

Optimal Distribution of N-Team Interacting Decision Makers with Hierarchical Command Inputs That Are Predicated on Order Statistics

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Abstract

The spectral analysis of simulated N-team of interacting decision makers with bounded rationality constraints of Oladejo, which assumes triangular probability density function of command inputs is hereby restructured and analysed, to have hierarchical command inputs that are predicated on order statistics distributions. The results give optimal distributions.

Keywords

Decision Maker, Command and Control, Bounded Rationality Constraints, Hierarchical, Conditional Probability Density Function, Situation Report, Computational Intelligence and Information Management

1. Introduction

The Double-Team of decision Maker Model that Boettcher and Levis developed [1] was extended and generalized by Oladejo [2] to N-Teams of Decision Makers with N^n Bounded Rationality Constraints. A Spectral Analysis of the model was provided by Oladejo [3], where command inputs were based on uniform distribution. This work considered the same model with modified command inputs that were hierarchical and whose distribution was predicated on order statistics. In this new work, the command inputs were categorized according to the superiority of the commander who was next higher in rank to the officer making input. The analytical procedure of optimizing convoluted strategies was used to derive the optimal distribution functions of the hierarchical command inputs. In the previous work there was a mixture of vertical and horizontal signal commu-

communications, whereas in this recent research consideration given to all signal communications must be ascending serially and vertically. This procedure is endemic with optimal sequencing [4], optimal choices [5], optimization of vending outlets [4] [6] [7], optimal transportation security [8], optimal maintenance policy [9] and optimized system models [10] [11] [12] [13].

The Model: The model of N-Team interacting decision makers with bounded rationality constraints was decomposed into coherent component strategies which were then used to analyze the system of specified C³I.

Symbols

X^i is the input signal having exponential tempo/inter arrival times of the i^{th} team.

π is the portioning algorithm of inputs to respective DM.

u is the internal decision.

f is the algorithm for process u^i to obtain the battle scenario.

q^i is the i^{th} team input regulator.

z^i is the initial situational assessment.

\bar{z}^i is the final situational assessment.

$g_C^{A^i}, g_C^{B^i}$ are the internal coordination strategies using SITREP z 's and v^i , the command input to the i^{th} team.

z' is situation assessment from other organizational members (SITREP or Feedback).

v is the final choice of decision.

v^i is the command input.

\bar{v}^i is the output of DM i .

\bar{v} is the final choice or response.

Y is the output or desired result.

h_j is the processing algorithm for the final choice leading to Y .

h^i is the algorithm for process v^i .

$P(v|z)$ is response selection strategy that maps z to Y in the absence of v' and determines choices of h_j .

$P(\bar{v}|\bar{z}, v')$ is response selection strategy that maps z and v' to v , and it also determines choices of h_j .

$H(x) = -\sum_x p(x) \log_2 p(x)$ is entropy of inputs, where $p(x)$ is probability or uncertainty associated with N random variables, X .

$T(x, z', v', Y)$ is mutual information or transmission or throughput between inputs x, z', v' and output Y .

G_t is the throughput.

G_b is the blockage.

G_n is the noise.

G_c is the internal coordination.

\mathbb{H} is the entropy of a random variable with probability p for taking one or two values, which also gives the amount of information transmitted.

A is the set of situational assessment options or strategies.

B is the set of available choices of tactics, modus operandi to remedy problem.
 ψ is the convolution of strategies which is the assumed combined effects of process activities.

where

$$j_{11} = 1, 2, \dots, U^1, j_{21} = 1, 2, \dots, V^1, j_{12} = 1, 2, \dots, U^2, j_{22} = 1, 2, \dots, V^2, \dots, \\ j_{1N} = 1, 2, \dots, U^N, \dots, j_{2N} = 1, 2, \dots, V^N$$

This is the structure of the generalized developed model as shown below:

The generalized developed model of **Figure 1** is as shown:

$$\begin{aligned} & \underline{DMt(t = 1, 2, \dots, N)} \\ G^t = & T(X^t, Z^{t(N-1)}, \dots, Z^{t1}, V^{(N-1)1} Z^{t1}, \dots, Z^{(N-1)t}; Y^t) + H(X^t, Z^{t(N-1)}, Z^{t1}, V^{(N-1)t}) \\ & - G_t^t + H(u^t) + H_z(v^t) + \sum_{i=1}^{U^t} [p_i g_c^i(pX^t) + \alpha_i \mathbb{H}(p_i)] + H(Z^t, Z^{t1}, \dots, Z^{(N-1)t}) \\ & + g_c^{A^t}(p(Z^t), p(Z^{t1}), \dots, p(Z^{t(N-1)})) + g_c^{B^t}(p(Z^t), p(V^{(N-1)t})) \\ & + \sum_{j=1}^{V^t} [p_j g_c^j(p(Z^t/V^t)) + \alpha_j \mathbb{H}(p_j)] + H(Y^t) + H(Z^t) + H(\bar{Z}^t) + H(Z^t, V^t) \\ & + T(X^t : Z^{t1}, \dots, Z^{t(N-1)}) + TZ^{-t}(X^t; Z^{t1}, \dots, Z^{t(N-1)} : V^{(N-1)t}) \end{aligned} \tag{1}$$

2. Analysis

Strategies which are probability density functions (pdfs) are given as follows:

$U^i(x^i) \sim$ exponential inter arrival times or tempo of operations:

$$\frac{1}{\theta} e^{-t/\theta}, t, \theta > 0$$

$f^i(u^i) \sim$ normal because of a defined goal (*i.e.* goal is focus):

$$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}, -\infty < x, \mu < \infty, \sigma^2 > 0 \tag{2}$$

$A(z^i) \sim$ uniform distribution: $\frac{1}{N}$ or $\frac{x}{b-a}$, $x = 0, 1, 2, \dots, N$ or

$a < x < b$, respectively (3)

$$v^t \sim \text{triangular } \frac{\bar{V}}{C}, 0 < VC, F = R = \frac{v^{-2}|^c}{2C|_0}, v^{-1} = 2R, (\text{i.e for } v^{-1} = c) \tag{4}$$

g_c^k is internal coordination strategy of corresponding algorithm which depends on the distribution of their respective inputs.

$g_c^{A^1} \sim$ Bernoulli: $p^x q^{1-x}$, $x = 0, 1$ (appropriate strategies are employed. Hence, $\sum x_i$ is binomial and all samples are appropriate. These are the internal coordination strategies among situational assessments represented by set A^1).

$g_c^{B^1} \sim$ geometric: pq^x , $x = 1, 2, \dots$, depending on x (the strategies would result in a success hence $\sum x_i$ is negative binomial that all samples are successful.

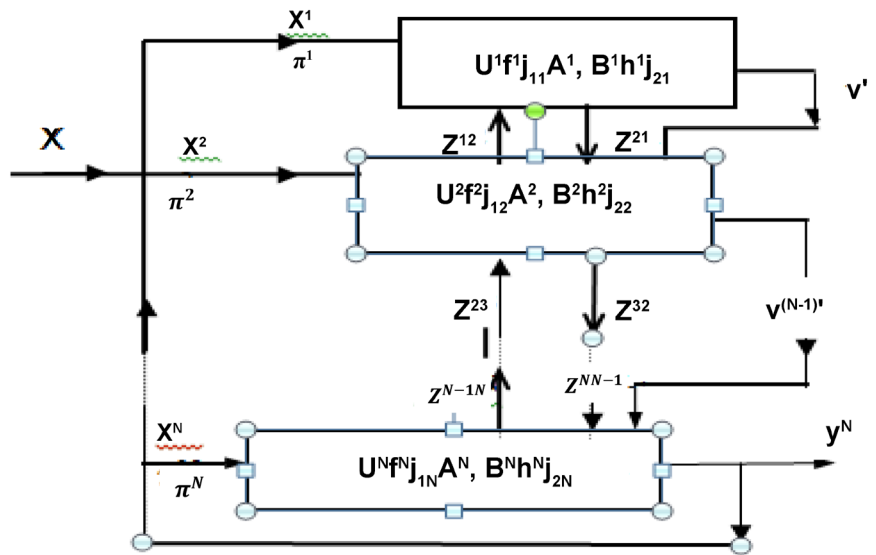


Figure 1. N-Team of interacting decision makers with bounded rationality.

These are the internal coordination strategies among final choices and command inputs represented by set B^l).

$h(v^j) \sim$ Weibull (due to reliability of subsystem before reaching final stage:
 $\alpha\beta X^{\beta-1} e^{-\alpha X^\beta}$, $X > 0$, $\alpha, \beta > 0$).

$\mu(X) \sim$ expo (this is due to random occurrences).

$F(u) \sim$ normal (i.e. geared towards a goal).

$A(z) \sim$ uniform (simple random sample, equally likely samples).

$B(z, \bar{v} | \bar{v}^j) \sim$ conditional jpdf {(uniform \times triangular) \times (g_C^A g_C^B) yielding interaction due to z 's to determine choices of v depending on v^j from superior command}.

$h(\bar{v}) \sim$ Weibull (output Y depends on survivability or reliability of other systems components).

$v^j \sim$ order Statistic-Hierarchical command inputs.

Now if command inputs v^j has pdf Ordered statistics: $f(y_n)$, $f(y_1)$, $f(y_k)$, $f(y_i, y_j)$

$$g_n(y_n) = n[F(y_n)]^{n-1} f(y_n), a < y_n < b \tag{5}$$

$$g_1(y_1) = n[1 - F(y_1)]^{n-1} f(y_1), a < y_1 < b \tag{6}$$

Let $g(y_i) = i$ be hierarchical transformations of signal inputs

$$g_k(y_k) = \frac{n!}{(k-1)(n-k)!} [F(y_k)]^{k-1} [1 - F(y_k)]^{n-k} f(y_k), a < y_k < b \tag{7}$$

$$g_{ij}(y_i, y_j) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} [F(y_i)]^{i-1} [F(y_j) - F(y_i)]^{j-i-1} \times [1 - F(y_j)]^{n-j} f(y_i) f(y_j), a < y_k < b \tag{8}$$

and zero elsewhere.

3. Methodology

An analytical approach was used to derive the optimal distributions of the convoluted strategies. Since the various events are independent the convolution of strategies was obtained by their product. The derivative of this convoluted strategy was equated to zero then solved.

$B(\bar{z}, \bar{v} | \bar{v}') \sim$ discrete conditional jpdfs is given by (uniform \times order statistic) $\times (g_C^A \cdot g_C^B)$.
 $B(\bar{z}, \bar{v} | \bar{v}')$ of first/initial command input.

3.1. Derivation of pdf

$$B(\bar{z}, \bar{v} | \bar{v}') = \frac{1}{N} \cdot N [1 - F(y_1)]^{N-1} f(y_1) \binom{N}{x} P^x q^{N-x} \binom{N-1}{x-1} P^x q^{N-x} \quad (9)$$

$$= [1 - F(y_1)]^{N-1} p \left[\binom{N}{x} P^x q^{N-x} \right]^2 = [1 - F(y_1)]^{N-1} p \left[\binom{N}{y_1} P^{y_1} q^{N-y_1} \right]^2 \cdot \frac{y_1}{N} \quad (10)$$

$$p \equiv f(y_1), y_1 = x, 1 < y_1 < N \quad (11)$$

$$F(y) = \int f(y) dy \quad \text{or} \quad (12)$$

$$F(y) = \sum f(y), q = 1 - p \quad (13)$$

$B(\bar{z}, \bar{v} | \bar{v}')$ of final/last command input.

$$B(\bar{z}, \bar{v} | \bar{v}') = \frac{1}{N} \cdot N [1 - F(y_n)]^{N-1} f(y_n) \binom{N}{y_n} P^{y_n} q^{N-y_n} \binom{N-1}{y_n-1} P^{y_n} q^{N-y_n}$$

$$= [1 - F(y_n)]^{N-1} p \left[\binom{N}{y_n} P^{y_n} q^{N-y_n} \right]^2 = \frac{y_n}{N}, p \equiv f(y_n), a < y_n < b \quad (14)$$

$B(\bar{z}, \bar{v} | \bar{v}')$ of intermediate k^{th} command input.

$$B(\bar{z}, \bar{v} | \bar{v}') = \frac{1}{N} \cdot \binom{N}{y_k} [F(y_k)]^{k-1} [1 - F(y_k)]^{N-k} \cdot p \left[\binom{N}{y_k} P^{y_k} q^{N-y_k} \right]^2 \cdot \frac{y_k}{N}$$

$$\frac{1}{N} \cdot \binom{N}{y_k} [F(y_k)]^{k-1} [1 - F(y_k)]^{N-k} \cdot p \left[\binom{N}{y_k} P^{y_k} q^{N-y_k} \right]^2 \cdot \frac{y_k}{N},$$

$$p \equiv f(y_k), a < y_k < b \quad (15)$$

$B(\bar{z}, \bar{v} | \bar{v}')$ of interacting i^{th} and j^{th} command input.

$$B(\bar{z}, \bar{v} | \bar{v}') = \frac{N!}{(i-1)!(j-i-1)!(N-j)!} [F(y_i)]^{i-1} [F(y_j) - F(y_i)]^{j-1}$$

$$\cdot [1 - F(y_j)]^{N-j} f(y_i) f(y_j) \cdot \binom{N}{y_i} P^{y_i} q^{N-y_i} \binom{N-y_i}{y_j-1} P^{y_j} q^{N-y_j-y_i}$$

$$= \frac{N!}{(i-1)!(j-i-1)!(N-j)!} [F(y_i)]^{i-1} [F(y_j) - F(y_i)]^{j-i-1}$$

$$\cdot [1 - F(y_j)]^{N-j} p_i p_j \left[\binom{N}{y_i} P_i^{y_i} q_i^{N-y_i} \right]^2 \cdot \frac{y_i}{N}$$

$$\cdot \left[\binom{N-y_i}{y_j-1} P_i^{y_i} q_i^{N-y_j-y_i} \right]^2 \cdot \frac{y_j}{N-y_i}$$

$$p_i \equiv f(y_i), p_j \equiv f(y_j), a < y_i < y_j < b \tag{16}$$

3.2. Optimal Probabilities

$\frac{d}{dp} B(\bar{z}, \bar{v} | \bar{v}') = 0$ to get optimal prob of command inputs. These derivatives yield:

$$\begin{aligned} & (N-1)[1-F(y_1)]^{N-2} (-p)(p) \left[\binom{N}{y_1} p^{y_1} q^{N-y_1} \right]^2 \frac{y_1}{N} \\ & + [1-F(y_1)]^{N-1} \cdot \left[\binom{N}{y_1} p^{y_1} q^{N-y_1} \right]^2 \frac{y_1}{N} + 2p[1-F(y_1)]^{N-1} \\ & \cdot \frac{y_1}{N} \left[\binom{N}{y_1} p^{y_1} q^{N-y_1} \right] \left[\binom{N}{y_1} y_1 p^{y_1-1} q^{N-y_1} - (N-y_1) p^{y_1} q^{N-y_1-1} \right] = 0 \end{aligned} \tag{17}$$

$$\begin{aligned} & - \frac{p^2(N-1) \frac{y_1}{N} + \frac{y_1}{N} + 2p \left[\binom{N}{y_1} p^{y_1} q^{N-y_1} \right]^{-1}}{1-F(y_1)} \\ & \cdot \left[\binom{N}{y_1} y_1 p^{y_1-1} q^{N-y_1} - (N-y_1) p^{y_1} q^{N-y_1-1} \right] \frac{y_1}{N} = 0 \\ & 1 - \frac{p^2(N-1)}{1-F(y_1)} + 2p \left[\binom{N}{y_1} p^{y_1} q^{N-y_1} \right]^{-1} \\ & \cdot \left[\binom{N}{y_1} y_1 p^{y_1-1} q^{N-y_1} - (N-y_1) p^{y_1} q^{N-y_1-1} \right] = 0 \end{aligned} \tag{18}$$

Simplifying Equation (18) to get

$$1 - \frac{p^2(N-1)}{1-F(y_1)} + 2p \left(\frac{y_1}{p} - \frac{N-y_1}{q} \right) = 0 \tag{19}$$

$$1 + 2p \left(\frac{y_1}{p} - \frac{N-y_1}{q} \right) = \frac{p^2(N-1)}{1-F(y_1)} \tag{20}$$

$$1 + \frac{2p}{pq} (qy_1 - p(N-y_1)) = \frac{p^2(N-1)}{1-F(y_1)} \tag{21}$$

$$q + 2(qy_1 - p(N-y_1)) = \frac{p^2q(N-1)}{1-F(y_1)} = q + 2(y_1(q+p) - Np) \tag{22}$$

$$1-F(y_1) = \frac{p^2q(N-1)}{q + 2(y_1(q+p) - Np)} \tag{23}$$

$$\therefore F(y_1) = 1 - \frac{(N-1)p^2q}{q + 2(y_1(q+p) - Np)} = 1 - \frac{(N-1)p^2q}{q + 2(y_1 - Np)} \tag{24}$$

Similarly

$$F(y_N) = 1 - \frac{(N-1)p^2q}{q + 2(y_N(q+p) - Np)} = 1 - \frac{(N-1)p^2q}{q + 2(y_N - Np)} \tag{25}$$

$B(\bar{z}, \bar{v} / v')$ for intermediate k^{th} command inputs

$$B(\bar{z}, \bar{v} / v') = \frac{1}{N} \binom{N}{y_i} [F(y_k)]^{k-1} [1-F(y_k)]^{N-k} p \left[\binom{N}{y_i} p^{y_k} q^{N-y_k} \right]^2 \frac{y_k}{N}$$

$$\frac{d}{dp} B(\bar{z}, \bar{v} / v') = 0 \quad (26)$$

$$\begin{aligned} & \frac{d}{dp} B(\bar{z}, \bar{v} / v') \\ &= \frac{y_k}{N} \binom{N}{y_i} \left\{ (k-1) [F(y_k)]^{k-2} (p) p [1-F(y_k)]^{N-k} \left[\binom{N}{y_k} p^{y_k} q^{N-y_k} \right]^2 \right. \\ & \quad + (N-k) [F(y_k)]^{k-1} [1-F(y_k)]^{N-k-1} (-p) p \left[\binom{N}{y_i} p^{y_k} q^{N-y_k} \right]^2 \\ & \quad + 2p [F(y_k)]^{k-1} [1-F(y_k)]^{N-k} \left[\binom{N}{y_i} y_k p^{y_k-1} q^{N-k} - (N-y_k) p^{y_k} q^{N-y_k-1} \right] \\ & \quad \left. \cdot \left[\binom{N}{y_i} p^{y_k} q^{N-y_k} \right] \right\} = 0 \end{aligned} \quad (27)$$

Simplifying Equation (27) to get

$$1 + \frac{(k-1)p^2}{F(y_k)} - \frac{(N-k)p^2}{1-F(y_k)} + 2p \left(\frac{y_k}{p} - \frac{N-y_k}{q} \right) = 0 \quad (28)$$

$$\frac{F(y_k)[1-F(y_k)] + (k-1)[1-F(y_k)]p^2 - (N-k)F(y_k)p^2}{F(y_k)[1-F(y_k)]} \quad (29)$$

$$\begin{aligned} & + \frac{2}{q} (qy_k - (N-y_k)p) = 0 \\ & F(y_k)[1-F(y_k)] + p^2(k-1)[1-F(y_k)] - p^2 - (N-k)F(y_k) \\ & = -\frac{2}{q} (y_k(q+p) - Np) F(y_k) [1-F(y_k)] \end{aligned} \quad (30)$$

$$\begin{aligned} & F(y_k) - [F(y_k)]^2 + p^2(k-1) - p^2(k-1)F(y_k) + p^2F(y_k) - p^2(N-k)F(y_k) \\ & = -\frac{2}{q} [y_k(q+p) - Np] F(y_k) + \frac{2}{q} [y_k(q+p) - Np] [F(y_k)]^2 \end{aligned} \quad (31)$$

$$\begin{aligned} & F(y_k) - [F(y_k)]^2 + p^2(k-1) - kp^2F(y_k) + p^2F(y_k) - Np^2F(y_k) + kp^2F(y_k) \\ & = -\frac{2}{q} [y_k(q+p) - Np] F(y_k) - \frac{2}{q} [y_k(q+p) - Np] [F(y_k)]^2 \end{aligned} \quad (32)$$

$$\begin{aligned} & \left\{ 1 - \frac{2}{q} [y_k(q+p) - Np] \right\} [F(y_k)]^2 \\ & + \left\{ -1 + p^2(k-1) + p^2(N-k) - \frac{2}{q} [y_k(q+p) - Np] \right\} F(y_k) - p^2(k-1) = 0 \end{aligned} \quad (33)$$

$$\begin{aligned} & \left\{ 1 - \frac{2}{q} [y_k(q+p) - Np] \right\} [F(y_k)]^2 \\ & + \left\{ (N-1)p^2 - 1 - \frac{2}{q} [y_k(q+p) - Np] \right\} F(y_k) - p^2(k-1) = 0 \end{aligned} \quad (34)$$

By quadratic equation: $ax^2 + bx + c = 0$, solution: $(x - \alpha)(x - \beta) = 0$,

$$(\alpha, \beta) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Let

$$F(y_i) = x \tag{35}$$

$$\begin{aligned} \therefore F(y_k) &= \frac{1 - (N-1)p^2 + \frac{2}{q}[y_k(q+p) - Np]}{2\left\{1 - \frac{2}{q}[y_k(q+p) - Np]\right\}} \\ &\pm \frac{\sqrt{\left\{1 - (N-1)p^2 + \frac{2}{q}[y_k(q+p) - Np]\right\}^2 + 4\left\{1 - \frac{2}{q}[y_k(q+p) - Np]\right\}(k-1)p^2}}{2\left\{1 - \frac{2}{q}[y_k(q+p) - Np]\right\}} \end{aligned} \tag{36}$$

$$\begin{aligned} F(y_{ij}) &= \frac{N!}{(i-1)!(j-i-1)!(N-j)!} [F(y_i)]^{i-1} p_i \left[\binom{N}{y_i} p_i^{y_i} q^{N-y_i} \right]^2 \frac{y_i}{N} \\ &\cdot [F(y_j) - F(y_i)]^{j-i-1} [1 - F(y_j)]^{N-j} p_j \left[\binom{N}{y_j} p_j^{y_j} q^{N-y_j-y_i} \right]^2 \frac{y_j}{N-y_i} \\ &\frac{d}{dp} B(\bar{z}, \bar{v} / v') = 0 \end{aligned} \tag{37}$$

Let

$$\phi_1 = [F(y_i)]^{i-1} p_i \left[\binom{N}{y_i} p_i^{y_i} q_i^{N-y_i} \right]^2 \frac{y_i}{N} \tag{38a}$$

$$\phi_2 = [F(y_j) - F(y_i)]^{j-i-1} \tag{38b}$$

$$\phi_3 = [1 - F(y_j)]^{N-j} p_j \left[\binom{N-y_i}{y_j-1} p_j^{y_j} q_j^{N-y_j-y_i} \right]^2 \frac{y_j}{N-y_i} \tag{38c}$$

$$\frac{d}{dp} B(\bar{z}, \bar{v} / v') = \phi_1' \phi_2 \phi_3 + \phi_1 \phi_2' \phi_3 + \phi_1 \phi_2 \phi_3' = 0 \tag{39}$$

$$\begin{aligned} \phi_1' &= (i-1)[F(y_i)]^{i-2} p_i^2 \left[\binom{N}{y_i} p_i^{y_i} q^{N-y_i} \right]^2 + [F(y_i)]^{i-1} \left[\binom{N}{y_i} p_i^{y_i} q^{N-y_i} \right]^2 \\ &\quad + 2p_i [F(y_i)]^{i-1} \left[\binom{N}{y_i} p_i^{y_i} q^{N-y_i} \right] \left[(N-y_i) p_i^{y_i} q^{N-y_i-1} \right] \frac{y_i}{N} \\ &= \frac{y_i}{N} \cdot \left\{ \frac{(i-1)p_i^2}{F(y_i)} + 1 + 2p_i \left[\frac{y_i}{p_i} - \frac{(N-y_i)}{q_i} \right] \right\} [F(y_i)]^{i-1} \left[\binom{N}{y_i} p_i^{y_i} q_i^{N-y_i} \right]^2 \frac{y_i}{N} \\ &= \frac{y_i}{N} \left\{ (i-1)[F(y_i)]^{-1} p_i^2 + 1 + \frac{2}{q_i} [y_i(q_i + p) - Np_i] \right\} [F(y_i)]^{i-1} \left[\binom{N}{y_i} p_i^{y_i} q^{N-y_i} \right]^2 \end{aligned} \tag{40}$$

$$\phi_2' = (j-i-1)[F(y_j) - F(y_i)]^{j-i-2} (p_j - p_i) \tag{41}$$

$$\begin{aligned} \phi'_3 &= (N-j)[1-F(y_i)]^{N-j-1}(-p_j)p_j \left[\left(\frac{N-y_i}{y_j-1} \right) p_i^{y_i} q_j^{N-y_j-y_i} \right]^2 \\ &+ [1-F(y_j)]^{N-j} \left[\left(\frac{N-y_i}{y_j-1} \right) p_j^{y_j} q_j^{N-y_j-y_i} \right]^2 \end{aligned} \quad (42)$$

$$\begin{aligned} &+ 2[1-F(y_j)]^{N-j} p_j \left[\left(\frac{N-y_i}{y_j-1} \right) p_j^{y_j} q_j^{N-y_j-y_i} \right] \\ &\cdot \left[\left(\frac{N-y_i}{y_j-1} \right) (y_j p_j^{y_j} q_j^{N-y_j-y_i} - (N-y_j-y_i) p_j^{y_j} q_j^{N-y_j-y_i}) \right] \cdot \frac{y_j}{N-y_i} \\ &= \left\{ -\frac{(N-j)p_j^2}{1-F(y_j)} + 1 + 2p_j \left[\frac{y_j}{p_j} - \frac{N-y_j-y_i}{q_j} \right] \right\} \\ &\cdot [1-F(y_j)]^{N-j} \left[\left(\frac{N-y_i}{y_j-1} \right) p_j^{y_j} q_j^{N-y_j-y_i} \right]^2 \cdot \frac{y_j}{N-y_i} \end{aligned} \quad (43)$$

$$\begin{aligned} &= \frac{y_j}{N-y_i} \left\{ -(N-j)p_j^2 [1-F(y_j)]^{-1} + 1 + \frac{2}{q_j} [y_j(q_j+p_j) - Np_j] \right\} \\ &\cdot [1-F(y_j)]^{N-j} \left[\left(\frac{N-y_i}{y_j-1} \right) p_j^{y_j} q_j^{N-y_j-y_i} \right]^2 \end{aligned}$$

$$\begin{aligned} &\frac{dB}{dp}(\bar{z}, \bar{v} \setminus v') \\ &= \left\{ \frac{(i-1)p_i^2}{F(y_i)} + 1 + \frac{2}{p_j} [y_j(q_i+p_i) - Np_i] \right\} B(\bar{z}, \bar{v} / v') \\ &+ (j-i-1)(p_j-p_i) [F(y_j)-F(y_i)]^{-1} B(\bar{z}, \bar{v} / v') \\ &+ \left\{ -(N-j)p_j^2 [1-F(y_j)]^{-1} + 1 - \frac{2}{q_j} [y_j(q_j+p_j) - Np_j] \right\} B(\bar{z}, \bar{v} / v') \\ &= (i-1)p_i^2 [F(y_i)]^{-1} + 1 - \frac{2}{q_i} [y_i(q_i+p_i) - Np_i] \\ &+ (j-i-1)(p_j+p_i) [F(y_j)-F(y_i)]^{-1} \\ &- (N-j)p_j^2 [1-F(y_j)]^{-1} + 1 - \frac{2}{q_j} [y_j(q_j+p_j) - Np_j] \\ &= 0 \end{aligned} \quad (44)$$

Simplifying Equation (44) to get

$$\begin{aligned} &(i-1)p_i^2 [F(y_j)-F(y_i)] [1-F(y_i)] \\ &+ \left\{ 1 + \frac{2}{q_i} [y_i(q_i+p_i) - Np_i] \right\} F(y_i) [F(y_j)-F(y_i)] [1-F(y_j)] \\ &+ (j-i-1)(p_j-p_i) F(y_i) [1-F(y_j)] - (N-j)p_j^2 F(y_i) [F(y_j)-F(y_i)] \\ &+ \left\{ 1 - \frac{2}{q_j} [y_j(q_j+p_j) - Np_j] \right\} [F(y_i)-F(y_j)-F(y_i)] = 0 \end{aligned} \quad (45)$$

$$\begin{aligned}
 & (i-1)p_i^2 \left[F(y_j) - F(y_j)^2 - F(y_i) + F(y_i)F(y_j) \right] \\
 & + \left\{ 1 - \frac{2}{q_i} [y_i(q_i + p_i) - Np_i] \right\} F(y_i) \left[F(y_j) - F(y_i)F(y_j)^2 \right] [1 - F(y_j)] \\
 & + (j-i-1)(p_j - p_i)F(y_i) [1 - F(y_j)] - (N-j)p_j^2 F(y_i) [F(y_j) - F(y_i)] \quad (46) \\
 & + \left\{ 1 - \frac{2}{q_j} [y_j(q_j + p_j) - Np_j] \right\} \left[F(y_i) - F(y_j) - F(y_i)F(y_j)^2 \right] \\
 & \cdot \left[F(y_i)^2 + F(y_j)F(y_i)^2 \right] = 0
 \end{aligned}$$

$$\begin{aligned}
 & \left\{ (N-j)p_j^2 - 1 + \frac{2}{q_j} [y_j(q_j + p_j) - Np_j] - 1 - \frac{2}{q_i} [y_i(q_i + p_i) - Np_i] \right\} (F(y_i))^2 \\
 & + [(j-i-1)(p_j - p_i) - (i-1)p_i^2] F(y_i) - [(i-1)p_i^2 (F(y_j))^2 - F(y_i)] \\
 & + [(i-1)p_i^2] (F(y_j))^2 - F(y_j) + \left\{ 1 + \frac{2}{q_i} [y_i(q_i + p_i) - Np_i] \right\} \\
 & \cdot \left[F(y_j)(F(y_i))^2 - [F(y_i)(F(y_j))^2] \right] + \left\{ (i-1)p_i^2 + 1 + \frac{2}{q_i} [y_i(q_i + p_i) - Np_i] \right. \\
 & \left. - [(j-i-1)(p_j - p_i) - (N-j)p_j^2 - 1 - \frac{2}{q_j} [y_j(q_j + p_j) - Np_j]] \right\} F(y_i)F(y_j) \Big] = 0 \quad (47)
 \end{aligned}$$

Then from Equation (35)

$$\begin{aligned}
 & \left\{ (N-j)p_j^2 - 1 + \frac{2}{q_j} [y_j(q_j + p_j) - Np_j] - 1 - \frac{2}{q_i} [y_i(q_i + p_i) - Np_i] \right\} (F(y_i))^2 \\
 & + [(j-i-1)(p_j - p_i) - (i-1)p_i^2] F(y_i) + 1 = 0 \quad (48)
 \end{aligned}$$

$$\begin{aligned}
 \therefore F(y_i) = & \frac{-(j-i-1)(p_j - p_i) - (i-1)p_i^2 \pm \sqrt{[(j-i-1)(p_j - p_i) - (i-1)p_i^2]^2 - 4\{(N-j)p_j^2 - 1\}}}{2 \left\{ (N-j)p_j^2 - 1 + \frac{2}{q_j} [y_j(q_j + p_j) - Np_j] - 1 - \frac{2}{q_i} [y_i(q_i + p_i) - Np_i] \right\}} \\
 & + \frac{\frac{2}{q_j} [y_j(q_j + p_j) - Np_j] - 1 - \frac{2}{q_i} [y_i(q_i + p_i) - Np_i] [(i-1)p_i^2] [(F(y_j))^2 - F(y_j)] - 1}{2 \left\{ (N-j)p_j^2 - 1 + \frac{2}{q_j} [y_j(q_j + p_j) - Np_j] - 1 - \frac{2}{q_i} [y_i(q_i + p_i) - Np_i] \right\}} = 0 \quad (49)
 \end{aligned}$$

Simplifying the above quadratic equations to get

$$-[(i-1)p_i^2] [(F(y_j))^2 - F(y_j)] - 1 = 0 \quad (50)$$

$$\Rightarrow (i-1)p_i^2 [(F(y_j))^2 - F(y_j)] + 1 = 0 \quad (51)$$

$$\therefore F(y_j) = \frac{(i-1)p_i^2 \pm \sqrt{(i-1)p_i^2 - 4(i-1)p_i^2}}{2(i-1)p_i^2} \quad (52)$$

But

$$F(y_j)(F(y_i))^2 - F(y_i)(F(y_j))^2 = F(y_i)F(y_j)[F(y_i) - F(y_j)] \quad (53)$$

Then

$$\begin{aligned} & \left\{ 1 + \frac{2}{q_i} [y_i(q_i + p_i) - Np_i] \right\} [F(y_i) - F(y_j)] \\ &= -(i-1)p_i^2 + \frac{2}{q_i} [y_i(q_i + p_i) - Np_i] - (j-i-1)(p_j - p_i) \\ & \quad - (N-j)p_j^2 - 1 - \frac{2}{q_j} [y_j(q_j + p_j) - Np_j] \\ \therefore F(y_j) - F(y_i) &= \frac{(i-1)p_i^2 + 1 + \frac{2}{q_i} [y_i(q_i + p_i) - Np_i] - (j-i-1)(p_j - p_i)}{1 + \frac{2}{q_i} [y_i(q_i + p_i) - Np_i]} \\ & \quad + \frac{-(N-j)p_j^2 - 1 - \frac{2}{q_j} [y_j(q_j + p_j) - Np_j]}{1 + \frac{2}{q_i} [y_i(q_i + p_i) - Np_i]} \\ &= 1 + \frac{(i-1)p_i^2 - (j-i-1)(p_j - p_i) - (N-j)p_j^2 - 1 - \frac{2}{q_j} [y_j(q_j + p_j) - Np_j]}{1 + \frac{2}{q_i} [y_i(q_i + p_i) - Np_i]} \end{aligned} \quad (54)$$

4. Results

$$F(y_1) = 1 - \frac{(10-1)(0.3)^2(0.7)}{0.7 + 2(1-3)} = 1 - \frac{0.567}{-3.3} = 1 + 0.171818 = 1.172 \quad \text{unrealistic}$$

$$F(y_3) = 1 - \frac{9(0.3)^2(0.7)}{0.7} = 1 - 0.81 = 0.19$$

$$F(y_{10}) = 1 - \frac{(10-1)(0.3)^2(0.7)}{0.7 + 2(10-3)} = 1 - \frac{0.567}{14.7} = 1 - 0.0386 = 0.9614$$

These results indicate:

- 1) $F(y_1)$ not feasible due to no (or low) command input(s).
- 2) $F(y_3)$ feasible.
- 3) $F(y_{10})$ feasible. As the hierarchy increases correspondingly the distribution increases. This can be taken as workload.

Example 1: Given $p_k = 0.3, k = 5, N = 10, y_k = k$

$$\begin{aligned} F(y_k) = F(y_5) &= \frac{1 - 9(0.09) + \frac{2}{0.7}(5 - 10(0.3))}{2 \left\{ 1 + \frac{2}{0.7}(5 - 10(0.3)) \right\}} \\ & \pm \frac{\sqrt{\left\{ -1 + 9(0.09) - \frac{2}{0.7}(5 - 10(0.3)) \right\}^2 + 4 \left(1 + \frac{2}{0.7}(5 - 10(0.3)) \right) 4(0.09)}}{2 \left\{ 1 + \frac{2}{0.7}(5 - 10(0.3)) \right\}} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1 - 0.81 + \frac{4}{0.7} \pm \sqrt{\left(-0.19 + \frac{4}{0.7}\right)^2 + 4\left(1 + \frac{4}{0.7}(0.36)\right)}}{2\left\{1 + \frac{4}{0.7}\right\}} \\
 &= \frac{0.12 + 5.7143 \pm \sqrt{(5.5243)^2 + 4(3.0572)}}{2(6.7143)} \\
 &= \frac{5.8343 \pm \sqrt{30.5179 + 12.288}}{13.4286} = \frac{5.8343 \pm \sqrt{42.7467}}{13.4286} = \frac{5.8343 \pm 6.5381}{13.4286} \\
 &= \frac{12.3724}{13.4286}, -\frac{0.7467}{13.4286} = 0.92135, -0.05241 \text{ (unrealistic)}
 \end{aligned}$$

Example 2: Given $p_i = 0.3, p_j = 0.5, i = 4, j = 7, k = 5, N = 10, y_k = K$

$$\begin{aligned}
 F(y_4) = F(y_4) &= \frac{-(2(0.2) - 3(0.09))}{2\left\{3(0.25) - 1 + \frac{2}{0.5}[7 - 5] - 1 - \frac{2}{0.7}[4 - 3]\right\}} \\
 &\pm \frac{\sqrt{\left[2(0.2) - 3(0.09)\right]^2 + 4\left\{3(0.25) - 1 + \frac{2}{0.5}[7 - 5] - 1 - \frac{2}{0.7}[4 - 3]\right\}}}{2\left\{3(0.25) - 1 + \frac{2}{0.5}[7 - 5] - 1 - \frac{2}{0.7}[4 - 3]\right\}} \\
 &= \frac{-(0.4 - 0.27) \pm \sqrt{(0.4 - 0.27)^2 + 4\{0.75 - 1 + 4(2) - 1 - 2.8571\}}}{2\{0.75 - 1 + 4(2) - 1 - 2.8571\}} \\
 &= \frac{-0.13 \pm \sqrt{15.5885}}{7.7858} = \frac{-0.13 \pm 3.9482}{7.7858} \\
 &= \frac{3.8182}{7.7858} = 0.49041, -\frac{4.0782}{7.7858} \text{ (unrealistic)}
 \end{aligned}$$

$$\begin{aligned}
 F(y_j) = F(y_7) &= \frac{3(0.09) \pm \sqrt{(3(0.09))^2 + 4[3(0.09)]}}{2(3(0.09))} \\
 &= \frac{0.27 \pm \sqrt{0.0729 + 1.08}}{0.54} = \frac{0.27 \pm \sqrt{0.1529}}{0.54} = \frac{0.27 \pm 1.073732}{0.54} \\
 &= \frac{1.343732}{0.54} = 2.4884, -\frac{0.803732}{0.54} = -1.4884 \text{ (both unrealistic)} \\
 &\approx 0 \text{ certainly unrealistic}
 \end{aligned}$$

$$\begin{aligned}
 F(y_j) = F(y_i) &= \frac{3(0.09) + 1 + \frac{2}{0.7}[4 - 3] - 2(0.2) - 3(0.25) - 1 - \frac{2}{0.5}[7 - 5]}{1 + \frac{2}{0.7} - 2(0.2) - 1 - \frac{2}{0.5}(2)} \\
 &= \frac{-6.0229}{-5.5429} = 1.0866 \\
 &\approx 0 \text{ certainly unrealistic}
 \end{aligned}$$

5. Discussion

The distribution of events at various stages of interactions were convolutions to obtain system distribution. Thereafter, analytical approach was used at various

stages to obtain optimal distribution of the command inputs.

6. Conclusion

Optimal distribution values obtained can be taken as system efficiency of the model, which can be used for system control.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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