Open Journal of Philosophy
2011. Vol.1, No.2, 57-60

Scientific Research
DOI:10.4236/ojpp.2011.12010

New Zeno and Actual Infinity

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Received July 9th, 2011; revised August 15th, 2011; accepted August 25th, 2011.

In 1964 José Benardete invented the "New Zeno Paradox" about an infinity of gods trying to prevent a traveller from reaching his destination. In this paper it is argued, contra Priest and Yablo, that the paradox must be resolved by rejecting the possibility of actual infinity. Further, it is shown that this paradox has the same logical form as Yablo's Paradox. It is suggested that constructivism can serve as the basis of a common solution to New Zeno and the paradoxes of truth, and a constructivist interpretation of Kripke's theory of truth is given.

Keywords: New Zeno, Yablo's Paradox, Infinity, Constructivism

The New Zeno Paradox

I wish to discuss the "New Zeno Paradox" invented by José Benardete, criticise the proposed solutions of Graham Priest and Stephen Yablo, and argue that the paradox should be resolved by rejecting the possibility of actual infinity. The paradox goes as follows:

"A man decides to walk one mile from A to B. A god waits in readiness to throw up a wall blocking the man's further advance when the man has travelled 1/2 mile. A second god (unknown to the first) waits in readiness to throw up a wall of his own blocking the man's further advance when the man has travelled 1/4 mile. A third god...&c. *ad infinitum*. It is clear that this infinite sequence of mere *intentions* (assuming the contrary to fact conditional that each god would succeed in executing his intentions if given the opportunity) logically entails the consequence that the man will be arrested at point A; he will not be able to pass beyond it, even though not a single wall will in fact be thrown down in his path." (Benardete, 1964)

Following Yablo (2000), we will assume that the man does not stop before B unless a barrier is raised to prevent him from proceeding. From this, together with the conclusion that no walls are raised, follows that the man *does move* beyond A—a contradiction.

According to Yablo, it is the combination of the god's intentions that is impossible. When the man tries to move away from A it will turn out that some of the gods won't be able to fulfil their intentions. Either some of the gods will raise their wall, even though walls before theirs have been raised, or some of the gods will refrain from raising their wall even though none of the prior walls were raised. Yablo writes:

"If there's a paradox here, it lies in the difficulty of combining individually operational subsystems into an operational system. But is this any more puzzling than the fact that although I can pick a number larger than whatever number you pick, and vice versa, we can't be combined into a system producing two numbers each larger than the other?" (Yablo, 2000)

The answer to that question is: Yes, it is much more surprising! For the two situations are not alike. If Yablo and I both try to pick the higher number, he who gets to choose last will succeed. If I choose after Yablo I will be able to fulfil my intention no matter what Yablo does (I can simply choose the number that is one higher than Yablo's). This resembles the situation described in the paradox, as each god has a course of action (raising the wall or refraining from doing so) available that is consistent with his intentions, no matter what the gods before him have done. But I can be prevented from fulfiling my intention of picking the higher number by someone (Yablo) acting after me. That is not the case for the gods. For each given god, it is irrelevant to his purposes what the succeeding gods do.

Actually, the intentions of a given god (let us call him Zeus and let him be placed $1/2^n$ of the way from A to B) are consistent with any combination of actions that the other gods can perform (that is, any combination of raisings and non-raisings of walls by the other gods irrespective of their intentions), and at the time when Zeus must decide what to do, he will have all the information needed to ensure that his intention is fulfiled. For the intention of Zeus is given by the biconditional

(I_n) *a wall shall be raised at the* $1/2^n$ *point*, *if and only if for all* $m > n$ *no wall has been raised at the* $1/2^m$ *point*,

which makes no reference to the actions of the gods after Zeus, and is such that Zeus has full power to determine the truth value of the left hand side at a time when the truth value of the right hand side is determined. The contradictory combined intention of the gods

(*I*) *for all* $n \geq 1$, *a wall shall be raised at the* $1/2^n$ *point*, *if and only if for all* $m > n$ *no wall has been raised at the* $1/2^m$ *point*, is not Zeus' intention.

So if Yablo is right that the solution is that it will turn out that some of the gods fail in their attempts to fulfil their intentions, we lack an explanation *why*. Yablo's "explanation" is that "logic stops them". But that does not explain why the individual gods fail.

If an individual fails in achieving some goal, "being stopped by logic" will only be a sufficient explanation if that goal is self-contradictory. And if a group of individuals have goals that are self-consistent separately and contradictory combined, so that there is at least one of these individuals who will not achieve his goal, his failure will have a more concrete explanation, i.e. an explanation that makes reference to contingent states of affairs that contradict the goal of this individual. If, for example, Achilles and the Tortoise race each other and both intend to be the first to reach the finishing line, their goals are contradictory combined, and so "being stopped by logic" can explain their failure in reaching both their goals. But if the Tortoise is the one to fail his goal, there is also a more concrete explanation for this, namely that Achilles got to the finishing line first (or at the exact same time as the Tortoise).

Assume that Zeus is one of the gods who fail their goal. The goal of Zeus is not self-contradictory, and so his failure will not be sufficiently explained by saying that he was "stopped by logic". The goals of all the gods are contradictory combined, but there can be given no concrete explanation for the failure of Zeus. No combination of raised and non-raised walls of the other gods will serve to explain why Zeus couldn't make the truth value of the left hand side of (I_n) equal to the right hand side.

This of course generalises to all the gods. So if the man moves away from A, something inexplicable will happen (i.e. an event without a cause will happen). And that is also the case if the man can't move away from A: Either there will be no raised walls and so the man's failure will be inexplicable, or some walls will be raised and then the actions of the associated gods will be inexplicable.

So given that Yablo is right, the situation described in the paradox will necessarily result in an inexplicable state of affairs. That is unacceptable, and so his solution must be rejected.

So where does that leave us? Let us examine the premises. Letting "*Rx*" mean that the man reaches point *x* and "*Bx*" mean that a barrier is raised at point *x*, where *x* ranges over the real numbers, and A is placed at $x = 0$ and B at $x = 1$, the premises are stated thus by Yablo (I have made some inessential changes):

(A1) $\forall x, y \in [0,1] : (Rx \wedge y < x) \rightarrow Ry$

(A2) $\forall x, y \in [0,1] : (By \wedge y \leq x) \rightarrow \sim Rx$

(A3) $\forall y \in [0,1] : (\forall x \in [0,1] : x \leq y \rightarrow \sim Bx) \rightarrow Ry$

(A4) $\forall x \in [0,1] : Bx \leftrightarrow (\exists n \in \mathbb{N} : x = 1/2^n \;\&\; R(x/2))$

Priest (1999) suggests that the paradox could be resolved by denying the possibility of motion, i.e. rejecting premises (A1), (A2) or (A3) or some combination thereof. Yablo shows that this won't work. He does so with the example of an infinite series of demons calling off YES's and NO's in reverse order, with demon *n* calling after demon $n + 1$. The *n*th demon calls at the time $t = 1/2^n$. The intention of each demon is to call YES iff all the earlier-calling demons have called NO. This amounts to using the same premises with "*Rx*" and "*Bx*" reinterpreted to mean "up to (and including) the time $t = x$ no demon has called YES" and "at $t = x$ a demon calls YES" respectively. In this version of the paradox motion plays no role, but the contradiction still ensues. And the first three premises have been reduced to truisms that can't be rejected with any degree of reasonability. The interpretations are as follows:

(A1) If no demon has called YES up to $t = x$ then no demon has called YES up to any earlier time.

(A2) If at $t = y$ a demon calls YES then there is no later time up to which no demon has called YES.

(A3) If no demon has called YES up to $t = y$ then no demon has called YES up to $t = y$.

So premise (A4) must be rejected. I agree with Yablo that far.

In order to analyze the situation in more detail, I will "split up" premise (A4), i.e. replace it with two premises whose conjunction implies (A4). Let *g* be a function from the set of natural numbers to the set of gods. Then the two new premises are

(A4') $\forall n, m \in \mathbb{N} : n \neq m \rightarrow g(n) \neq g(m)$

(A4") $\forall n \in \mathbb{N} : (\forall m \in \mathbb{N} : n \neq m \rightarrow g(n) \neq g(m))$
$\rightarrow B(1/2^n) \leftrightarrow R(1/2^{n+1})$

Premise (A4') says that there exists infinitely many gods. Premise (A4") expresses the individual god's ability to raise a barrier at his unique point iff the man gets half the way to his. If we assume the logical possibility of the existence of gods with the ability to raise arbitrarily thin walls arbitrarily fast (or just demons with the ability to call YES or NO arbitrarily fast) and base their decision of whether to do so on previous events, then (A4") can't be rejected without accepting the possibility of inexplicable states of affairs as argued above. So (A4') must be rejected instead. That amounts to rejecting the possibility of actual infinity.

That solves the paradox because if only potential infinity and not actual infinity can exist, the "closest" situations to the one described in the paradox are these:

- One god intends to stop the man the first time he arrives at a point in the set $\{1/2^n \mid n \in \mathbb{N}\}$.
- A potential infinity of gods are created one after the other and when each god is created he is assigned to a point on the route.

And they do not give rise to a paradox. The god in the first situation simply has an inconsistent intention, and so *his* failure to fulfil it can be sufficiently explained by saying that "logic stops him". In the second situation there will only exist a finite number of gods *at the time when the man begins his journey*. So one of these gods will be the first on the route, and he will raise his wall while none of the others will.

The Logical Essence of New Zeno

I will use the rest of this paper to provide further support for this conclusion; that New Zeno should be solved by rejecting actual infinity. I will do this by first carrying out a deeper logical analysis of the paradox than the one above. This analysis will reveal a close affinity to the semantic paradoxes, in particular the one named after Yablo. And then (in the next section) I will reach the conclusion through an appeal to Priest's Principle of Uniform Solution.

One step towards identifying the "logical essence" of New Zeno has been taken with Yablo's modification into a simpler form not involving movement and the demonstration that this modified paradox has the same logical structure as the original. Another step can be taken by making an alternative and simpler formalization of the modified paradox, where only the predicate *B* and not *R* is used. Still using "*Bx*" to mean that a demon calls YES at $t = x$, all the premises (A1) - (A4) can be replaced with just this one:

$$\forall x \in \{1/2^n \mid n \in \mathbb{N}\} : Bx \leftrightarrow (\forall y < x : \sim By)$$

But instead of using the reals and the natural numbers in a naive way, where it is not clear what properties of the metric and order relations on these sets are necessary to achieve the contradiction, the premises of the paradox can be given as a set of formulas from which the contradiction can be deduced using only standard first order predicate logic. Let *B* and $\in A$ be unary predicates, written prefix and postfix respectively, and $<$ a binary predicate, written infix. Then the set of formulas consists of these four:

(B1) $\exists x : x \in A$

(B2) $\forall x, y, z \in A : x < y \wedge y < z \rightarrow x < z$

(B3) $\forall x \in A \exists y \in A : y < x$

(B4) $\forall x \in A : Bx \leftrightarrow (\forall y < x : \sim By)$

The contradiction is derived as follows. From (B1) by existential specification we have $a \in A$, and then from (B4) $Ba \leftrightarrow (\forall y < a : \sim By)$ follows by universal specification. Assume Ba and then deduce $\forall y < a : \sim By$. From (B3) follows $\exists y \in A : y < a$ and hence by existential specification again we have $b < a$. From this follows in conjunction with (B2) and $\forall y < a : \sim By$ that $\forall y < b : \sim By$ and $\sim Bb$ hold. As we also get $Bb \leftrightarrow (\forall y < b : \sim By)$ from (B4) this implies the contradiction. Discarding the assumption, we have $\sim Ba$, from which it follows together with $Ba \leftrightarrow (\forall y < a : \sim By)$ that $\exists y < a : By$ holds, and then the use of existential specification yet again produces Bb. From this a contradiction can be derived analogously to how it was derived from Ba.

What we see is that in addition to (B4) the only thing that is required is a non-empty set equipped with a transitive and non-wellfounded order relation. The metric on the reals and the linearity of the standard order relation on the same do not play any role in the derivation of the contradiction; their purpose is only to make the premises plausible. The metric serves to make it plausible that the traveller can cover the distance from point A to point B in a finite amount of time. The linearity makes it plausible that the right hand side of (I_n) is a given at the time where god number *n* must decide whether or not to make the left hand side true.

What is interesting about the fact that (B1) - (B4) is the "logical essence" of the paradox is that it reveals that New Zeno has the same logical structure as Yablo's Paradox (Yablo, 1993). This paradox results from this infinite list of sentences:

Y1: For all $n > 1$ the sentence Yn is not true
Y2: For all $n > 2$ the sentence Yn is not true
Y3: For all $n > 3$ the sentence Yn is not true
⋮

This set of sentences satisfies (B1) - (B4) when the predicates are interpreted as follows: "*Bx*" means that *x* is true, "$x \in A$" means that *x* is a sentence on the list and "$x < y$" means that *x* is further down on the list than *y*.

The (semi)formal proof of contradiction above then corresponds to this informal proof: There is some sentence on the list, call it "*a*". *a* is true iff all sentences further down on the list are false. Assume that *a* is true. Then all sentences further down on the list are false. There is a sentence further down on the list, call it "*b*". All sentences further down than *b* are false. *b* is false. *b* is true iff all sentences further down on the list than *b* are false. Contradiction! So *a* is false. There is a true sentence further down than *a*, call it "*b*". *b* is true. Contradiction follows analogously.

The Principle of Uniform Solution

According to Priest's so-called Principle of Uniform Solution, paradoxes with the same logical structure should be solved in a similar way. When it has now been demonstrated that New Zeno and Yablo's Paradox have the same logical structure, this puts a significant restriction on what can be accepted as possible solutions to these two paradoxes. On the one hand, the denial-of-the-possibility-of-motion solution and the logic-stops-them solution to New Zeno can not be transfered to Yablo's Paradox, and on the other hand, the denial of either semantical closure, the validity of the Tarskian T-schema or classical logic —the conjunction of which are normally considered the source of the semantic paradoxes—will not solve New Zeno.

I claim that constructivism can serve as such a unifying solution. Three aspects of constructivism, relevant to this paper, can be distinguished. The first is the thesis that any infinity is merely a potiential such, in the sense that its elements are created in time and at any given time is finite in number. The second is mentalism with regard to certain abstract entities, e.g. mathematical objects. And the third is the rejection of *tertium non datur*. It is the first of these three aspects which can unify the solutions. The other two will become relevant in the following, but do not apply to New Zeno and are therefore not part of what unifies. The second is a way to explain why the first applies to abstract entities. And the third, even though often considered *the* central thesis of constructivism, is but a side-effect of the two others.

Constructivism qua the first aspect solves New Zeno as explained in the first section. Yablo's Paradox can be solved with Kripke's theory of truth (Kripke, 1975)[1]. And that theory can, I will argue, be interpreted as a constructivistic theory.

In (Beall, 2007, chapter 1) Beall presents Kripke's theory through a metaphor about books. It goes as follows. Imagine a world initially consisting only of non-semantic facts. In this world, there is a writer with two very large books. They carry the titles *The True* and *The False*. In the beginning they are empty, but the writer sets out to fill them so that they accurately reflect their titles. In the first book, he records every fact of the world, and in the second, he records every state of affairs that fails to obtain in the world. For instance, he writes "Snow is white" in the first book and "Snow is green" in the second. After having done so, he realises that his work is not complete. For now there are more facts than when he started. By writing in the books, he has added facts to the world, namely facts about what is written in the books, and he did not include these facts in *The True*, nor did he include non-obtaining facts about the books in *The False*. So in each book, he puts the heading "Chapter 1" over what he has written so far and starts writing the more comprehensive chapters 2 of each book. Chapter 2 of *The True* is a complete record of all facts about the world outside the books as well as about chapter 1 of each of the books. He uses the predicate "is true" in the meaning "is a sentence written in *The True*" and similarly the predicate "is false" in the meaning "is a sentence written in *The False*". So "'Snow is white' is true" and "'Snow is green' is false" both appear in this chapter. Because "Snow is green" is in *The False*, it is determined that this sentence will never be in *The True*, no matter how many chapters are written, so the writer can put "'Snow is green' is true" in chapter 2 of *The False*.

There are sentences that the writer puts in neither of the two chapters 2. One of them is "''''Snow is white' is true' is true' is true". This is because the sentence "''''Snow is white' is true' is true" is not in chapter 1 of either of the books, so whether it will be written in *The True* is not yet determined. Another is "This sentence is false". And for the same reason. The sentence referred to by "This sentence" namely "This sentence is false". itself, is not in chapter 1 of either of the books. After having written the two new chapters, there are again new facts, so the writer also compiles increasingly comprehensive chapters 3, 4, 5, etc. Of the two mentioned sentences, the first eventually gets into one of the books (*The True*, in chapter 4), while the second never does. It is "ungrounded".

According to this metaphor, Kripke's theory introduces an element of temporality in semantics; the sentence "'Grass is green' is true" is made true *later* than the sentence "Grass is green". I believe we should take this temporality seriously and not just metaphorically. Imagine, in analogy to the ideal ma-

[1]Much criticism can be directed at Kripke's theory, so whether the paradox can really be solved by the theory is a controversial issue. I will not go into that discussion here, but simply assume that Kripke's theory is adequate.

thematician appealed to by writers in intuitionistic mathematics (for example by Brouwer in his (1933)), an ideal linguist who was to construct the set of truths and the set of falsities given only non-semantical facts. We can then see Kripke's theory as a set of rules for the linguist to follow in this undertaking. Were he to follow these rules, he would of course proceed just as the writer in the metaphor except that we must abstract from the concrete writing of books.

So a constructivist theory of truth that can be an interpretation and justification of Kripke's theory is that the truth values of sentences are the mental constructions of an ideal linguist. Let me immediately prevent a possible misunderstanding: I am not claiming some form of anti-realism or idealism in a general sense. The existence of mind-independent facts is not rejected. We can consistently believe that it is a mind-independent *fact* that grass is green, while claiming that the *truth* of the sentence "Grass is green" is mind-dependent. For the truth of this sentence is not determined by the fact of the greenness of grass alone; an equally important role is played by the rules of the English language. These rules are a human construct, and the act of applying them to sentences to assign a truth value is a mental one.

Mental constructions happen in time, and when the language, to whose sentences truth values are to be assigned, itself contains predicates for the truth values, these mental constructions cannot be carried out in any order. According to the correspondence theory of truth, a sentence is true if it corresponds to or represents a fact. For something to represent something else, the represented must in some sense be logical prior to the representing. So when not only the representing but also the represented is a sentence, i.e. when a sentence is about sentences, temporality appears in semantics; the semantics of some sentences must be prior to the semantics of other sentences. Only after the sentence "Grass is green" is made true, is there a fact to which the sentence "'Grass is green' is true" can correspond. This I believe to be the lesson of Kripke's theory (although perhaps not of Kripke).

In this theory, Yablo's Paradox is solved in exactly the same way as the Liar Paradox. None of the sentences Y1, Y2, Y3, … are ever put in either of the books and hence none of them are true or false. This is because each of the sentences depend for their truth value on an infinity of other sentences in the list, and at no point in time do all these sentences and their truth values exist so as to determine the truth value of any given sentence on the list.

It is in other words premise (B3) that should be rejected, both in the case of New Zeno and in the case of Yablo's Paradox: Rejecting the possibility of an actual infinity of gods and sentences/truth values makes for a uniform solution to these paradoxes.

Conclusion

The New Zeno Paradox and Yablo's Paradox are of similar logical form and hence, according to the Principle of Uniform Solution, they should have similar solutions. This requirement disqualifies many proposed solutions to each of the paradoxes. On the other hand it serves to make the constructivist solution to New Zeno proposed in this paper and Kripke's solution to the paradoxes of truth under a constructivist interpretation lend reciprocal support to each other.

Of the three proposed solutions to New Zeno; the denial-of-the-possibility-of-motion solution, the logic-stops-them solution and the denial-of-actual-infinity solution, the first falls for the Principle of Uniform Solution applied to New Zeno and the paradox of the demons calling YES and NO, and the second falls for the same principle applied to the latter paradox and Yablo's Paradox. Only constructivism can, it seems to me, be a basis for a common solution to all these paradoxes.

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