Investigation of Probability Generating Function in an Interdependent *M*/*M*/1/:(∞; *GD*) Queueing Model with Controllable Arrival Rates Using Rouche's Theorem

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ABSTRACT

Present paper deals a $M/M/1/:(\infty; GD)$ queueing model with interdependent controllable arrival and service rates wherein customers arrive in the system according to poisson distribution with two different arrivals rates-slower and faster as per controllable arrival policy. Keeping in view the general trend of interdependent arrival and service processes, it is presumed that random variables of arrival and service processes follow a bivariate poisson distribution and the server provides his services under general discipline of service rule in an infinitely large waiting space. In this paper, our central attention is to explore the probability generating functions using Rouche's theorem in both cases of slower and faster arrival rates of the queueing model taken into consideration; which may be helpful for mathematicians and researchers for establishing significant performance measures of the model. Moreover, for the purpose of high-lighting the application aspect of our investigated result, very recently Maurya [1] has derived successfully the expected busy periods of the server in both cases of slower and faster arrival rates, which have also been presented by the end of this paper.

Keywords: Interdependent Queueing Model; Bivariate Poisson Process; Controllable Arrival Rates; Probability Generating Function; Laplace Transform; Rouche's Theorem; Performance Measures

1. Introduction

The probability generating function approach plays a vital role in the study of queueing problems as it is crucially useful in performance analysis of a wide range of queueing models. As an example, the probability generating function approach facilitates to determine the expected busy and idle periods and system size distribution. In the queueing literature, it has been enthusiastically observed that most of the previous researchers [2-7] and references therein have presumed that the parameters of arrival and service rates in the queueing systems are independent to each other. However, it is not so in general because we find many queueing situations in our real life where the arrival and service rates are correlated with an elevated extent. We remark here that the arrival rate of a variety of queueing systems is usually controlled in order to reduce the queue length. Queueing models with controllable arrival rates have been studied by a few noteworthy researchers [3,8-10] which reveals the fact that there is still an increasing demand of analyzing an interdependent queueing models with controllable arrival rates. Srinivasa Rao *et al.* [8] have confined to obtain the average system size and average waiting time of an

$M/M/1/\infty$ interdependent queueing model with controllable arrival rates under steady state conditions. Of late, Pal [4] considered the same queueing model which was already examined by Srinivasa Rao *et al.* [8] with a version of its limited waiting space and he succeeded to investigate the cost per unit time of a served customer in the system. Recently, Thiagarajan M. and Srinivasan A. [9] focused their attention to explore the $M^X/M/1/\infty$ interdependent queueing model with bulk arrivals and controllable arrival rates. In this sequential work, we consider here an interdependent $M/M/1/:(\infty; GD)$ queueing model incorporating bivariate Poisson process and controllable arrival rates in order to investigate the probability generating functions in faster and slower arrival rates.

2. Description of the Model

In the present study, we consider an interdependent $M/M/1/:(\infty; GD)$ queueing model with bivariate Poisson process and controllable arrival rates. The arrival pattern of customers are controlled by the system that it allows two arrival rates λ_1 and λ_2; $\lambda_1 > \lambda_2$. Without

loss of generality we assume that whenever the system size attains a fixed number S, the arrival rate reduces to λ_2 from λ_1 and the arrival rate λ_2 remains unchanged till the system size is greater than $R; 0 \leq R < S$. But as soon as the system size reduces to R, the arrival rate λ_2 changes back to λ_1 and the same pattern of change of arrival rates is repeated during the complete busy period of the system. Moreover, we assume that both $\{X(t)\}$ and $\{Y(t)\}$ representing respectively the arrival and service processes are interdependent and these discrete random variables follow a bivariate Poisson distribution [11] with their probability mass function $P\{X(t) = x, Y(t) = y\}$ defined as given below

$P\{X(t) = x, Y(t) = y\}$

$= e^{-(\lambda_2 + \mu - \nu)t}$

$\times \dfrac{\sum_{k=0}^{\min(\lambda_i, \mu)} (x, y)(\nu t)^k \left[(\lambda_i - \nu)t\right]^{x-k} \left[(\mu - \nu)t\right]^{y-k}}{k!(x-k)!(y-k)!}$

with following feasible conditions:
$$x, y = 1, 2, 3 \cdots; \lambda_i > 0; i = 1, 2$$

and $\mu > 0; 0 < \nu < \min(\lambda_i, \mu)$.

Here μ is the mean service rate and ν is the covariance between arrival and service processes.

3. Postulates of the Model

In addition to our assumptions in previous section-2 of the model, we have here underlying postulates for the purpose of our current study and analysis:

Postulate 3.1: The probability that there is one arrival and no service completion during a small interval of time Δt is $(\lambda_i - \mu)\Delta t + O\Delta t$; when the system has arrival rate $\lambda_i; i = 1, 2$.

Postulate 3.2: The probability that there is neither arrival and nor service completion during a small interval of time Δt is $1 - (\lambda_i - \mu - 2\nu)\Delta t + O\Delta t$, when the system has arrival rate $\lambda_i; i = 1, 2$.

Postulate 3.3: The probability that there is no arrival and one service completion during a small interval of time Δt is $(\mu - \nu)\Delta t + O\Delta t$, whatever be the arrival rate $\lambda_i; i = 1, 2$.

Postulate 3.4: The probability that there is one arrival and one service completion during a small interval of time Δt is $\nu\Delta t + O\Delta t$; whatever be the arrival rate $\lambda_i; i = 1, 2$.

4. Differential-Difference Equations

Before proceeding further, we use symbol $P_n(t_i); i = 1, 2$ be the probability that there are n customers in the system at time $t_i; i = 1, 2$ when system allows the arrival rate $t_i; i = 1, 2$.

Now it is fairly easy to observe that $P_n(t_1)$ exists when $n = 0, 1, \cdots, R-1, R$; however both $P_n(t_1)$ and

$P_n(t_2)$ exist when $n = R+1, R+2, \cdots S-1$; but only $P_n(t_2)$ exists when $n = S, S+1 \cdots$

We further assume that the initial system size is 1 and $R+1$ respectively when system has arrival rate $\lambda_i; i = 1, 2$. Let $P_0'(t_1)$ and $P_{R+1}'(t_2)$ be the busy period density respectively when the system has arrival rate $\lambda_i; i = 1, 2$.

Now in view of an absorbing barrier at empty system during its faster arrival rate λ_1 the governing differential difference equations of the system size for the model are as following:

$$P_0'(t_1) = (\mu - \nu)P_1(t_1) \tag{4.1}$$

$$P_1'(t_1) = -(\lambda_1 + \mu - 2\nu)P_1(t_1) + (\mu - \nu)P_2(t_1) \tag{4.2}$$

$$P_n'(t_1) = -(\lambda_1 + \mu - 2\nu)P_n(t_1) + (\mu - \nu)P_{n+1}(t_1) + (\lambda_1 - \nu)P_{n-1}(t_1); n = 2, 3, \cdots R-1 \tag{4.3}$$

The differential difference equation for the system size $n = R$ is as following:

$$P_R'(t_1) = (\lambda_1 - \nu)P_{R-1}(t_1) - (\lambda_1 + \mu - 2\nu)P_R(t_1) + (\mu - \nu)P_{R+1}(t_1) \tag{4.4}$$

Moreover, the differential difference equations for the system size $n = R+1, R+2, \cdots S-2$ are as following:

$$P_n'(t_1) = (\lambda_1 - \nu)P_{n-1}(t_1) - (\lambda_1 + \mu - 2\nu)P_n(t_1) + (\mu - \nu)P_{n+1}(t_1) \tag{4.5}$$

And the differential difference equation governing the state $n = S+1$ is as follows

$$P_{S+1}'(t_1) = (\lambda_1 - \nu)P_{S-2}(t_1) - (\lambda_1 + \mu - 2\nu)P_{S-1}(t_1) \tag{4.6}$$

Similarly, in view of an absorbing barrier at $(R+1)^{th}$ system size during its slower arrival rate λ_2, we have the differential difference equations governing for the system size $n = R+1, R+2$ as following:

$$P_{R+1}'(t_2) = (\mu - \nu)P_{R+2}(t_2) \tag{4.7}$$

$$P_{R+2}'(t_2) = (\lambda_2 + \mu - 2\nu)P_{R+2}(t_2) + (\mu - \nu)P_{R+3}(t_2) \tag{4.8}$$

As in the earlier case of faster arrival rate, it is fairly easy to obtain the differential difference equations governing the states for $n = R+3, R+4 \cdots, S-1, S$ in slower arrival rate of the model which are given as following

$$P_n'(t_2) = (\lambda_1 - \nu)P_{n-1}(t_2) - (\lambda_2 + \mu - 2\nu)P_n(t_2) + (\mu - \nu)P_{n+1}(t_2) \tag{4.9}$$

$$P_S'(t_2) = (\lambda_1 - \nu)P_{S-1}(t_1) + (\lambda_2 - \nu)P_{S-1}(t_2) - (\lambda_2 + \mu - 2\nu)P_S(t_2) + (\mu - \nu)P_{S+1}(t_2) \tag{4.10}$$

Moreover, the differential difference equations governing the states of the system for $n = S+1, S+2, \cdots$. in slower arrival rate are as follows

$$P_n'(t_2) = (\lambda_2 - v)P_{n-1}(t_2)$$
$$-(\lambda_2 + \mu - 2v)P_n(t_2) + (\mu - v)P_{n+1}(t_2) \tag{4.11}$$

5. Determination of Probability Generating Function in Faster Arrival Rate

We define following probability generating function for the busy period of server in faster arrival rate:

$$H(z, t_1) = \sum_{k=0}^{S-1} P_k(t_1)z^k \tag{5.1}$$

and we use symbol $\overline{H}(z, s_1)$ for the Laplace transform of $H(z, t_1)$ in following equation:

$$\overline{H}(z, s_1) = \int_0^\infty e^{-st} H(z, t_1)\,dt; |z| \leq 1 \tag{5.2}$$

Multiplying by z^k for $k = 0, 1, 2, \cdots S-1$ in respective differential difference Equations (4.1) to (4.6) and then summing over k and simplifying we obtain following partial differential equation

$$z\frac{\partial}{\partial t_1}H(z, t_1)$$
$$= \left[(\mu - v) - (\lambda_1 + \mu - 2v)z + (\lambda_1 - v)z^2\right]$$
$$\times H(z, t_1) - \left[(\mu - v) - (\lambda_1 + \mu - 2v)z\right. \tag{5.3}$$
$$\left. + (\lambda_1 - v)z^2\right]P_0(t_1) + (\mu - v)z^{R+1}P_{R+1}(t_2)$$

Taking Laplace transform of both sides of partial differential difference Equation (5.3), it is fairly easy to obtain

$$\overline{H}(z, s_1)$$
$$= \frac{(1-z)-z^2}{(\mu - v) - (\lambda_1 + \mu - 2v + s_1)z + (\lambda_1 - v)z^2} \tag{5.4}$$
$$\times \left[(\mu - v) - (\lambda_1 - v)z\overline{P_0}(s_1)\right.$$
$$\left. + (\mu - v)z^{R+1}\overline{P_{R+1}}(s_2)\right]$$

As we know the fact that the $\overline{H}(z, s_1)$ converges in the region of the unit circle; $|z| \leq 1$ and $\mathrm{Re}(s_1) > 0$ whenever the denominator of RHS of Equation (5.4) has zeros in the unit circle; $|z| \leq 1$.

The two zeros of the denominators are as following:

$$z_i(0) = \frac{(\lambda_1 + \mu - 2v + s_1)}{2(\lambda_1 - v)}$$
$$\pm \frac{\sqrt{(\lambda_1 + \mu - 2v + s_1)^2 - 4(\mu - v)(\lambda_1 - v)}}{2(\lambda_1 - v)} \tag{5.5}$$

From Equation (5.5), we can observe here that

$$|z_1(0)| \neq |z_2(0)|$$

Moreover, we have

$$\sum_{i=1}^2 z_i(0) = \frac{(\lambda_1 + \mu - 2v + s_1)}{(\lambda_1 - v)} \tag{5.6}$$

$$\prod_{i=1}^2 z_i(0) = \frac{(\mu - v)}{(\lambda_1 - v)} \tag{5.7}$$

On making use of Rouche's theorem in Equation (5.4), it is fairly easy to evaluate $\overline{P_0}(s_1)$ as following:

$$\overline{P_0}(s_1) = \frac{\{z_1(0)\}^2 + (\mu - v)\overline{P_{R+1}}(s_2)\{z_1(0)\}^{R+1}}{s_1 z_1(0)} \tag{5.8}$$

In view of Equation (5.8), $\overline{H}(z, s_1)$ from Equation (5.5) yields as following:

$$\overline{H}(z, s_1) = \frac{(\mu - v)z^{R+1}\overline{P_{R+1}}(s_2)}{(\lambda_1 - v)\{z - z_1(0)\}\{z_2(0) - z\}}$$
$$+ \frac{z^2 - (1-z)\left[(\mu - v) - (\lambda_1 - v)z\right]}{(\lambda_1 - v)\{z - z_1(0)\}\{z_2(0) - z\}} \tag{5.9}$$
$$\times \left[\{z_1(0)\}^2 + (\mu - v)\{z_1(0)\}^{R+1}\overline{P_{R+1}}(s_2)\right]$$

Equation (5.9) can be used in view of Gross and Harris [12] to explore the expected busy periods $E(T_1)$ in faster arrival rate as discussed by Maurya [1] and it is expressed as following:

$$E(T_1) = \frac{1-R}{(\mu - \lambda)} + (\mu - v)\left[\frac{R-1}{\mu - \lambda} - 1\right] \tag{5.10}$$

6. Determination of Probability Generating Function in Slower Arrival Rate

In this section, we define following probability generating function for the busy period of slower arrival rate:

$$W(z, t_2) = \sum_{k=R+1}^\infty P_k(t_2)z^k \tag{6.1}$$

and we use symbol $\overline{W}(z, s_2)$ for the Laplace transform of $W(z, t_2)$ in following equation:

$$\overline{W}(z, s_2) = \int_0^\infty e^{-st} W(z, t_2)\,dt; |z| \leq 1 \tag{6.2}$$

Multiplying through differential difference Equations (4.7) to (4.11) by appropriate power of z^k for $k = R+1, R+2, \cdots S-1, S, S+1, \cdots$ and then summing over k and proceeding as in earlier case of faster arrival rate λ_1, it is fairly easy to obtain $\overline{W}(z, s_2)$ as following:

$$\overline{W}(z, s_2) = \frac{z^{R+2}}{(\lambda_2 - v)\{z - z_1(1)\}\{z_2(1) - z\}} - \frac{(1-z)}{(\lambda_2 - v)\{z - z_1(1)\}\{z_2(1) - z\}} \tag{6.3}$$
$$\times \left[\{(\mu - v) - (\lambda_2 - v)z\}\overline{P_{R+1}}(s_2) + (\lambda - v)z^{S-1}\overline{P_{S-1}}(s_1)\right]$$

where $z_i(1)$ is given by following equation:

$$z_i(1) = \frac{(\lambda_2 + \mu - 2\nu + s_2)}{2(\lambda_2 - \nu)}$$
$$\pm \frac{\sqrt{(\lambda_2 + \mu - 2\nu + s_2)^2 - 4(\mu - \nu)(\lambda_2 - \nu)}}{2(\lambda_2 - \nu)} \quad (6.4)$$

It is remarkable that $z_i(1)$ in Equation (6.4) possesses following three properties:

$$|z_1(1)| \neq |z_2(1)| \quad (6.5)$$

$$\sum_{i=1}^{2} z_i(1) = \frac{(\lambda_2 + \mu - 2\nu + s_2)}{(\lambda_2 - \nu)} \quad (6.6)$$

$$\prod_{i=1}^{2} z_i(1) = \frac{(\mu - \nu)}{(\lambda_2 - \nu)} \quad (6.7)$$

Applying Rouche's theorem in Equation (6.3), we may have $\overline{W}(z, s_2)$ as following:

$$\overline{W}(z, s_2) = \frac{z^{R+2}}{(\lambda_2 - \nu)\{z - z_1(1)\}\{z_2(1) - z\}}$$
$$- \frac{(\mu - \nu)(1 - z)z\left[\{z_1(1)\}^{R+2} + 1\right]}{(\lambda_2 - \nu)\{z - z_1(1)\}\{z_2(1) - z\}} \quad (6.8)$$
$$\times \left[(\lambda_1 - \nu)\overline{P_{S-1}}(s_1)\{z_1(1)\}^{S+1}\right]$$

We remark here that using equation (6.8) in the light of Gross and Harris [12], Maurya [1] succeeded to evaluate $E(T_2)$; the expected busy period in slower arrival rate which is expressed as following:

$$E(T_2) = \frac{1}{(\mu - \lambda_2)} \times \left[\frac{(\lambda_2 - \nu)s(\lambda_2 - \nu)^R}{2(\lambda_1 - \nu)(\mu - \nu)^R} + R + 2\right] \quad (6.9)$$

7. Conclusion

The probability generating function is the most important mathematical technique to examine the transient and steady state behavior of queueing models, particularly to explore many significant performance measures in study of wide range of queueing models which has been been evidenced by recent work of Maurya [1,6,7] and references therein and therefore it plays considerably a vital role in analyzing queueing problems. In the present paper, we have successfully investigated the probability generating functions for two different cases of slower and faster arrival rates of an interdependent $M/M/1/:(\infty; GD)$ queueing model with controllable arrival rates taking into account that the two parameters of arrival and service rates follow the bivariate Poisson process. In order to emphasize the application aspect of the investigated result in the present paper, it is much relevant to remark here

that by using the probability generating function approach, recently Maurya [1] considered the same queueing model and explored the expected busy periods of the server in both cases of faster and slower arrival rates. Moreover, we remark here that cost per unit time in both cases of faster and slower arrival rates of the queueing model taken into present consideration can be determined. In addition to this, it is highly expected that the research of the present investigation can be further extended by incorporating the concept of some other versions of control policies, server breakdown and multi-phase services.

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