac{\partial}{\partial\lambda}\left(\sigma_e^2\left(B'B\right)^{-1}\right) \\
&\quad + \bar{J}_T \otimes \frac{\partial}{\partial\lambda}\left(\sigma_e^2\left(B'B\right)^{-1}\right) \\
&= \sigma_e^2\left[E_T \otimes \frac{\partial}{\partial\lambda}\left(\left(B'B\right)^{-1}\right) + \bar{J}_T \otimes \frac{\partial}{\partial\lambda}\left(\left(B'B\right)^{-1}\right)\right]
\end{aligned} \tag{24}$$

Based on assumption 1, it is easy to establish from Equation (24) that:

$$\begin{aligned}
\left.\frac{\partial\Omega}{\partial\lambda}\right|_{H_0^a} &= \sigma_e^2\left[\left(E_T + \bar{J}_T\right) \otimes M\right] \\
&= \sigma_e^2\left[I_T \otimes M\right]
\end{aligned} \tag{25}$$ [5]

$$\begin{aligned}
&(\mathrm{B})\left.\frac{\partial\Omega}{\partial\sigma_e^2}\right|_{H_0^a} \\
&= \frac{\partial}{\partial\sigma_e^2}\Big\{E_T \otimes \\
&\quad \cdot\left[\sigma_e^2 I_N + \sigma_v^2\left(i_N i_N'\right)\right] + \bar{J}_T \otimes \left[\sigma_e^2 I_N + T\sigma_\mu^2 I_N + \sigma_v^2\left(i_N i_N'\right)\right]\Big\} \\
&= E_T \otimes \frac{\partial}{\partial\sigma_e^2}\left(\sigma_e^2 I_N + \sigma_v^2 i_N i_N'\right) \\
&\quad + \bar{J}_T \otimes \frac{\partial}{\partial\sigma_e^2}\left(\sigma_e^2 I_N + T\sigma_\mu^2 I_N + \sigma_v^2 i_N i_N'\right) \\
&= E_T \otimes I_N + \bar{J}_T \otimes I_N = \left(E_T + \bar{J}_T\right) \otimes I_N
\end{aligned}$$

$$\left.\frac{\partial\Omega}{\partial\sigma_e^2}\right|_{H_0^a} = I_T \otimes I_N \tag{26}$$

$$\begin{aligned}
&(\mathrm{C})\left.\frac{\partial\Omega}{\partial\sigma_\mu^2}\right|_{H_0^a} \\
&= \frac{\partial}{\partial\sigma_\mu^2}\Big\{E_T \otimes \left[\sigma_e^2 I_N + \sigma_v^2\left(i_N i_N'\right)\right] \\
&\qquad + \bar{J}_T \otimes \left[\sigma_e^2 I_N + T\sigma_\mu^2 I_N + \sigma_v^2\left(i_N i_N'\right)\right]\Big\} \\
&= E_T \otimes \frac{\partial}{\partial\sigma_\mu^2}\left(\sigma_e^2 I_N + \sigma_v^2 i_N i_N'\right) \\
&\quad + \bar{J}_T \otimes \frac{\partial}{\partial\sigma_\mu^2}\left(\sigma_e^2 I_N + T\sigma_\mu^2 I_N + \sigma_v^2 i_N i_N'\right) \\
&= \bar{J}_T \otimes \frac{\partial}{\partial\sigma_\mu^2}\left(\sigma_e^2 I_N + T\sigma_\mu^2 I_N + \sigma_v^2 i_N i_N'\right) \\
&= \bar{J}_T \otimes \frac{\partial}{\partial\sigma_\mu^2}\left(T\sigma_\mu^2 I_N\right)
\end{aligned}$$

$$\left.\frac{\partial\Omega}{\partial\sigma_\mu^2}\right|_{H_0^a} = T\bar{J}_T \otimes I_N \tag{27}$$

$$\begin{aligned}
&(\mathrm{D})\left.\frac{\partial\Omega}{\partial\sigma_v^2}\right|_{H_0^a} \\
&= \frac{\partial}{\partial\sigma_v^2}\Big\{E_T \otimes \left[\sigma_e^2 I_N + \sigma_v^2 i_N i_N'\right] \\
&\qquad + \bar{J}_T \otimes \left[\sigma_e^2 I_N + T\sigma_\mu^2 I_N + \sigma_v^2 i_N i_N'\right]\Big\} \\
&= E_T \otimes \frac{\partial}{\partial\sigma_v^2}\left(\sigma_e^2 I_N + \sigma_v^2 i_N i_N'\right) \\
&\quad + \bar{J}_T \otimes \frac{\partial}{\partial\sigma_v^2}\left(\sigma_e^2 I_N + T\sigma_\mu^2 I_N + \sigma_v^2 i_N i_N'\right) \\
&= E_T \otimes \frac{\partial}{\partial\sigma_v^2}\left(\sigma_v^2 i_N i_N'\right) + \bar{J}_T \otimes \frac{\partial}{\partial\sigma_v^2}\left(\sigma_v^2 i_N i_N'\right) \\
&= E_T \otimes i_N i_N' + \bar{J}_T \otimes i_N i_N' = \left(E_T + \bar{J}_T\right) \otimes i_N i_N'
\end{aligned}$$

[5]We also replace $\left(E_T + \bar{J}_T\right)$ by I_T where $E_T = I_T - \bar{J}_T$.

$$\left.\frac{\partial\Omega}{\partial\sigma_v^2}\right|_{H_0^a} = I_T \otimes i_N i_N' \quad (28)$$

Proposition 2: Based on proposition 1 and assumptions 2 and 3, we can write the derivations of $\left.\Omega^{-1}\frac{\partial\Omega}{\partial\phi}\right|_{H_0^a}$ for the parameters, λ, σ_e^2, σ_μ^2 and σ_v^2, respectively, as:

$$\left.\Omega^{-1}\frac{\partial\Omega}{\partial\lambda}\right|_{H_0^a} = \sigma_e^2\left[\left(E_T \otimes A_1 + \bar{J}_T \otimes A_2\right)M\right]$$

$$\left.\Omega^{-1}\frac{\partial\Omega}{\partial\sigma_e^2}\right|_{H_0^a} = \left(E_T \otimes A_1 + \bar{J}_T \otimes A_2\right)$$

$$\left.\Omega^{-1}\frac{\partial\Omega}{\partial\sigma_\mu^2}\right|_{H_0^a} = \left(T\bar{J}_T \otimes A_2\right)$$

$$\left.\Omega^{-1}\frac{\partial\Omega}{\partial\sigma_v^2}\right|_{H_0^a} = \left[\left(E_T \otimes A_1 + \bar{J}_T \otimes A_2\right)\left(i_N i_N'\right)\right]$$

Proof:

These derivatives are quite straightforward to show particularly using the information in proposition 1.

Proposition 3: Based on propositions 1 and 2 and assumptions 2 and 3, we can write the derivations of $\left.\Omega^{-1}\frac{\partial\Omega}{\partial\phi}\Omega^{-1}\right|_{H_0^a}$ for the parameters, λ, σ_e^2, σ_μ^2 and σ_v^2, respectively, as:

$$\left.\Omega^{-1}\frac{\partial\Omega}{\partial\lambda}\Omega^{-1}\right|_{H_0^a} = \sigma_e^2\left(E_T \otimes A_1 M A_1 + \bar{J}_T \otimes A_2 M A_2\right)$$

$$\left.\Omega^{-1}\frac{\partial\Omega}{\partial\sigma_e^2}\Omega^{-1}\right|_{H_0^a} = \left[E_T \otimes \left(A_1\right)^2 + \bar{J}_T \otimes \left(A_2\right)^2\right]$$

$$\left.\Omega^{-1}\frac{\partial\Omega}{\partial\sigma_\mu^2}\Omega^{-1}\right|_{H_0^a} = \left[T\bar{J}_T \otimes \left(A_1\right)^2\right]$$

$$\left.\Omega^{-1}\frac{\partial\Omega}{\partial\sigma_v^2}\Omega^{-1}\right|_{H_0^a} = E_T \otimes A_1\left(i_N i_N'\right)A_1 + \bar{J}_T \otimes A_2\left(i_N i_N'\right)A_2$$

Proof:

These derivatives are straightforward to show using the information in proposition 2.

Proposition 4: Following propositions 1 - 3, we can easily calculate the partial derivates $\frac{\partial L}{\partial\phi}$ for λ, σ_e^2, σ_μ^2 and σ_v^2, respectively, evaluated at the restricted MLE:

$$\left.\frac{\partial L}{\partial\lambda}\right|_{H_0^a} = D\left(\tilde{\lambda}\right) = \frac{\tilde{\sigma}_e^2\tilde{\sigma}_v^2}{2}\left[\left(T-1\right)\tilde{M} + \tilde{M}_D\right] + \frac{\tilde{\sigma}_e^2}{2}\left[\tilde{\Xi}_* + \tilde{\Xi}_{**}\right]$$

where $\tilde{M} = \frac{i_N' M i_N}{\tilde{\sigma}_e^2\left(\tilde{\sigma}_e^2 + N\tilde{\sigma}_v^2\right)}$; $\tilde{M}_D = \frac{i_N'\tilde{D}^{-1}M\tilde{D}^{-1}i_N}{1 + \tilde{\sigma}_v^2\sum_i \frac{1}{\tilde{\mathfrak{g}}_i}}$

$$\tilde{\Xi}_* = \tilde{u}_*'\left(E_T \otimes M\right)\tilde{u}_*;\ \tilde{\Xi}_{**} = \tilde{u}_{**}'\left(\bar{J}_T \otimes M\right)\tilde{u}_{**}$$

$$\tilde{u}_* = \left(E_T \otimes I_N\right)\Omega^{-1}\tilde{u} = \left(E_T \otimes A_1\right)\tilde{u}$$

$$\tilde{u}_{**} = \left(E_T \otimes I_N\right)\Omega^{-1}\tilde{u} = \left(\bar{J}_T \otimes A_2\right)\tilde{u}$$

$$\left.\frac{\partial L}{\partial\sigma_e^2}\right|_{H_0^a} = D\left(\tilde{\sigma}_e^2\right) = \frac{1}{2}\left[\left(\tilde{\Sigma}_* + \tilde{\Sigma}_{**}\right) - \operatorname{tr}\left[\left(T-1\right)A_1 + A_2\right]\right]$$

where $\tilde{\Sigma}_* = \tilde{u}_*'\tilde{u}_*$; $\tilde{\Sigma}_{**} = \tilde{u}_{**}'\tilde{u}_{**}$ and $\tilde{\sigma}_1^2 = \tilde{\sigma}_e^2 + N\tilde{\sigma}_v^2$.

$$\left.\frac{\partial L}{\partial\sigma_\mu^2}\right|_{H_0^a} = D\left(\tilde{\sigma}_\mu^2\right) = \frac{T}{2}\left[\tilde{\Sigma}_{**} - \operatorname{tr}\left(A_2\right)\right]$$

$$\left.\frac{\partial L}{\partial\sigma_v^2}\right|_{H_0^a} = D\left(\tilde{\sigma}_v^2\right)$$

$$= \frac{N}{2}\left[N^{-1}\left(\tilde{\tilde{\Sigma}}_* + \tilde{\tilde{\Sigma}}_{**}\right) - \operatorname{tr}\left[\left(T-1\right)A_1 + A_2\right]\right]$$

where $\tilde{\tilde{\Sigma}}_* = \tilde{u}_*'\left(E_T \otimes i_N i_N'\right)\tilde{u}_*$ and

$\tilde{\tilde{\Sigma}}_{**} = \tilde{u}_{**}'\left(\bar{J}_T \otimes i_N i_N'\right)\tilde{u}_{**}$.

Proof:

See the appendix for further simplifications and proofs of the partial derivatives.

Recall that we define $\phi' = \left(\sigma_e^2, \sigma_\mu^2, \sigma_v^2, \lambda\right)$, therefore, under H_0^a, $\tilde{\phi}' = \left(\tilde{\sigma}_e^2, \tilde{\sigma}_\mu^2, \tilde{\sigma}_v^2, 0\right)$ can be defined as the solution obtained after maximization of the first order condition and $\tilde{u} = y + X\tilde{\beta}_{MLE}$ is the corresponding residual under H_0^a. Note that all the parameters $\left[i.e.\ \tilde{\phi}' = \left(\tilde{\sigma}_e^2, \tilde{\sigma}_\mu^2, \tilde{\sigma}_v^2, 0\right)\right]$ were evaluated when $D\left(\tilde{\phi}\right) = 0$ at the restricted MLE except $D\left(\tilde{\lambda}\right)$. This is because we are testing whether λ is statistically different from zero. Thus, the partial derivatives under H_0^a are rewritten in vector form as:

$$D\left(\tilde{\phi}\right) = \begin{pmatrix} D\left(\tilde{\sigma}_e^2\right) \\ D\left(\tilde{\sigma}_\mu^2\right) \\ D\left(\tilde{\sigma}_v^2\right) \\ D\left(\tilde{\lambda}\right) \end{pmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\tilde{\sigma}_e^2\tilde{\sigma}_v^2}{2}\left[\left(T-1\right)\tilde{M} + \tilde{M}_D\right] + \frac{\tilde{\sigma}_e^2}{2}\left[\tilde{\Xi}_* + \tilde{\Xi}_{**}\right] \end{bmatrix}$$

Also, using the method developed by [27], we obtain the information matrix under H_0^a. The information matrix is given by:

$$I_{\phi\phi} = E\left[-\frac{\partial^2 L}{\partial\phi_i \partial\phi_j}\right] = \frac{1}{2} tr\left[\Omega^{-1}\frac{\partial\Omega}{\partial\phi_i}\Omega^{-1}\frac{\partial\Omega}{\partial\phi_j}\right] \quad (29)$$

Proposition 5: Using the formular expressed in Equation (29) and information in proposition 2, we can derive respective elements in $I_{\phi\phi}$ under H_0^a for the vector of parameters $\phi' = \left(\sigma_e^2, \sigma_\mu^2, \sigma_v^2, \lambda\right)$ as follows:

$$E\left[-\frac{\partial^2 L}{\partial^2 \lambda}\right] = \frac{1}{2}\operatorname{tr}\left[\left(\sigma_e^2\left(E_T \otimes A_1 + \bar{J}_T \otimes A_2\right)\right)^2\right]$$

$$E\left[-\frac{\partial^2 L}{\partial^2 \sigma_e^2}\right] = \frac{1}{2}\operatorname{tr}\left[\left(E_T \otimes A_1 + \bar{J}_T \otimes A_2\right)^2\right]$$

$$E\left[-\frac{\partial^2 L}{\partial^2 \sigma_\mu^2}\right] = \frac{1}{2}\operatorname{tr}\left[\left(T\bar{J}_T \otimes A_2\right)^2\right]$$

$$E\left[-\frac{\partial^2 L}{\partial^2 \sigma_v^2}\right] = \frac{1}{2}\left[\left(T-1\right)\operatorname{tr}\left\{\left(A_1 i_N i_N'\right)^2\right\} + \operatorname{tr}\left\{\left(A_1 i_N i_N'\right)^2\right\}\right]$$

$$E\left[-\frac{\partial^2 L}{\partial\lambda\partial\sigma_e^2}\right] = \frac{\sigma_e^2}{2}\left[\left(T-1\right)\operatorname{tr}\left(A_1^2 M\right) + \operatorname{tr}\left(A_2^2 M\right)\right]$$

$$E\left[-\frac{\partial^2 L}{\partial\lambda\partial\sigma_\mu^2}\right] = \frac{T\sigma_e^2}{2}\operatorname{tr}\left(A_2^2 M\right)$$

$$E\left[-\frac{\partial^2 L}{\partial\lambda\partial\sigma_v^2}\right]$$
$$= \frac{1}{2}\left[\left(T-1\right)\operatorname{tr}\left(A_1 i_N i_N' A_1 M\right) + \operatorname{tr}\left(A_1 i_N i_N' A_1 M\right)\right]$$

$$E\left[-\frac{\partial^2 L}{\partial\sigma_e^2\partial\sigma_v^2}\right] = \frac{1}{2}\left[\left(T-1\right)\operatorname{tr}\left(A_1^2 i_N i_N'\right) + \operatorname{tr}\left(A_2^2 i_N i_N'\right)\right]$$

$$E\left[-\frac{\partial^2 L}{\partial\sigma_e^2\partial\sigma_\mu^2}\right] = \frac{T}{2}\operatorname{tr}\left(A_2^2\right)$$

$$E\left[-\frac{\partial^2 L}{\partial\sigma_\mu^2\partial\sigma_v^2}\right] = \frac{1}{2}\operatorname{tr}\left(A_2 i_N i_N' A_2\right)$$

Given these information under H_0^a, the LM statistic is given by,[6]

$$LM_\lambda = D\left(\tilde{\phi}\right)' \cdot \left(I_{\tilde{\phi}\tilde{\phi}}\right)^{-1} \cdot D\left(\tilde{\phi}\right) \quad (30)$$

Under H_0^a, LM_λ is distributed as χ_1^2. The statistic expressed in (30) is the LM test statistic, which tests for no cross-sectional dependence in a two-way random effects model.

[6]Details of derivations of the information matrix can be provided on request.

Decision Criteria:

The LM statistic is a scalar and the value obtained when the test is performed on the two-way error components model is compared with the critical value for the chi-squared distribution—χ_1^2. The intention is to ascertain whether to reject the null hypothesis, H_0^a, that there is no cross-sectional dependence problem in a two-way random effects model. Essentially, if LM_λ is less than the critical value for the chi-squared distribution, then, we do not reject the null hypothesis implying that there is no cross-sectional dependence; otherwise, we reject it.

4. Concluding Remarks

This paper provides a framework for testing for no cross-sectional dependence assuming the presence of random individual and time effects. Thus, several important issues have not been incorporated. These include testing other hypotheses earlier specified, that is; $H_0^b: \lambda = 0 \middle| \sigma_e^2 > 0; \sigma_\mu^2 > 0; \sigma_v^2 = 0$ which tests for cross-sectional dependence assuming the presence of random individual effect only (while ignoring the presence of time effects; and $H_0^c: \lambda = 0 \middle| \sigma_e^2 > 0; \sigma_\mu^2 = 0; \sigma_v^2 = 0$ which tests for cross-sectional dependence ignoring the presence of both the random country and time effects. Also, the empirical applications section involving Monte Carlo experiments is also not yet considered. These are some of the suggestions for future research.

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Appendix

(A) Derivation of the VCV Matrix for the Unrestricted Model

Here, $\lambda \neq 0$ and the VCV matrix of u can be derived as follows.

Recall Equation (10),

$$u = \left(I_T \otimes B^{-1}\right)e + \left(i_T \otimes I_N\right)\mu + \left(I_T \otimes i_N\right)v$$

Using assumption (1), the VCV matrix can be expressed as:

$$\begin{aligned} E(uu') &= \left(I_T \otimes B^{-1}\right)E(ee')\left(I_T \otimes B^{-1}\right)' \\ &\quad + \left(i_T \otimes I_N\right)E(\mu\mu')\left(i_T \otimes I_N\right)' \\ &\quad + \left(I_T \otimes i_N\right)E(vv')\left(I_T \otimes i_N\right)' \end{aligned} \tag{A.1}$$

Let $E(uu')$ in this case be represented by Ω, and by further simplification, (A.1) becomes:

$$\begin{aligned} \Omega &= \left(I_T \otimes \sigma_e^2 (B'B)^{-1}\right) + \left(i_T i_T' \otimes \sigma_\mu^2 I_N\right) \\ &\quad + \left(I_T \otimes \sigma_v^2 (i_N i_N')\right) \end{aligned} \tag{A.2}$$

$$\begin{aligned} \Omega &= \left(I_T \otimes \sigma_e^2 (B'B)^{-1}\right) + \left(J_T \otimes \sigma_\mu^2 I_N\right) \\ &\quad + \left(I_T \otimes \sigma_v^2 (i_N i_N')\right) \end{aligned} \tag{A.3}$$

where $J_T = i_T i_T'$ and it is a matrix of ones of dimension T. To obtain the spectral decomposition of (A.3), we use the [24] method which involves replacing J_T by $T\bar{J}_T$ and I_T by $E_T + \bar{J}_T$ where $E_T = I_T - \bar{J}_T$ and $\bar{J}_T = J_T / T$ in (A.3). This is done as follows:

$$\begin{aligned} \Omega &= \sigma_e^2\left[\left(E_T + \bar{J}_T\right) \otimes (B'B)^{-1}\right] \\ &\quad + \sigma_\mu^2\left(T\bar{J}_T \otimes I_N\right) + \sigma_v^2\left[\left(E_T + \bar{J}_T\right) \otimes (i_N i_N')\right] \\ &= \sigma_e^2\left[E_T \otimes (B'B)^{-1} + \bar{J}_T \otimes (B'B)^{-1}\right] \\ &\quad + T\sigma_\mu^2\left(\bar{J}_T \otimes I_N\right) + \sigma_v^2\left[E_T \otimes (i_N i_N') + \bar{J}_T \otimes (i_N i_N')\right] \\ &= E_T \otimes \sigma_e^2 (B'B)^{-1} + E_T \otimes \sigma_v^2 (i_N i_N') \\ &\quad + \bar{J}_T \otimes \sigma_e^2 (B'B)^{-1} + \bar{J}_T \otimes T\sigma_\mu^2 I_N + \bar{J}_T \otimes \sigma_v^2 (i_N i_N') \end{aligned}$$

$$\begin{aligned} \Omega &= E_T \otimes \left[\sigma_e^2 (B'B)^{-1} + \sigma_v^2 (i_N i_N')\right] \\ &\quad + \bar{J}_T \otimes \left[\sigma_e^2 (B'B)^{-1} + T\sigma_\mu^2 I_N + \bar{J}_T \otimes \sigma_v^2 (i_N i_N')\right] \end{aligned} \tag{A.4}$$

Using the [25] method of inversion, therefore, the inverse of Equation (A.5) can be expressed as:

$$\begin{aligned} \Omega^{-1} &= E_T \otimes \left[\sigma_e^2 (B'B)^{-1} + \sigma_v^2 (i_N i_N')\right]^{-1} \\ &\quad + \bar{J}_T \otimes \left[\sigma_e^2 (B'B)^{-1} + T\sigma_\mu^2 I_N + \sigma_v^2 (i_N i_N')\right]^{-1} \\ &= E_T \otimes A_1 + \bar{J}_T \otimes A_2 \end{aligned} \tag{A.5}$$

where $A_1 = \left[\sigma_e^2 (B'B)^{-1} + \sigma_v^2 (i_N i_N')\right]^{-1}$ and

$A_2 = \left[\sigma_e^2 (B'B)^{-1} + T\sigma_\mu^2 I_N + \sigma_v^2 (i_N i_N')\right]^{-1}$

(B) Derivation of the VCV Matrix for the Restricted Model

Here, $\lambda = 0$ and as a consequence, $Var(\varepsilon_{ti}) = Var(e_{ti}) = \sigma_e^2$. Given this assumption, Equation (10) reduces to:

$$u = \left(I_T \otimes I_N\right)e + \left(i_T \otimes I_N\right)\mu + \left(I_T \otimes i_N\right)v \tag{B.1}$$

Then, using assumption (1), the VCV matrix can be expressed as:

$$\begin{aligned} E(uu') &= \left(I_T \otimes I_N\right)E(ee')\left(I_T \otimes I_N\right)' \\ &\quad + \left(i_T \otimes I_N\right)E(\mu\mu')\left(i_T \otimes I_N\right)' \\ &\quad + \left(I_T \otimes i_N\right)E(vv')\left(I_T \otimes i_N\right)' \end{aligned} \tag{B.2}$$

Thus, (A.4) under the unrestricted model reduces to:

$$\begin{aligned} \Omega &= \left(I_T \otimes \sigma_e^2 I_N\right) + \left(i_T i_T' \otimes \sigma_\mu^2 I_N\right) \\ &\quad + \left(I_T \otimes \sigma_v^2 (i_N i_N')\right) \end{aligned} \tag{B.3}$$

$$\begin{aligned} \Omega &= \left(I_T \otimes \sigma_e^2 (B'B)^{-1}\right) + \left(J_T \otimes \sigma_\mu^2 I_N\right) \\ &\quad + \left(I_T \otimes \sigma_v^2 (i_N i_N')\right) \end{aligned} \tag{B.4}$$

Just as before, we use the [24] method to obtain the spectral decomposition of (B.4) and following the same procedure as Appendix A, we have:

$$\begin{aligned} \Omega &= E_T \otimes \left[\sigma_e^2 I_N + \sigma_v^2 (i_N i_N')\right] \\ &\quad + \bar{J}_T \otimes \left[\sigma_e^2 I_N + T\sigma_\mu^2 I_N + \bar{J}_T \otimes \sigma_v^2 (i_N i_N')\right] \end{aligned} \tag{B.5}$$

Similarly, using the [25] method of inversion, Ω^{-1} can be expressed as:

$$\begin{aligned} \Omega^{-1} &= E_T \otimes \left[\sigma_e^2 I_N + \sigma_v^2 (i_N i_N')\right]^{-1} \\ &\quad + \bar{J}_T \otimes \left[\sigma_e^2 I_N + T\sigma_\mu^2 I_N + \bar{J}_T \otimes \sigma_v^2 (i_N i_N')\right]^{-1} \end{aligned}$$

$$\Omega^{-1} = E_T \otimes A_1 + \bar{J}_T \otimes A_2 \tag{B.6}$$

where $A_1 = \left[\sigma_e^2 I_N + \sigma_v^2 (i_N i_N')\right]^{-1}$ and

$A_2 = \left[\sigma_e^2 I_N + T\sigma_\mu^2 I_N + \bar{J}_T \otimes \sigma_v^2 (i_N i_N')\right]^{-1}$.

(C) Derivation of the Partial Derivatives $\frac{\partial L}{\partial \phi}$

$$\frac{\partial L}{\partial \phi} = -\frac{1}{2} tr\left[\Omega^{-1} \frac{\partial \Omega}{\partial \phi}\right] + \frac{1}{2}\left[u'\Omega^{-1} \frac{\partial \Omega}{\partial \phi} \Omega^{-1} u\right]$$

$$\left.\frac{\partial L}{\partial \lambda}\right|_{H_0^a} = -\frac{1}{2}tr\left[\Omega^{-1}\frac{\partial \Omega}{\partial \lambda}\right] + \frac{1}{2}\left[\tilde{u}'\Omega^{-1}\frac{\partial \Omega}{\partial \lambda}\Omega^{-1}\tilde{u}\right]$$
$$= -\frac{1}{2}tr\left[\sigma_e^2\left(E_T \otimes A_1 M + \bar{J}_T \otimes A_2 M\right)\right]$$
$$+\frac{1}{2}\tilde{u}'\left(\sigma_e^2\left(E_T \otimes A_1 M A_1 + \bar{J}_T \otimes A_2 M A_2\right)\right)\tilde{u}$$
$$= -\frac{1}{2}tr\left[\sigma_e^2\left(E_T \otimes A_1 M + \bar{J}_T \otimes A_2 M\right)\right]$$
$$+\frac{1}{2}\tilde{u}'\left(\sigma_e^2\left(E_T \otimes A_1 M A_1 + \bar{J}_T \otimes A_2 M A_2\right)\right)\tilde{u}$$
$$= -\frac{\sigma_e^2}{2}tr\left[\left((T-1)A_1 + A_2\right)M\right]$$
$$+\frac{\sigma_e^2}{2}\left[\tilde{u}'\left(E_T \otimes A_1 M A_1\right)\tilde{u} + \tilde{u}'\left(\bar{J}_T \otimes A_2 M A_2\right)\tilde{u}\right] \quad \text{(C.1)}$$

Note that:

$$tr(A_1) = \frac{N}{\sigma_e^2}\left[1 - \frac{\sigma_v^2}{\sigma_e^2 + N\sigma_v^2}\right]$$

$$tr(A_2) = \sum_i \frac{1}{g_i} - \frac{\sigma_v^2 \sum_i \frac{1}{g_i^2}}{1 + \sigma_v^2 \sum_i \frac{1}{g_i}};$$

where $g_i = \sigma_e^2 + T\sigma_\mu^2$.

Therefore,

$$tr\left[\left((T-1)A_1 + A_2\right)M\right] = (T-1)tr\left(A_1 M\right) + tr\left(A_2 M\right)$$

By some algebraic simplifications, we can write that:

$$tr\left[\left((T-1)A_1 + A_2\right)M\right]$$
$$= -\left[(T-1)\frac{\sigma_v^2}{\sigma_e^2}\frac{i_N' M i_N}{\sigma_e^2 + N\sigma_v^2} + \sigma_v^2 \frac{i_N' D^{-1} M D^{-1} i_N}{1 + \sigma_v^2 \sum_i \frac{1}{g_i}}\right]$$

Note further that:

$$\tilde{u}'\left(E_T \otimes A_1 M A_1\right)\tilde{u}$$
$$= \tilde{u}'\left[\left(E_T \otimes A_1\right)\left(E_T \otimes M\right)\left(E_T \otimes A_1\right)\right]\tilde{u}$$

in which case,

$$\tilde{u}_* = \left(E_T \otimes I_N\right)\Omega^{-1}\tilde{u} = \left(E_T \otimes A_1\right)\tilde{u},$$

therefore;

$$\tilde{u}'\left(E_T \otimes A_1 M A_1\right)\tilde{u} = \tilde{u}_*'\left(E_T \otimes M\right)\tilde{u}_* = \tilde{\Xi}_*$$

Similarly,

$$\tilde{u}'\left(\bar{J}_T \otimes A_2 M A_2\right)\tilde{u}$$
$$= \tilde{u}'\left[\left(\bar{J}_T \otimes A_2\right)\left(\bar{J}_T \otimes M\right)\left(\bar{J}_T \otimes A_2\right)\right]\tilde{u}$$

in which case,

$$\tilde{u}_{**} = \left(\bar{J}_T \otimes I_N\right)\Omega^{-1}\tilde{u} = \left(\bar{J}_T \otimes A_1\right)\tilde{u},$$

therefore;

$$\tilde{u}'\left(\bar{J}_T \otimes A_2 M A_2\right)\tilde{u} = \tilde{u}_{**}'\left(\bar{J}_T \otimes M\right)\tilde{u}_{**} = \tilde{\Xi}_{**}.$$

As a consequence, we can write (C.1) as:

$$\left.\frac{\partial L}{\partial \lambda}\right|_{H_0^a} = D\left(\tilde{\lambda}\right)$$
$$= \frac{\tilde{\sigma}_e^2\tilde{\sigma}_v^2}{2}\left[(T-1)\tilde{M} + \tilde{M}_D\right] + \frac{\tilde{\sigma}_e^2}{2}\left[\tilde{\Xi}_* + \tilde{\Xi}_{**}\right] \quad \text{(C.2)}$$
$$= 0$$

where $\tilde{M} = \dfrac{i_N' M i_N}{\tilde{\sigma}_e^2\left(\tilde{\sigma}_e^2 + N\tilde{\sigma}_v^2\right)}$; $\tilde{M}_D = \dfrac{i_N' \tilde{D}^{-1} M \tilde{D}^{-1} i_N}{1 + \tilde{\sigma}_v^2 \sum_i \frac{1}{\tilde{g}_i}}$

$$\left.\frac{\partial L}{\partial \sigma_e^2}\right|_{H_0^a} = -\frac{1}{2}tr\left[\Omega^{-1}\frac{\partial \Omega}{\partial \sigma_e^2}\right] + \frac{1}{2}\left[\tilde{u}'\Omega^{-1}\frac{\partial \Omega}{\partial \sigma_e^2}\Omega^{-1}\tilde{u}\right]$$
$$= -\frac{1}{2}tr\left[E_T \otimes A_1 + \bar{J}_T \otimes A_2\right]$$
$$+\frac{1}{2}\tilde{u}'\left[E_T \otimes A_1^2 + \bar{J}_T \otimes A_2^2\right]\tilde{u}$$
$$= -\frac{1}{2}tr\left[E_T \otimes A_1 + \bar{J}_T \otimes A_2\right]$$
$$+\frac{1}{2}\tilde{u}'\left[\left(E_T \otimes A_1^2\right) + \left(\bar{J}_T \otimes A_2^2\right)\right]\tilde{u} \quad \text{(C.3)}$$

Using the information leading to (C.2), we can prove that:

$$tr\left[(T-1)A_1 + A_2\right]$$
$$= (T-1)\frac{N}{\sigma_e^2}\left[1 - \frac{\sigma_v^2}{\sigma_e^2 + N\sigma_v^2}\right] + \sum_i \frac{1}{g_i} - \frac{\sigma_v^2 \sum_i \frac{1}{g_i^2}}{1 + \sigma_v^2 \sum_i \frac{1}{g_i}}$$

And also with the representations that:

$$\tilde{u}'\left(E_T \otimes A_1^2\right)\tilde{u} = \tilde{u}'\left[\left(E_T \otimes A_1\right)\left(E_T \otimes A_1\right)\right]\tilde{u},$$

in which case,

$$\tilde{u}_* = \left(E_T \otimes I_N\right)\Omega^{-1}\tilde{u} = \left(E_T \otimes A_1\right)\tilde{u},$$

therefore;

$$\tilde{u}'\left(E_T \otimes A_1^2\right)\tilde{u} = \tilde{u}_*'\tilde{u}_* = \tilde{\Sigma}_*;$$

and in the same vein,

$$\tilde{u}'\left(\bar{J}_T \otimes A_2^2\right)\tilde{u} = \tilde{u}_{**}'\tilde{u}_{**} = \tilde{\Sigma}_{**}.$$

Given this information therefore, (C.3) becomes:

$$\left.\frac{\partial L}{\partial \sigma_e^2}\right|_{H_0^a} = D\left(\tilde{\sigma}_e^2\right)$$

$$= \frac{1}{2}\left[\left(\tilde{\Sigma}_* + \tilde{\Sigma}_{**}\right) - \left(\frac{N(T-1)}{\sigma_e^2}\left[1 - \frac{\sigma_v^2}{\sigma_e^2 + N\sigma_v^2}\right] + \sum_i \frac{1}{g_i} - \frac{\sigma_v^2 \sum_i \frac{1}{g_i^2}}{1 + \sigma_v^2 \sum_i \frac{1}{g_i}}\right)\right]$$

$$= 0 \tag{C.4}$$

$$\left.\frac{\partial L}{\partial \sigma_\mu^2}\right|_{H_0^a} = -\frac{1}{2} tr\left[\Omega^{-1}\frac{\partial \Omega}{\partial \sigma_\mu^2}\right] + \frac{1}{2}\left[\tilde{u}'\Omega^{-1}\frac{\partial \Omega}{\partial \sigma_\mu^2}\Omega^{-1}\tilde{u}\right]$$

$$= -\frac{T}{2}\left[tr\left(\bar{J}_T \otimes A_2\right) + \frac{1}{2}\tilde{u}'\left(\bar{J}_T \otimes A_2^2\right)\tilde{u}\right] \tag{C.5}$$

$$= -\frac{T}{2}\left[tr\left(A_2\right) + \frac{1}{2}\tilde{u}'\left(\bar{J}_T \otimes A_2^2\right)\tilde{u}\right]$$

Using the information that:

$$\tilde{u}'\left(\bar{J}_T \otimes A_2^2\right)\tilde{u} = \tilde{u}'_{**}\tilde{u}_{**} = \tilde{\Sigma}_{**};$$

and

$$tr\left(A_2\right) = \sum_i \frac{1}{g_i} - \frac{\sigma_v^2 \sum_i \frac{1}{g_i^2}}{1 + \sigma_v^2 \sum_i \frac{1}{g_i}};$$

(C.5) becomes,

$$\left.\frac{\partial L}{\partial \sigma_\mu^2}\right|_{H_0^a} = D\left(\tilde{\sigma}_\mu^2\right)$$

$$= \frac{T}{2}\left[\tilde{\Sigma}_{**} - \left(\sum_i \frac{1}{g_i} - \frac{\sigma_v^2 \sum_i \frac{1}{g_i^2}}{1 + \sigma_v^2 \sum_i \frac{1}{g_i}}\right)\right] = 0 \tag{C.6}$$

$$\left.\frac{\partial L}{\partial \sigma_v^2}\right|_{H_0^a}$$

$$= -\frac{1}{2} tr\left[\Omega^{-1}\frac{\partial \Omega}{\partial \sigma_v^2}\right] + \frac{1}{2}\left[\tilde{u}'\Omega^{-1}\frac{\partial \Omega}{\partial \sigma_v^2}\Omega^{-1}\tilde{u}\right]$$

$$= -\frac{1}{2} tr\left[\left(E_T \otimes A_1 + \bar{J}_T \otimes A_2\right)\left(i_N i'_N\right)\right]$$

$$+ \frac{1}{2}\left[\tilde{u}'\left(E_T \otimes A_1\left(i_N i'_N\right)A_1\right)\tilde{u} + \tilde{u}'\left(\bar{J}_T \otimes A_2\left(i_N i'_N\right)A_2\right)\tilde{u}\right]$$

$$= -\frac{N}{2} tr\left[(T-1)A_1 + A_2\right]$$

$$+ \frac{1}{2}\left[\tilde{u}'_*\left(E_T \otimes i_N i'_N\right)\tilde{u}_* + \tilde{u}'_{**}\left(\bar{J}_T \otimes i_N i'_N\right)\tilde{u}_{**}\right]$$

$$\left.\frac{\partial L}{\partial \sigma_v^2}\right|_{H_0^a}$$

$$= D\left(\tilde{\sigma}_v^2\right)$$

$$= \frac{N}{2}\left[N^{-1}\left(\tilde{\tilde{\Sigma}}_* + \tilde{\tilde{\Sigma}}_{**}\right) - \left(\frac{(T-1)}{\sigma_e^2}\left[1 - \frac{\sigma_v^2}{\sigma_e^2 + N\sigma_v^2}\right] + \sum_i \frac{1}{g_i} - \frac{\sigma_v^2 \sum_i \frac{1}{g_i^2}}{1 + \sigma_v^2 \sum_i \frac{1}{g_i}}\right)\right]$$

$$= 0 \tag{C.7}$$

where $\tilde{\tilde{\Sigma}}_* = \tilde{u}'_*\left(E_T \otimes i_N i'_N\right)\tilde{u}_*$ and $\tilde{\tilde{\Sigma}}_{**} = \tilde{u}'_{**}\left(\bar{J}_T \otimes i_N i'_N\right)\tilde{u}_{**}$.