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Asymptotic Analysis for U-Statistics and Its Application to Von Mises Statistics

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Abstract

Let $X, \bar{X}, X_1, \cdots, X_N$ be i.i.d. random variables taking values in a measurable space ($\mathfrak{X}$, $\mathfrak{B}$). Let ϕ_1: $\mathfrak{X} \to \mathbb{R}$ and ϕ: $\mathfrak{X}^2 \to \mathbb{R}$ be measurable functions. Assume that ϕ is symmetric, *i.e.* $\phi(x,y)=\phi(y,x)$, for any $x, y \in \mathfrak{X}$. Consider U-statistic $T = \frac{1}{N}\sum_{1\le i<j\le N}\phi(X_i, X_j) + \frac{1}{\sqrt{N}}\sum_{1\le i\le N}\phi_1(X_i)$, assuming that $\mathbb{E}\phi_1(X) = 0$, $\mathbb{E}\varphi(x, X) = 0$ for all $x \in \mathfrak{X}$, $\mathbb{E}\phi^2(x, X) < \infty$, $\mathbb{E}\phi_1^2(X) < \infty$. We will provide bounds for $\Delta_N = \sup_x |F(x) - F_0(x) - F_1(x)|$, where F is a distribution function of T and F_0, F_1 are its limiting distribution function and Edgeworth correction respectively. Applications of these results are also provided for von Mises statistics case.

Keywords: U-Statistics, Von Mises Statistics, Symmetric Statistics

1. Introduction

Consider the measurable space ($\mathfrak{X}$, $\mathfrak{B}$, μ), with measure $\mu = \mathcal{L}(\mathfrak{X})$. Let $L = L^2(\mathfrak{X}, \mathfrak{B}, \mu)$ denote the real Hilbert space of square integrable real functions. Let $\mathbb{Q}: L^2 \to L^2$ denote the Hilbert-Schmidt operator associated with the kernel ϕ and defined via

$$\mathbb{Q}f(x) = \int_{\mathfrak{X}}\phi(x,y) f(y)\mu(\mathrm{d}y) = \mathbb{E}\phi(x, X) f(X),$$

Let $q_1, q_2 \cdots$ be its eigenvalues. Without loss of generality we shall assume that $|q_1| \ge |q_2| \ge \cdots$.

Let $\{e_j : j \ge 1\}$ denote an orthonormal complete system of eigenfunctions of $\mathbb{Q}$ of the corresponding eigenvalues $q_1, q_2 \cdots$. Then

$$\sigma^2 = \mathbb{E}\phi^2(\bar{X}, X) = \sum_{j\ge1} q_j^2, \phi(x,y) = \sum_{j\ge1} q_j e_j(x) e_j(y) \quad (1.1)$$

since $\mathbb{Q}$ is a Hilbert-Schmidt operator and the kernel ϕ is degenerate. The series in (1.1) converges in L^2 $(\mathfrak{X}^2, \mathfrak{B}^2, \mu\times\mu)$. Consider the subspace $L^2(\phi, \phi_1) \subset L^2$ $(\mathfrak{X}^2, \mathfrak{B}^2, \mu\times\mu)$ generated by ϕ_1 and eigenfunctions e_j corresponding to nonzero eigenvalues q_j. Introducing, if necessary, an eigenvalues $e_0 : \mathbb{Q}e_0 = 0$,we can assume that $e_0, e_1, \cdots$ is an orthonormal basis in $L^2(\phi, \phi_1)$. Thus, we have

$$\phi_1(x) = \sum_{j\ge0} a_j e_j(x) \text{ in } L_2, \quad \beta_2 = \mathbb{E}\phi_1^2(X) = \sum_{j\ge0} a_j^2, \quad (1.2)$$

with $a_j = \mathbb{E}\phi_1(X) e_j(X)$ and $\mathbb{E}e_j(X) = 0$, for all *j*. Therefore $(e_j(X))_{j\ge0}$ is an orthonormal system of random variables with zero means.

Hilbert space $\ell_2 \subset \mathbb{R}^\infty$ consists of $x = (x_1, x_2 \cdots) \in \mathbb{R}^\infty$, such that

$$|x|^2 =_{def} \langle x, x\rangle, \langle x, y\rangle = \sum_{j\ge0} x_j y_j, |x| < \infty.$$

Consider the random vector

$$X =_{def} (e_0(X), e_1(X), e_2(X), \cdots), \quad (1.3)$$

which takes values in $\mathbb{R}^\infty$. Since $(e_j(X))_{j\ge0}$ is a system of mean zero uncorrelated random variables with variances 1, the random vector X has mean zero and $cov(e_i, e_j) = \delta_{ij}$ and δ_{ij} is Kronecker's symbol. Using (1.1) and (1.2), we can write

$$\phi(X, \bar{X}) = \langle \mathbb{Q}X, X\rangle, \quad \phi_1(X) = \langle a, X\rangle, \quad (1.4)$$

where we define $\mathbb{Q}x = (0, q_1x_1, q_2x_2, \cdots)$ for $x \in \mathbb{R}^\infty$ and $a = (a_j)_{j\ge0} \in \mathbb{R}^\infty$. The equalities (1.4) allow us to assume that the measurable space $\mathfrak{X}$ is $\mathbb{R}^\infty$. Let X

be a random vector taking values in $\mathbb{R}^\infty$ with mean zero and covariance $\operatorname{cov}(X_i, X_j) = \delta_{ij}$ and that

$$\phi(X, \bar{X}) = \langle \mathbb{Q}X, \bar{X} \rangle, \quad \phi_1(X) = \langle a, X \rangle. \qquad (1.5)$$

Without loss of generality we shall assume that the kernels $\phi(x, y)$ and $\phi_1(x)$ are linear functions in each of their arguments ([2]).

Introduce the definitions:

$$\beta_s = \mathbb{E}\,|\,\phi_1(X)\,|^s, \quad \gamma_s = \mathbb{E}\,|\,\phi(X, \bar{X})\,|^s, \quad \sigma^2 = \gamma_2,$$
$$\gamma_{s,r} = \mathbb{E}(\mathbb{E}\{|\,\phi(X, \bar{X})\,|^s |\, X\})^r,$$

and assume that

$$\beta_2 < \infty, \quad 0 < \sigma^2 < \infty.$$

for the statistic T we can write

$$\mathbb{E}T^2 = \beta_2 + \frac{N-1}{2N}\sigma^2.$$

The statistic T is called degenerate since $\sigma^2 > 0$ ensures that the quadratic part of the statistic is not asymptotically negligible and therefore statistic T is not asymptotically normal. More precisely, the asymptotic distribution of T is non-Gaussian and is given by the distribution of the random variable

$$T_0 = \frac{1}{2}\sum_{j\ge 1} q_j(\eta_j^2 - 1) + \sum_{j\ge 0} a_j \eta_j, \qquad (1.6)$$

η_j is a sequence of i.i.d. standard normal variables, $a_0, a_1, \cdots$ denotes a sequence of square summable weights and $|\,q_1\,| \ge |\,q_2\,| \ge \cdots$ denote eigenvalues of the Hilbert-Schmidt operator, say $\mathbb{Q}$, associated with the kernel ϕ.

Consider the concentration functions of statistic T_*

$$Q(T_*; \lambda) = \sup_x \mathbb{P}\{x \le T_* \le x + \lambda\}, \quad \lambda \ge 0, \qquad (1.7)$$

$$T_* = \sum_{1\le i<k\le N} \varphi(X_j, X_k) + f_1(X_1, \cdots, X_M) + f_2(X_{M+1}, \cdots, X_N),$$

where $f_1 = f_1(X_1, \cdots, X_M)$ is an arbitrary statistic depending only on $X_1, \cdots, X_M$, $f_2 = f_2(X_{M+1}, \cdots, X_N)$ is as well arbitrary but independent of $X_1, \cdots, X_M$. Note that the class of statistics T_* is slightly more general than the class of statistics T. We shall denote $c, c_1, \cdots$ constants. If a constant depends on, say s, we shall write $c(s) = c_s$.

Consider the distibution functions

$$F(x) = \mathbb{P}\{T \le x\}, \quad F_0(x) = \mathbb{P}\{T_0 \le x\},$$
$$\Delta_N = \sup_x |\,\Delta_N(x)\,|, \quad \Delta_N(x) = F(x) - F_0(x) - F_1(x),$$

$F_1(x)$ denotes an Edgeworth correction.The Edgeworth correction $F_1(x) = F_1(x; \mathcal{L}(X), \phi_1, \phi)$ is defined as a function of bounded variation satisfying $F_1(-\infty) = 0$ and with the Fourier-Stieltjes transform given by

$$\hat{F}_1(t) = \frac{(it)^3}{6\sqrt{N}} \mathbb{E}(\phi_1(X) + \phi(X, G))^3 e\{tT_0\}.$$

Let us notice that F_1 vanishes if $\phi_1 = 0$ or if

$$\mathbb{E}\phi_1^3(X) = \mathbb{E}\phi_1^2(X)\phi(X, x) = \mathbb{E}\phi_1(X)\phi^2(X, x) = \mathbb{E}\phi^3(X, x) = 0, \qquad (1.8)$$

holds for all $x \in \mathfrak{X}$. Using the technique presented in this work we may obtain the result for approximation bound of order $O(N^{-1})$ for U-statistic distribution function which has an order $|\,q_9\,|^{-\alpha}$ (see Theorem 3, 2) below) or $|\,q_{13}\,|^{-\alpha}$ (see Theorem 3, 1) below) with respect to dependence on first nine or thirteen eigenvalues of operator $\mathbb{Q}$, respectively.

2. Auxiliary Results

Consider the vector $G = (\eta_0, \eta_1 \cdots)$ with values in $\mathbb{R}^\infty$, where $\eta_0, \eta_1, \cdots$ standard normal variables. Let us formulate lemma in which equalities for the moments of determinants of random matrices consisting of the scalar products such as $\langle \mathbb{Q}G_i, G_j \rangle$ are obtained. Analogue of this lemma is proved in [1] for matrices consisting of the scalar products such as $\langle G_i, G_j \rangle$ where *G*-Gaussian (0, $\sigma^2\mathbb{I}$) vector.

Lemma 1. *Let* $G_1, \cdots, G_s, G_1', \cdots, G_s'$ *be random elements in a Hilbert space* H *such that* $G_i = (\eta_0, \eta_1 \cdots)$, *where* $\eta_0, \eta_1, \cdots$ *standard normal variables. Let* $|\,q_1\,| \ge |\,q_2\,| \ge \cdots$ *be the eigenvalues of Hilbert-Schmidt operator* $\mathbb{Q}$. $W = (det\mathbb{A})^2$, *where* $\mathbb{A} = \mathbb{A}(G) = \{a_{ij}(G)\}_{i,j=1}^s$, $a_{ij}(G) = \phi(G_i, G_j') = \langle \mathbb{Q}G_i, G_j \rangle$.

Then

$$\mathbb{E}W = (s!)^2 \sum_{1\le i_1<\ldots<i_s<\infty} (q_{i_1} \ldots q_{i_s})^2, \quad (\mathbb{E}W^2)^{1/2} \le c(s)\mathbb{E}W.$$

Nondegeneracy condition

We shall assume that random vector Z, a kernel ϕ, parameters c, c_1, s and p satisfy the nondegeneracy condition if

$$\mathbb{P}\{W(\bar{Z}) > \delta\} \ge p, \quad \delta = q_1^2 \ldots q_9^2,$$
$$\mathbb{P}\{|\,\phi(Z_i, \bar{Z}_j)\,| \le c\} \ge c_1, \quad 1 \le i, \quad j \le s, \qquad (2.1)$$

where $W(\bar{Z}) = (detA)^2$, $A = \{a_{ij}\}_{i,j=1}^s, a_{ij} = \varphi(Z_i, \bar{Z}_j)$,

Z_i, $\bar{Z}_j$ are independent copies of Z.

Here parameter p is small and parameter c_1 is close to 1. Let $\mathcal{N}(\delta, p)$ denote the set of all vectors Z satisfying the nondegeneracy condition.

Notice that G satisfies the nondegeneracy condition. Let vectors G and X have equal means and covariances, then

$$\mathbb{E}\phi_1(G) = \mathbb{E}\phi(G,x) = 0\,,\ \ \mathbb{E}\phi_1^2(G) = \mathbb{E}\phi_1^2(X)\,,$$
$$\mathbb{E}\phi_1(G)\phi(G,x) = \mathbb{E}\phi_1(X)\phi(X,x)\,,$$
$$\mathbb{E}\phi(G,x)\varphi(G,y) = \mathbb{E}\phi(X,x)\phi(X,y)\,.$$

The following Lemma 2 means that increase of n yields equivalence of nondegeneracy conditions fulfillments for sum and Gaussian vector.

Lemma 2. *Let* $G \in \mathcal{N}(4q_1^2\cdots q_9^2, 1-p)$ *be a Gaussian random vector and* $\mathbb{P}\{W(\overline{G}) > 4q_1^2\cdots q_9^2\} \ge 1-p$. *Then for* $m \ge c_s\,|\,q_1\cdots q_9\,|^{-3}\ p^{-1}(|\,q_1\cdots q_9\,|^{-3}\ p^{-1}\gamma_{2,3/2} + \gamma_3)$ *we have* $S_m \in \mathcal{N}(q_1^2\cdots q_9^2, 1-2p)$, *where* $S_m = m^{-1/2}(X_1 + \cdots + X_m)$ *is random sum*.

Further, it is necessary to bound the characteristic function of the statistic T_*. That will be done in Lemmas 3, 4 and Theorem 1.

The following Lemma 3 has a similar proof to Lemma 6.5 from [2].

By $\tau, \tau_1, \tau_2\cdots$ we shall denote independent copies of a symmetric random variable τ with nonnegative characteristic function and such that

$$1 \le \mathbb{E}\tau^2 \le 2\,,\ \ \mathbb{P}\{|\,\tau\,| \le 2\} = 1\,. \qquad (2.2)$$

Lemma 3. *Let* $s \in \mathbb{N}$ *and* $L \in \mathbb{Z}_+$. *Assume that vector* $Y \in \mathcal{N}((q_1\cdots q_9)^2, p)$ *takes values in* R^∞. *Write*

$$\Lambda = \sum_{j=1}^{sL}\tau_j Y_j\,,\ \ \overline{\Lambda} = \sum_{j=1}^{sL}\overline{\tau}_j\overline{Y}_j\,,\ \ q = [\,pL/4\,]\,,$$

where Y_j *and* $\overline{Y}_j$ *are independent copies of Y. Then*

$$\mathbb{E}e\{t\phi(\Lambda,\overline{\Lambda}\} \le c_d(s)(pL)^{-d} + \sup_{\mathbb{A}}\mathbb{E}e\{t\langle\mathbb{A}U,V\rangle\}\,,$$
$$t\in\mathbb{R}\,,\ \ d \ge 0\,,$$

where $\sup_{\mathbb{A}}$ *denotes the supremum over all* $s\times s$ *nonrandom matrices* $\mathbb{A}$ *such that* $(det\mathbb{A})^2 > q_1^2\cdots q_9^2$.

U and V denote independent vectors in $\mathbb{R}^s$ *which are sums of n independent copies of* $W = (\tau_1,\cdots,\tau_s)$.

In the following lemma the bound from above for the characteristic function $\mathbb{E}e\{t\langle\mathbb{A}U,V\rangle\}$ is received. This results was proved in [1]. The received estimation contains the determinant of matrix in right-hand side of inequality. This fact allows to use eigenvalues of operator $\mathbb{Q}$ for the estimation of characteristic function.

Lemma 4. *Let A be a nondegenerate* $s\times s$ *matrix. Let* $X \in \mathbb{R}^s$ *denote a random vector with covariance* C. *Assume that there exists a constant* c_s *such that*

$$\mathbb{P}\{|\,X\,| \le c_s\} = 1\,,\ \ |\,A\,| \le c_s\,,\ \ |\,C^{-1}\,| \le c_s\,. \qquad (2.3)$$

Let U and V denote independent random vectors which are sums of n independent copies of X. *Then*

$$|\,\mathbb{E}e\{t\langle AU,V\rangle\}\,| \le c(s)\,|\,detA\,|^{-1}\ \mathcal{M}^{2s}(t;N)\quad\text{for}\ \ |\,t\,| > 0\,,$$

where $\mathcal{M}(t;N) = 1\big/\sqrt{|\,t\,|\,N} + \sqrt{|\,t\,|}$ for $|\,t\,| > 0$.

Using our Lemmas 3 and 4 we may obtain a bound for characteristic function for statistic T_*.

Theorem 1. *Let* $m\in\mathbb{N}$. *Assume that the sum* $T = (2m)^{-1/2}(\tilde{X}_1 + \cdots + \tilde{X}_m) \in \mathcal{N}((q_1\cdots q_9)^2, p)$. *Then, for any statistic* T_* *we have*

$$|\,\mathbb{E}e\{tT_*\}\,| \ll_s \frac{1}{|\,q_9\,|^9}\mathcal{M}^{2s}(tm; pM/m)\,.$$

The proof of this theorem is similar to proof of Theorem 6.2 in [2].

Write :

$$\psi(t) = |\,\mathbb{E}_9 e\{tT^9\}\,|\,. \qquad (2.4)$$

In following lemma a multiplicative inequality for characteristic function of T^9 is given. This inequality yields the desired bound $\mathcal{O}(N^{-1})$ for an integral of the characteristic function of a *U*-statistic. Similar result was proved in Lemma 7.1 in [2]

Lemma 5. *Let* $d \ge 0$ *and* $s \in \mathbb{N}$. *Assume that* $Y = (2m)^{-1}\sum_{k=1}^{k=m}\tilde{X}_k \in \mathcal{N}((q_1\cdots q_9)^2, p)$. *Then there exist constants* $c_1(s,d)$ *and* $c_2(s,d)$ *such that the event*

$$D = \{\psi(t-\gamma)\psi(t+\gamma) \le c_1(s,d)\frac{1}{|\,q_9\,|^9\,v}\mathcal{M}^s(\gamma m; pM/m)\}\,, \qquad (2.5)$$

satisfies

$$\mathbb{P}\{D\} \ge 1 - c_2(s,d)(pM/m)^{-d}\,. \qquad (2.6)$$

For $A \ge t_0$, $t_1 \ge 0$ define the integrals

$$I_0 = \int_{-t_1}^{t_1}|\,\hat{\Psi}(t)\,|\,\mathrm{d}t\,,\ \ I_1 = \int_{t_0\le|t|\le A}|\,\hat{\Psi}(t)\,|\frac{\mathrm{d}t}{|\,t\,|}\,,$$

where $\hat{\Psi} = \int_{\mathbb{R}} e\{tx\}\mathrm{d}\Psi(x)$ denotes the Fourier-Stieltjes transform of the distribution function $\Psi(x) = \mathbb{P}\{T_* \le x\}$. The estimation for these integrals is received in following lemma, which has a proof similar to Lemma 3.3 in [2].

Lemma 6. *Let* $m\in\mathbb{N}$. *Assume that the random vector* $Y = (2m)^{-1/2}(\tilde{X}_1 + \ldots + \tilde{X}_m) \in \mathcal{N}((q_1\cdots q_9)^2, p)$ *and* $s \ge 9$. *Let*

$$k = \frac{pM}{m}\,,\ \ t_0 = \frac{c_0(s)}{m}k^{-1+2/s}\,,\ \ t_1 = \frac{c_1(s)}{m}k^{-1/2}\,,$$
$$\frac{c_2(s)}{m} \le A \le \frac{c_3(s)}{m}\,,$$

where $c_j(s)$, $0 \le j \le 3$ *are some positive constants. Then*

$$I_0 \ll_s |\,q_9\,|^{-9}\ (pM)^{-1}\,,\ \ I_1 \ll_s \max\{1, |\,q_9\,|^{-18}\}m(pM)^{-1}\,. \qquad (2.7)$$

3. Approximation Accuracy Estimation

For $r \in \mathbb{Z}_+$ and functions f_i, introduce the statistic

$$T^{(r)} = \frac{1}{N} \sum_{1 \le i < j \le N} \phi(Z_i, Z_j) + \sum_{1 \le i \le N} f_i(Z_i) \quad (3.1)$$

where

$$Z_j = X_j \text{ for } 1 \le j \le r, \quad Z_j = G_j \text{ for } r < j \le N.$$

Write $l = [(N-2)/20]$ and put

$$\kappa(t) = \kappa(t, N, \phi, \mathcal{L}(X)) = \kappa_1(t) + \kappa_2(t), \quad (3.2)$$

where

$$\kappa_1(t) = \sup_L |\mathbb{E}e\{tN^{-1} \sum_{1 \le j < k \le l} \phi(X_j, X_k) + L(X_1, \cdots, X_l)\}|, \quad (3.3)$$

$$\kappa_2(t) = \sup_L |\mathbb{E}e\{tN^{-1} \sum_{1 \le j < k \le l} \phi(G_j, G_k) + L(G_1, \cdots, G_l)\}|, \quad (3.4)$$

where supremum is taken over all linear statistics L, that is, over all functions which can be represented as $L(x_1, \cdots, x_l) = \sum_{j=1}^{l} f_j(x_j)$ with some functions $f_1, \cdots, f_l$.

Consider the following Lemma 7, which has a similar proof as Lemma 4.2 in [2].

Lemma 7. *Let* $m \in \mathbb{N}$, $s \ge 9$ *and* $t_0 = m^{-1}(pN/m)^{-1+2/s}$. *Assume that the random vector* $Y = (2m)^{-1/2}(\tilde{X}_1 + \cdots + \tilde{X}_m)$ *satisfies the nondegeneracy condition. Then, for* $pN > m$, $m^{-1} \ge t_* \ge t_0$ *the distribution function* $F^{(r)}$ *of* $T^{(r)}$ *satisfies*

$$F^{(r)}(x) = \frac{1}{2} + \frac{i}{2\pi} V.P.\int_{-Nt_*}^{Nt_*} e\{-xt\} \hat{F}^{(r)}(t) \frac{dt}{t} + R, \quad (3.5)$$

where $|R| \ll_s (|q_9|^{-9} + max\{1, |q_9|^{-18}\}) m/(pN)$.

The Edgeworth correction $F_1(x) = F_1(x; \mathcal{L}(X), \phi_1, \phi)$ is defined as a function of bounded variation satisfying $F_1(-\infty) = 0$ and with the Fourier-Stieltjes transform given by

$$\hat{F}_1(t) = \frac{(it)^3}{6\sqrt{N}} \mathbb{E}(\phi_1(X) + \phi(X, G))^3 e\{tT_0\}. \quad (3.6)$$

Lemma 8. *Assume that the nondegeneracy condition is fulfilled.*

1) *Let* $s \ge 13$ *and* $m_0 \asymp |q_1 \cdots q_9|^{-3} p^{-1}(|q_1 \cdots q_9|^{-3} p^{-1}\gamma_{2,3/2} + \gamma_3)$. *Then*

$$\Delta_N \ll_s \frac{m_0(|q_9|^{-9} + max\{1, |q_9|^{-18}\})}{pN} + N^{-1}(\beta_3^2 + \sigma^2\gamma_{2,2})(\frac{1}{|q_s|^s} + \frac{1}{|q_s|^6}) + \frac{|q_9|^{-9}}{p^6 N} \cdot (\beta_4 + \beta_3^2 + \sigma^2 + \gamma_3 + \sigma^2\gamma_3 + \gamma_{2,2} + \sigma^2\gamma_{2,2}) \quad (3.7)$$

2) *Assume that the condition* (1.8) *holds and that* $s \ge 9$. *Then*

$$\Delta_N \ll_s \frac{m_0(|q_9|^{-9} + max\{1, |q_9|^{-18}\})}{pN} + N^{-1}(\beta_3^2 + \sigma^2\gamma_{2,2})(\frac{1}{|q_s|^s} + \frac{1}{|q_s|^6}) + \frac{|q_9|^{-9}}{p^4 N} \cdot (\beta_4 + \sigma^2 + \gamma_3 + \gamma_{2,2}). \quad (3.8)$$

To prove this lemma we need to make the same steps as in Lemma 4.1 in [2] replacing Theorem 6.2 by Theorem 1.

Now we can formulate a following Theorem 2, where bounds for Δ_N are obtained. This theorem were proved in [4]:

Theorem 2. 1) *Let* $s \ge 13$ $m_0 \asymp |q_1 \cdots q_9|^{-3} p^{-1}(|q_1 \cdots q_9|^{-3} p^{-1}\gamma_{2,3/2} + \gamma_3)$, $p_0 \asymp c(s)$. *Then*

$$\Delta_N \ll \frac{m_0(|q_9|^{-9} + max\{1, |q_9|^{-18}\})}{cN} + N^{-1}(\beta_3^2 + \sigma^2\gamma_{2,2})(\frac{1}{|q_{13}|^{13}} + \frac{1}{|q_{13}|^6}) + \frac{|q_9|^{-9}}{cN} \cdot (\beta_4 + \beta_3^2 + \sigma^2 + \gamma_3 + \sigma^2\gamma_3 + \gamma_{2,2} + \sigma^2\gamma_{2,2}), \quad (3.9)$$

2) *Assume that* (1.8) *holds and* $s \ge 9$. *Then*

$$\Delta_N \ll \frac{m_0(|q_9|^{-9} + max\{1, |q_9|^{-18}\})}{cN} + N^{-1}(\beta_3^2 + \sigma^2\gamma_{2,2})(\frac{1}{|q_9|^9} + \frac{1}{|q_9|^6}) + \frac{|q_9|^{-9}}{cN} \cdot (\beta_4 + \sigma^2 + \gamma_3 + \gamma_{2,2}). \quad (3.10)$$

4. An Extension of Bounds to Von Mises Statistics. Applications

Assuming that the kernels ϕ and ϕ_1 are degenerate, consider the von Mises statistic

$$M = \frac{1}{2N} \sum_{1 < i, j < N} \phi(X_i, X_j) + \frac{1}{\sqrt{N}} \sum_{1 \le i \le N} \phi_1(X_i). \quad (4.1)$$

Introducing the function $\psi(x) = (\phi(x, x) - \nu)/2$ with $\nu = E\phi(X, X)$, we can rewrite (4.1) as

$$M - \frac{\nu}{2} = T + \frac{1}{N} \sum_{1 \le i \le N} \psi(X_i) \quad (4.2)$$

In this section we shall extend the bounds to statistics of type (4.2), assuming that $E\psi(X) = 0$ and $\rho = E\psi^2(X) < \infty$.

Similarly to the case of T, we can represent the kernel ϕ (respectively, ϕ_1 and ψ) as a bilinear (respectively, linear) function, defined on R^∞. However in this case we have to assume that R^∞ has an additional coordinate since ψ can be linearly independent of ϕ_1 and of the eigenfunctions of $\mathbb{Q}$. To fix notation, we shall assume that R^∞ consists of vectors $x = (x_{-1}, x_0, x_1, \cdots)$. Then all representations and results of Section 2 concerning ϕ and ϕ_1 still hold, and for ψ we have $\psi(x) = \langle b, x\rangle$ with some $b = (b_{-1}, b_0, b_1, \cdots)$ such that $\sum_{j\ge -1} b_j^2 < \infty$. Write $\psi_0(x) = \sum_{j\ge 0} b_j x_j$.

Introduce the function F_* of bounded variation (provided that $q_3 \ne 0$) with the Fourier-Stieltjes transform

$$\hat{F}_*(t) = \frac{it}{\sqrt{N}} E\psi(G)e\{tT_0\} = \frac{it}{\sqrt{N}} E\psi_0(G)e\{tT_0\}$$

and such that $F_*(-\infty) = 0$. Bellow we shall show that (see Lemma 9.3 [2])

$$\hat{F}_*(t) = \frac{(it)^2}{\sqrt{N}} E\psi(X)\left(\varphi_1(X) + \varphi(X,G)\right)e\{tT_0\}. \quad (4.3)$$

Notice that $F_* = 0$ whenever $\phi_1 = 0$.

Write $H_1 = F_1 + F_*$, and let H denote the distribution function of $M - \nu/2$. Define

$$\delta_N = \sup_x |\delta_N(x)|, \quad \delta_N(x) = H(x) - F_0(x) - H_1(x).$$

Theorem 3. 1) *Assume that* $q_{13} \ne 0$. *Then we have*

$$\delta_N \le \frac{m_0(|q_9|^{-9} + max\{1, |q_9|^{-18}\})}{cN} + N^{-1}(\beta_3^2 + \sigma^2\gamma_{2,2})\left(\frac{1}{|q_{13}|^{13}} + \frac{1}{|q_{13}|^6}\right) + \frac{|q_9|^{-9}}{cN} \cdot (\beta_4 + \sigma^2 + \gamma_3 + \gamma_{2,2} + \rho). \quad (4.4)$$

2) *Assume that (1.8) is fulfilled and* $q_9 \ne 0$. *Then we have*

$$\Delta_N \ll \frac{m_0(|q_9|^{-9} + max\{1, |q_9|^{-18}\})}{cN} + N^{-1}(\beta_3^2 + \sigma^2\gamma_{2,2})\left(\frac{1}{|q_9|^9} + \frac{1}{|q_9|^6}\right) + \frac{|q_9|^{-9}}{cN} \cdot (\beta_4 + \sigma^2 + \gamma_3 + \gamma_{2,2} + \rho).$$

Proof. We shall use the following estimates. Write

$$\xi = \frac{t}{N}\sum_{1\le j\le N} \psi(X_j), \quad \zeta = \frac{t}{N}\sum_{1\le j\le N} \psi(G_j). \quad (4.5)$$

Expanding with remainder $\mathcal{O}(\xi)$, splitting the sum ξ in parts and conditioning , we have

$$\left|Ee\{tT + \xi\} - Ee\{tT\} - iE\xi e\{tT\}\right| \ll \chi N^{-1} t^2 \rho. \quad (4.6)$$

Proceeding similarly to the proof of Lemma 8.2 from [2], we obtain

$$\left|\hat{F}_*(t) - iE\zeta e\{R_{[1,N]}T\}\right| \ll \chi N^{-1} t^2 (\rho + \sigma^2). \quad (4.7)$$

Applying the Bergstrom-type identity

$$\mathbb{E}S = \mathbb{E}\mathcal{R}_{[1,N]}S + \sum_{j=1}^N \{\mathbb{E}\mathcal{R}_{[2,j]}S - \mathbb{E}\mathcal{R}_{[1,j]}S\},$$

$$\mathcal{R}_{[1,j]}S = S(G_1, \cdots, G_j, X_{j+1}, \cdots, X_N)$$

with $S = \xi e\{tT\}$ and proceeding similarly to the proof of Lemma 8.3 from [2], we get

$$\left|E\xi e\{tT\} - E\zeta e\{\mathcal{R}_{[1,N]}T\}\right| \ll \chi N^{-1}\left(t^2 + t^4\right)(\rho + \beta_4 + \beta_3^2 + \gamma_3 + \gamma_{2,2} + \gamma_2\Gamma_{2,2}). \quad (4.8)$$

Arguments similar to the proof of Lemma 8.5 from [2] allow proving

$$\left|\hat{F}_*(t)\right| \ll N^{-1.2}|t|\rho^{1/2}\prod_{j\le 1}\left(1 + 2t^2 q_j^2/25\right)^{-1/4}, \quad (4.9)$$

and, for $s \ge 3$,

$$\int_{|t|\ge\lambda}\left|\hat{F}_*(t)\right|\frac{dt}{|t|} \ll_s N^{-1/2}\rho^{1/2}|q_s|^{-s/2}\lambda^{1-s/2}, \quad \lambda > 0 \quad (4.10)$$

$$\int_R \left|\hat{F}_*(t)\right|\frac{dt}{|t|} \ll_s N^{-1/2}\rho^{1/2}|q_s|^{-1}. \quad (4.11)$$

The estimates (4.6)-(4.11) allow proceeding similarly to the proof of Theorem 2, using a lemma similar to Lemma 8. Proving such a lemma, we have to apply Lemma 8 to the distribution function H. This is possible since that statistic $M - \nu/2$ is a statistic of type (3.1). The estimates (4.10) and (4.11) allow application of the Fourier inversion to the function F_*. As a result, we arrive at

$$\int_{-Nt_*}^{Nt_*}\left|\hat{H}(t) - \hat{F}_0(t) - \hat{H}_1(t)\right|\frac{dt}{|t|}.$$

Here, however, we have $\hat{H}(t) = Ee\{tT + \xi\}$, and

$$\begin{aligned}\left|\hat{H}(t) - \hat{F}_0(t) - \hat{H}_1(t)\right| &\le \left|\hat{F}(t) - \hat{F}_0(t) - \hat{F}_1(t)\right| \\ &+ \left|Ee\{tT + \xi\} - Ee\{tT\} - iE\xi e\{tT\}\right| \\ &+ \left|\hat{F}_*(t) - iE\zeta e\{R_{[1,N]}T\}\right| \\ &+ \left|E\xi e\{tT\} - E\zeta e\{R_{[1,N]}T\}\right|.\end{aligned} \quad (4.12)$$

Therefore, using (4.6)-(4.8), we can proceed as in the proof of Lemma 11. As a final result we get bounds similar to those of Theorem 2, with the additional summand ρ.

5. References

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