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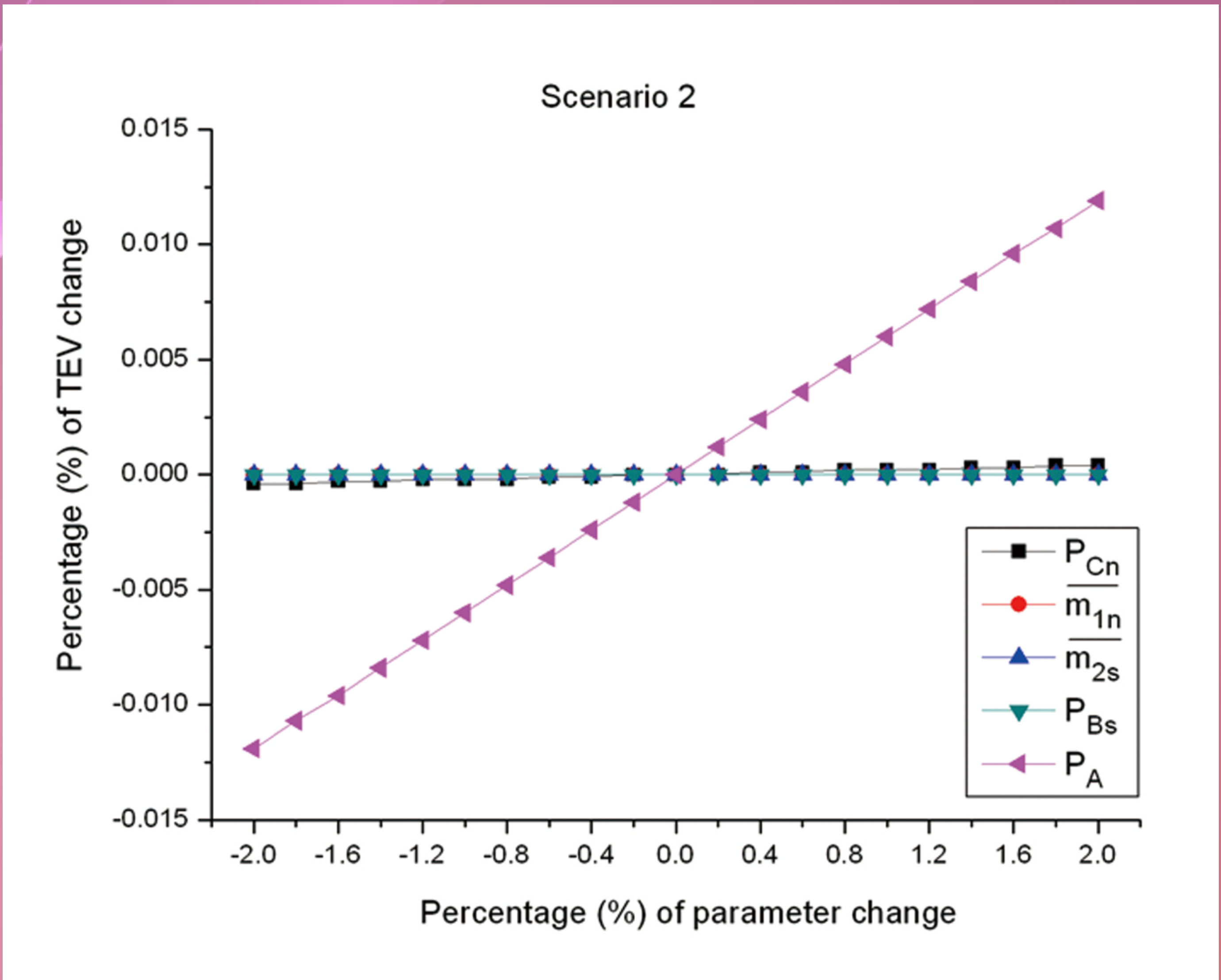


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The figure on the front cover is from the article published in Open Journal of Optimization, 2014, Vol. 3, No. 4, pp. 43-58 by Elena Sarri and George P. Papavassilopoulos.

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Scientific Research

# A Mathematical Model for Deriving Optimal Leasing Policies of a Satellite Operator

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## Abstract

**This paper presents a dynamic mathematical model of optimal leasing allocation of satellite bandwidth and services in terms of expected revenues and associated risk. This tool meets the need of a Satellite Operator to determine the optimal leasing policy of the available bandwidth. A methodology and a tool for techno-economic evaluation of satellite services are developed. The output of the tool enables the policy decisions to be customized by the attitude toward risk that the company wants to apply at each time period. The study is based on inputs concerning data and services from an existing Satellite Operator and addresses a real situation. Demand and pricing data have been gathered from the international market. The decision making tool is given in the set-up of a decision tree presenting quantified alternative leasing policies and risks. Sensitivity analysis is also performed to measure the efficiency of the model.**

## Keywords

**Decision Making Tool, Dynamic Model, Satellite Services, Techno-Economic Evaluation**

## 1. Introduction

The evolutionary penetration of the satellite services and demand in the telecommunication market during the last decades has created the need for decision making tools for a satellite operator [1] [2]. There is still an incremental growth to the telecommunication applications and services [3] [4]. A number of studies have proposed methodologies and tools for the techno-economic evaluation of different aspects of satellite communication dealing with physical, operational, design performance and risk parameters [5]. The need for in-depth analysis of the economic feasibility of new telecommunication solutions that take into account both technological and econo-

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metric characteristics of the applications, has become obvious, both in industrial and academic community. This study deals with the creation and assessment of an optimal business policy of a Satellite Operator. The ability of real time decisions has a significant effect to the viability and profitability of the company in the growing and competitive satellite market. The decision making model presented here is an applied tool for assessment purposes, which deals with real problems and can be used as such, by any Satellite Operator. Limited work has addressed the revenue modeling [6] in terms of market penetration, expected profits and risks, in order to provide a quantitative estimation to the Satellite Operator (after the launching) for the optimal business decision making.

The problem of determining the optimal leasing policy for satellite services, at different time periods is similar to an inventory control problem, with evolution in time, incorporating dynamic and stochastic elements [7] [8], modified accordingly to address the specific application [9]. The purpose of our work is to model and quantify this problem, using heuristic techniques, as well as to evaluate the outcome. The validation of this work results from its usage by the Greek Satellite Operator in the last five years and the development of a new methodology based on stochastic dynamic optimization, presented in [10] with similar results.

We present a model that compares different scenarios of combinations of customers asking to hire satellite capacity. There is a variety of services that a satellite can support. For each of these services requested by a customer, there is a different bandwidth demand, duration of lease, and a different price. The purpose of this work is to find the most profitable case for the incumbent operator. Each scenario leads to a decision tree. This maximization problem is described in terms of real and expected revenues, along with the corresponding probability of getting them. The final outcome leads to optimal enterprise steps that maximize revenues and enable the evaluation of different contingency plans. The decisions are taken successively in a time horizon and thus the presented tool incorporates time evolution. Consequently the model is dynamic.

At the first stage of the work, demand and pricing data have been gathered from the international market. These data provided by the operating Satellite firm i) are statistically processed to produce mean values and standard deviations for each service and each bandwidth demand when needed, and ii) are integrated into a mathematical model, implemented at the second stage, and taking into account all possible states and stages [11]. The model evaluates all probable revenues, along with their associated risks that could result from each alternative decision branch. The concept of the tool gives the benefit of accessing alternative courses of action, to the decision maker. Depending on the company policy at each time period, variant enterprise steps could be followed. Sensitivity analysis is included, in order to determine the impact of small errors in parameters estimates to the final decision. It is performed using both mathematical calculation of the shadow prices and heuristic calculation, providing consistent results.

The paper is organized as follows: Sections 2 and 3 describe the stages of work followed and the main concept of the model. Section 4 formulates the mathematical model and Section 5 presents the sensitivity analysis of the proposed model.

## 2. Stages of Work

The first stage of this study involves the recording and the evaluation of the pricing data coming from the international market of leasing satellite capacity as well as their statistical processing. Some charging schemes have been proposed for broadband networks while this work involves real pricing data [12]. A decision making tool that maximizes the profits of a satellite operator, is implemented in the second stage. The goal of the proposed model is to enable the decision maker, to determine the best possible scenario for the satellite operator, which is the scenario with the larger amount of income.

The hypothesis that a satellite operator has different lease demands from different customers is made. Each customer wants to hire satellite capacity with a specified bandwidth, for a given lease period, which has an associated cost. The model output provides guidelines on the combination of customers that is the most profitable for the operator.

A scenario can be customized by entering the characteristics of the customers. These are:

$S$ : the type of the requested service,

$k$ : the ascending number of customer $\left(k \in [1,10] \subseteq N\right)$,

$w_k$ (MHz): the capacity in MHz that the $k$-th costumer requires $\left(w_k \in [1,36] \subseteq \Re\right)$. If the required capacity is more than 36 MHz, then it can occupy one or more transponders completely. Therefore, for our calculations instead of $w_k$ we compute $a_k$, which is the remaining capacity that does not occupy a whole transponder: $w_k = n \cdot 36 + a_k$,

$t_{\text{start}_k}$ : the starting time (month) for which the $k$-th customer requires the leasing of capacity $w_k$, considering a horizon of five years $\left(t_{\text{start}_k} \in [1,60] \subseteq N\right)$,

$t_{\text{end}_k}$ : the ending time (month) for which the $k$-th customer requires the leasing of capacity $w_k$, $t_{\text{end}_k} \in [1,60] \subseteq N$, and

$C_k$ : the corresponding cost of lease, in Euros, for the $k$-th customer depending on the type of the requested service.

In the structure of the model, the possibility of beginning the hire in different time periods (different months) is included. **Table 1** shows an example of the input parameters of the model.

The indicative cost for each service, is calculated using the data collected from the global satellite market. From the gathered data we created a basic classification of the possible services that a satellite operator could offer. These services correspond to different bandwidth demands and are presented in **Table 2**. The statistical analysis includes the pricing for the equivalent capacity, per 36 MHz, per month and the duration of lease in months. For each service, the mean value, the standard deviation, and the percentage of difference between standard deviation and mean value of these measures are determined along with their cross-correlation. Similar quantities are calculated for the duration of hire.

**Table 1.** Input parameters.

| $K$ | $S$ | $t_{\text{start}_k}$ (month) | $t_{\text{end}_k}$ (month) | $t_k$ (month) | $w_k$ (MHz) | $C_k$ (KEuros) |
|---|---|---|---|---|---|---|
| 1 | A | 1 | 15 | 15 | 12 | 15 |
| 2 | B | 3 | 10 | 8 | 15 | 20 |
| 3 | C | 2 | 20 | 19 | 1 | 1 |
| 4 | D | 6 | 22 | 17 | 5 | 10 |
| 5 | E | 1 | 16 | 16 | 3 | 5 |
| 6 | F | 1 | 18 | 18 | 5 | 5 |
| 7 | G | 3 | 10 | 8 | 18 | 30 |
| 8 | H | 11 | 15 | 5 | 19 | 20 |
| 9 | I | 4 | 30 | 27 | 8 | 9 |
| 10 | J | 3 | 23 | 21 | 4 | 3 |

**Table 2.** Types of services.

| $s$ | Services |
|---|---|
| 1 | VSAT |
| 2 | Telephony |
| 3 | IP Gateway |
| 4 | Corporate |
| 5 | Broadcast |
| 6 | Video Contribution |
| 7 | Media company |
| 8 | Government |

## 3. Concept of the Model

The underlying concept of the problem of optimal allocation of the available satellite spectrum is based on a heuristic approach of the inventory control problem [7]. In the context of the proposed work, the model enables the comparison of 10 at most possible incoming customers with maximum required lease duration of 60 months, corresponding to a typical level of satellite demand. However, the model can easily be expanded to compare more customers requiring satellite capacity, for more than 5 years.

All the possible combinations of customers are calculated (see **Figure 1** Group A), where a description of these combinations is given in terms of feasibility to implement or not. Specifically, there is an indication of Possible, Not Possible or Negotiable Combination for each combination of customers.

A Negotiable Combination is the combination of customers that exceeds by 1 MHz at most the highest possible capacity that a transponder can serve *i.e.* 36 MHz, which can probably constitute an issue of negotiation between the provider and the consumer. A Possible Combination is the feasible combination of customers from the point of view of the maximum capacity of the transponder and a Not Possible Combination is a not feasible one. The proposed tool gives the benefit of sorting by the Description of combination in order to present all the Possible Combinations (see **Figure 1** Group B). The next step is to extract the Possible Combinations that are best to compare, by examining the most promising ones. This is done by selecting the combinations with the highest amount of total occupied capacity, which is the sum of the requested capacity by each customer of the combination (see **Figure 1** Group C). Obviously, the more bandwidth is occupied from a transponder, the more profit there will be for the firm. These are the "Real Revenues". This confines the search for the best case scenario to a small number of possible combinations.

The next step is to decide which of these combinations, with high occupied capacity, are more profitable. The criteria that are used to lead to the optimum combination are:

a) the amount of Real Revenues, representing the revenues that an operator will gain from hiring the capacity to the customers of each combination,

b) the calculation of Additional Expected Future Profits for the satellite operator, taking into consideration the standard deviation of the prices and consequently the corresponding risk.

Each combination has different time of maximum requested capacity hire. Therefore, in order to properly compare different scenarios, it is necessary to reduce them to the same time period *i.e.* to the same month of maximum hiring. We include to our calculations the additional possible income that can be acquired by this left over-free capacity that is called the "Remaining Capacity" (C Remaining). It is also possible that at specific months, not all the available capacity of the transponder of the satellite will be occupied with each combination. This leads to the undesirable effect of not having maximum occupancy of the transponder of the satellite, at each month. So the satellite operator could probably hire out this available capacity, to other possible future customers that are not included in the combination, and gain more revenues. This is called the "Empty Capacity" (C Empty). These moreover profits, consist the Additional Expected Future Profits.

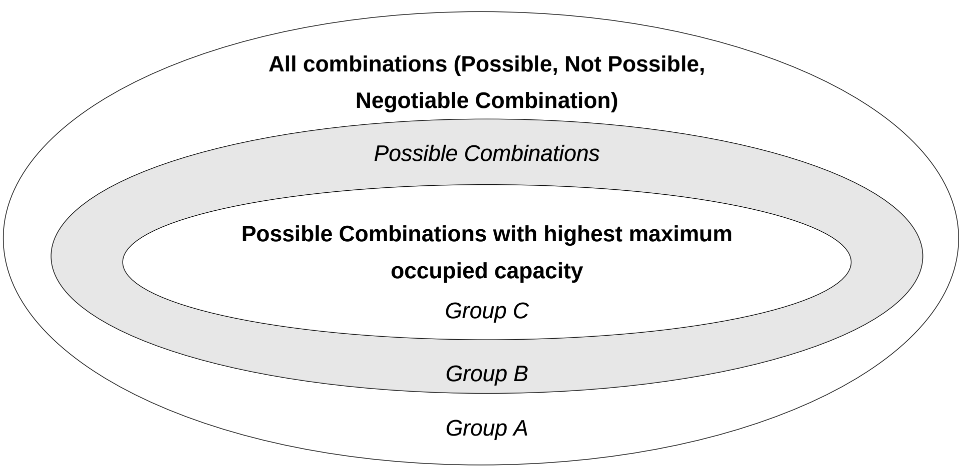


**Figure 1.** Possible combinations of customers.

Let us examine the following example involving two cases that a satellite operator may need to compare and decide which one leads to maximum revenues. These are Scenario 1, which includes the combination of the customer number 1, 2, 3, 4 and 5 (**Figure 2**) and Scenario 2, which includes the combination of the customer number 6, 7, 8, 9 and 10 (**Figure 3**), with the characteristics shown in **Table 1**.

The sum of the occupied capacity of the transponder by the customers of the combinations in Scenario 1 and Scenario 2 are shown in **Figure 4** and **Figure 5**, respectively. These figures represent the profile of the satellite transponder occupancy.

In Scenario 1, the maximum demand on the transponder's capacity occurs during the 22nd month, while in Scenario 2, it occurs during the 30th month. The additional possible income that can be acquired from this left over free capacity is calculated. This is the "Remaining Capacity" (C Remaining) and appears at the white region in **Figure 4**.

Each Scenario does not lead to maximum occupancy of the transponder of the satellite at each month. This "Empty Capacity" (C Empty) is shown as the lined region of **Figure 4** and **Figure 5**. For instance, in Scenario 1 (**Figure 4**) there are 21 MHz of unoccupied capacity during the 1st month, 20 MHz during the 2nd month, 5 MHz from the 3rd to the 5th month, 0 MHz between the 5th and the 10th month, etc. These expected profits are included to our calculations.

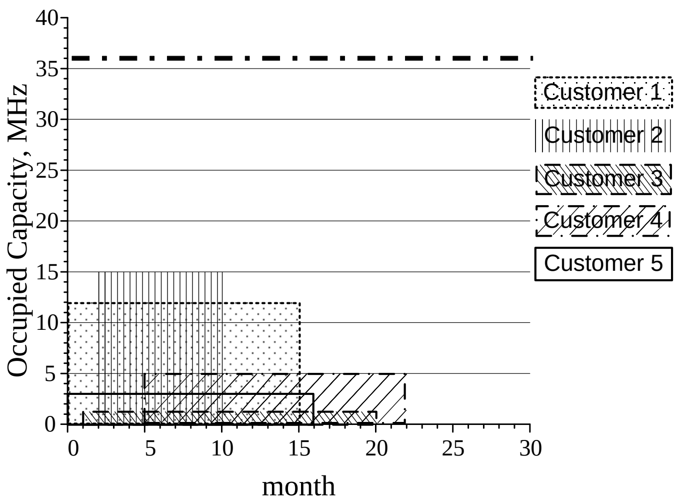


**Figure 2.** Scenario 1.

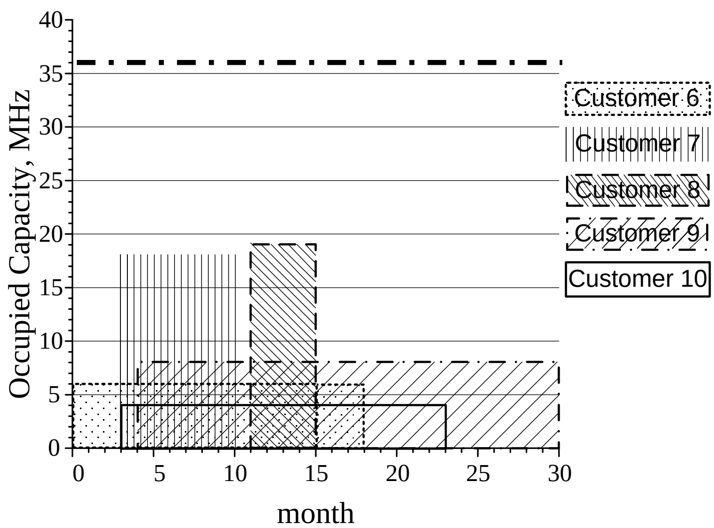


**Figure 3.** Scenario 2.

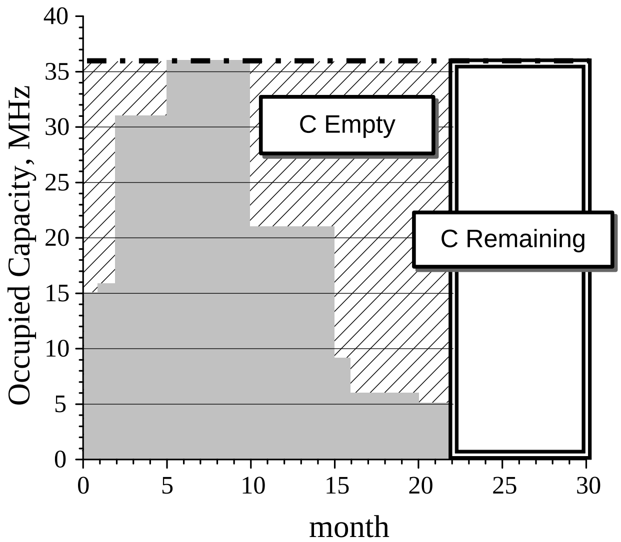


**Figure 4.** Scenario 3.

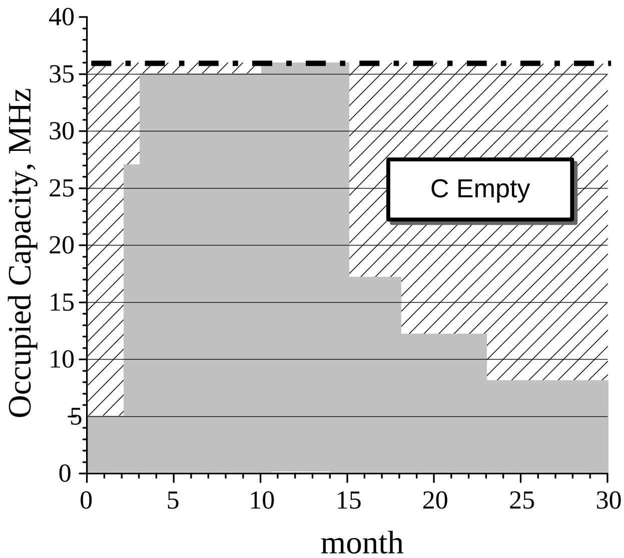


**Figure 5.** Scenario 4.

## 4. Structure of the Tool

**Figure 6** shows the general structure of the model. This was implemented in Microsoft. NET environment in order to meet the need for increased computational burden. All feasible combinations of $n$ objects per $n$, are $\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}=2^n$, with $\binom{n}{r}=\frac{n!}{r!(n-r)!}$. All feasible combinations of 10 customers per 10, starting from 2 customers per 2 and more, are $2^n-\frac{n!}{0!(n-0)!}-\frac{n!}{1!(n-1)!}=1.048.555$. It is meaningless to include calculations with only one customer, since this will not result to maximization of the profits for the operator.

The tool starts with the input as shown in **Table 1**, which is the demand of the customers that arrives to the satellite operator. Then all the combinations of customers are being calculated taking into account the evolution in time. The possible revenues for each scenario resulting from the leasing of the Remaining and the Empty Capacity are computed, using demand and pricing Parameters.

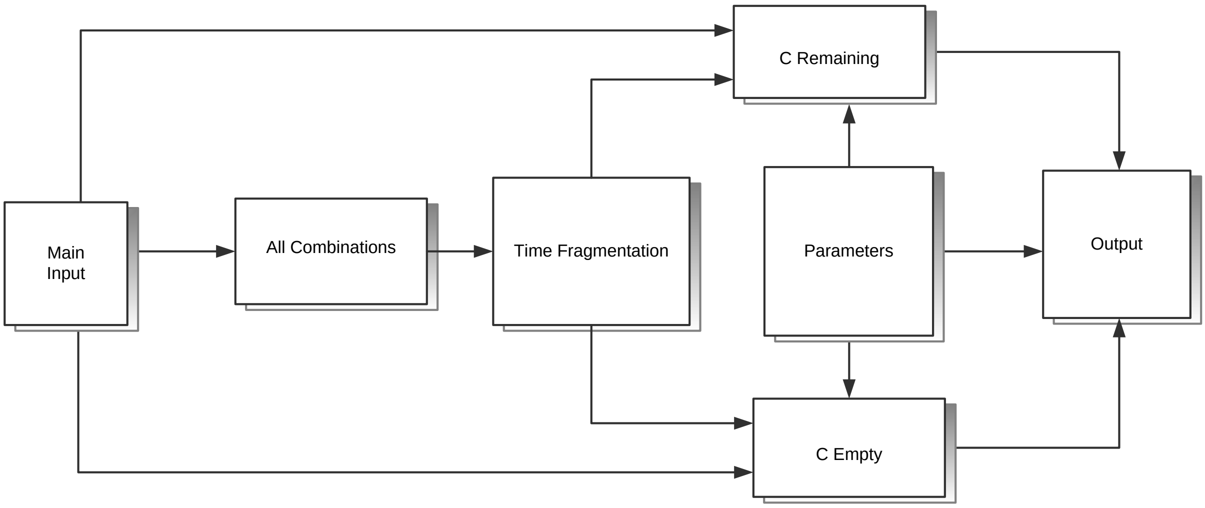


**Figure 6.** Structure of the tool.

## 4.1. Calculation of All Combinations of *k* Customers per *k*

All the possible combinations of the customers (PCf), $f \in [1, 1.048.555] \subseteq N$ are:

$$(\mathrm{PCf}): \begin{cases} w_\alpha \,\&\, w_\beta \,\&\, w_\gamma \,\&\, w_\delta \,\&\, w_\varepsilon \,\&\, w_\zeta \,\&\, w_\eta \,\&\, w_\theta \,\&\, w_\iota \,\&\, w_\kappa, & \alpha > \beta > \gamma > \delta > \varepsilon > \zeta > \eta > \theta > \iota > \kappa \\ 0, & \text{else} \end{cases}$$

$$\{\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \iota, \kappa\} \in [1, 10] \subseteq N \quad \text{and} \quad w_\beta \neq 0 \tag{1}$$

where $w_\alpha$, $w_\beta$, $w_\gamma$, $w_\delta$, $w_\varepsilon$, $w_\zeta$, $w_\eta$, $w_\theta$, $w_\iota$, $w_\kappa$ are auxiliary parameters for the requested capacity of each customer and each combination. We introduce the requirement for non zero value of $w_\beta$ in order to avoid the calculations of combinations with a single customer.

For our calculations we specify the following intermediate parameters:

$d_{ij}$ : the numbered month for which the *i*-th customer requires the lease of capacity wk, at the *j*-th month. This parameter divides into pieces, with length of one month, and numbers the time interval $t_k$

$$\left(d_{ij} \in [1, 60] \subseteq N, i \in [1, k] \subseteq N\right).$$

$w_{ij}$ : the capacity that the *i*-th costumer requires, at the *j*-th month

$$w_{ij} = \begin{cases} w_k, & j \in \left[t_{\mathrm{start}_k}, t_{\mathrm{end}_k}\right] \\ 0, & \text{else} \end{cases} \tag{2}$$

We assign $w_{j_{(\mathrm{PCf})}}$, as the total capacity for the (PCf) combination at the *j*-th month.

## 4.2. Calculation of Revenues from C Empty

For each scenario the possible revenues from the future leasing of the "Empty Capacity" up to the 36 MHz, are calculated.

The Empty Capacity $\left(\mathrm{C\ Empty}_{(\mathrm{PCf}),j}\right)$ for each combination of customers (PCf), at the *j*-th month is:

$$\mathrm{C\ Empty}_{(\mathrm{PCf}),j} = \begin{cases} 36 - \sum\limits_{\substack{i=a \\ j=t_{\mathrm{start}_i}}}^{\substack{t_{\mathrm{end}_i} \\ \kappa}} w_{ij(\mathrm{PCf})}, & t < \max\left\{t_{\mathrm{end}_{k(\mathrm{PCf})}}\right\} \\ 0, & t > \max\left\{t_{\mathrm{end}_{k(\mathrm{PCf})}}\right\} \end{cases}, \quad \begin{array}{c} i \in \{\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \iota, \kappa\}, \\ j \in \left[t_{\mathrm{start}_i}, t_{\mathrm{end}_i}\right]. \end{array} \tag{3}$$

where $w_{ij_{(\mathrm{PCf})}}$ is the capacity of the $i$-th customer of the $f$-th combination (PCf) at the $j$-th month, and $\max\left\{t_{\mathrm{end}_{k_{(\mathrm{PCf})}}}\right\}$ is the maximum ending month of each combination (PCf) which coincides with the $\max\left\{d_{i,j_{(\mathrm{PCf})}}\right\}$.

This calculation is categorized depending on the amount of bandwidth that is not used each month by the customers of each combination. This "Empty Capacity" could potentially be hired out and generate revenues. The selected ranges of capacity in MHz are shown in **Table 3**, resulting from the gathered data.

We are considering $C_n$ $\left(n \in \{0,8\} \subseteq N\right)$ as the central value of capacity for each range of the unused bandwidth, corresponding to a certain probability $P_{C_n}$, a mean value $\overline{m1_n}$ and a standard deviation $\sigma 1_n$. Each range, has a bandwidth of $\left[C_{n-1} + \frac{C_n - C_{n-1}}{2}, C_n + \frac{C_{n+1} - C_n}{2}\right]$, in MHz. The initial value for $n = 0$ corresponds to a capacity $C_0 = 0$ MHz and the ending value $n = 8$ corresponds to the highest possible capacity of the transponder, $C_8 = 36$ MHz.

We use the notation:

$\overline{m1_n}$ : the mean value price, per MHz, per month, asking for the $n$-th range of bandwidth,

$\sigma 1_n$ : the standard deviation of the value price, per MHz, per month, asking for the $n$-th range of bandwidth,

$P_{C_n}$ : the probability of appearance of a customer asking for the $n$-th range of bandwidth, and

$P_A$ : the probability of appearance of a customer asking for satellite services.

The selected capacity ranges along with the corresponding probability of appearance of a new incoming customer, have been statistically computed by the available gathered data (**Figure 7**), and constitute the Parameter data set (**Figure 6**).

A decision tree arises for each scenario, as shown in **Figure 8**.

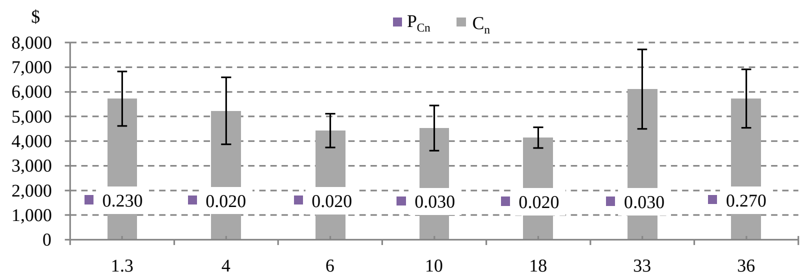


**Figure 7.** Pricing schemes (mean value price, standard deviation per MHz, per month, probability of appearance of a customer, in one month) for each range of Empty Capacity.

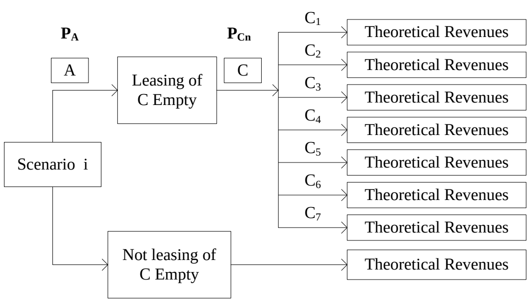


**Figure 8.** Revenues from C empty.

**Table 3.** Ranges of empty capacity.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $C_n$ (MHz) | 1.3 | 4 | 6 | 10 | 18 | 33 | 36 |

The Expected Values of theoretical income, resulting from the "Empty Capacity" for each combination, depending on the not-leased bandwidth at each time period, are calculated as:

$$\text{Exp Val C Empty}_{(\text{PCf})} = P_A \cdot \sum_{j=1}^{60} P_{C_n} \cdot \text{C Empty}_{(\text{PCf}),j} \cdot \overline{m1_n} \tag{4}$$

The standard deviation of the Expected Value, for each combination is:

$$\sigma_{\text{Exp Val C Empty}_{(\text{PCf})}} = P_A \cdot \sum_{j=1}^{60} P_{C_n} \cdot \overline{\sigma 1_n} \cdot \text{C Empty}_{(\text{PCf}),j} \tag{5}$$

The probable revenues for each combination, which are called Theoretical Revenues, depending on the number of MHz that are not used, are calculated as:

$$\text{The Rev C Empty}_{(\text{PCf})} = \sum_{j=1}^{60} \text{C Empty}_{(\text{PCf}),j} \cdot \overline{m1_n} \tag{6}$$

While the standard deviation of the Theoretical Revenues for each combination is:

$$\sigma_{\text{The Rev C Empty}_{(\text{PCf})}} = \sum_{j=1}^{60} \overline{\sigma 1_n} \cdot \text{C Empty}_{(\text{PCf}),j} \tag{7}$$

The Expected Values of Revenues are calculated as intermediate volumes, which are used only for comparison purposes among scenarios. Such revenues incorporate the corresponding risk, and measure the monetary value of each scenario. The values of Theoretical Revenues are the real amounts of money (in Euros) that can be acquired, following each branch of the decision tree.

### 4.3. Calculation of Revenues from C Remaining

We proceed analogously calculating all possible revenues, (Expected Values and Theoretical Revenues) that can result from the leasing of the "Remaining Capacity". This calculation is categorized depending on the type of service (s), using the statistically processed data (**Figure 9**), forming a decision tree (**Figure 10** up).

First, we calculate the remaining time $t_{\text{left}_{(\text{PCf})}}$, which is the time interval for each combination, between the maximum requested month of lease by the customers of the (PCf) combination, until the max requested time of all compared scenarios.

$$t_{\text{left}_{(\text{PCf})}} = \max\left\{t_{\text{end}_k}\right\} - \max\left\{d_{i,j_{(\text{PCf})}}\right\} \tag{8}$$

The following variables are introduced:

$P_{Bs}$: the probability of appearance of a customer, in one month, asking for service s.

$\overline{m2_s}$ : the mean value price per 36 MHz per month, asking for service s.

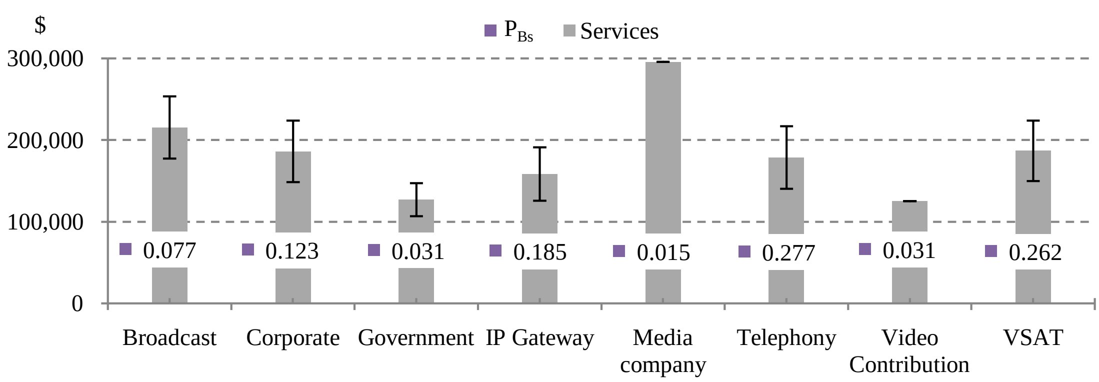


**Figure 9.** Pricing schemes (mean value price, standard deviation per 36 MHz, per month, probability of appearance of a customer, in one month) for each service.

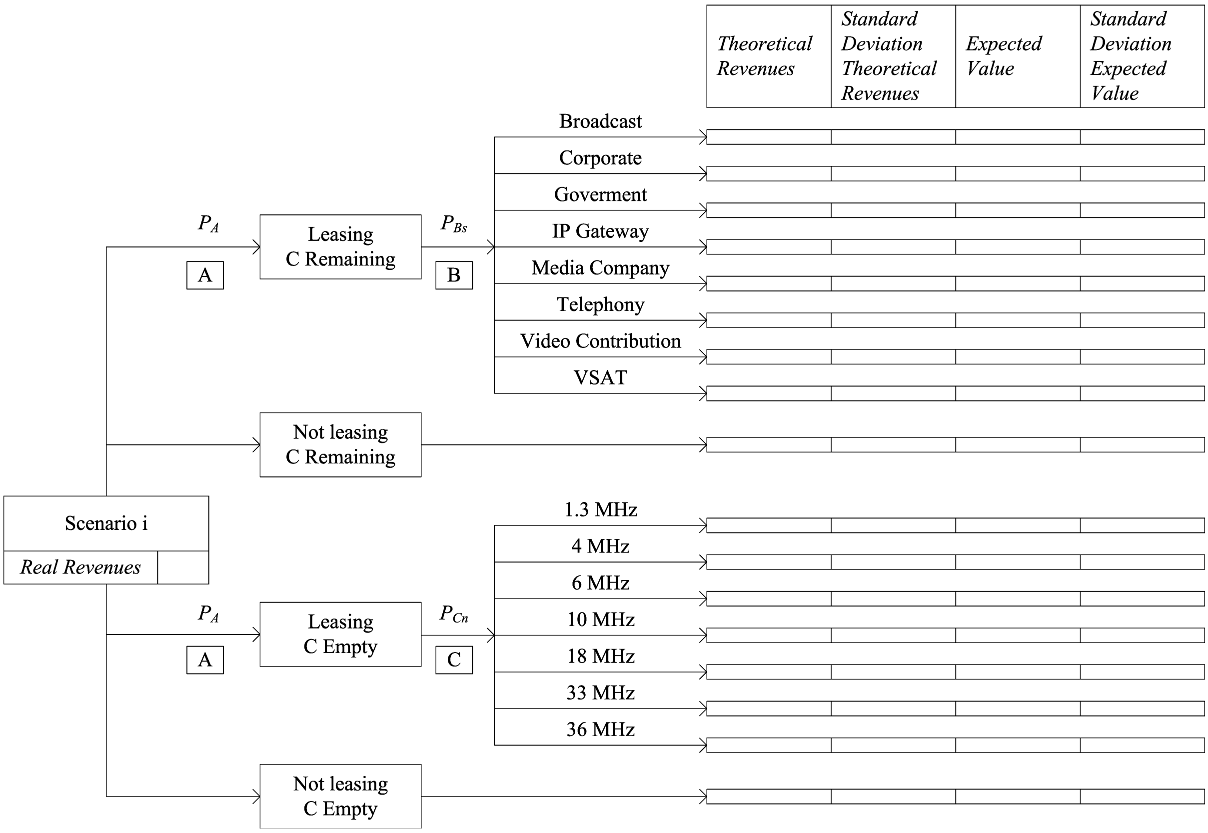


**Figure 10.** Output of the model-decision tree.

$\sigma 2_s$ : the standard deviation of the value price, per 36 MHz, per month, asking for service s.

These variables also constitute the Parameter data set and have been statistically calculated by the available gathered data.

The Expected Values of theoretical income for the Remaining Capacity depending on the type of service are:

$$\text{Exp Val C Rem}_{(\text{PCf}),s} = P_A \cdot P_{Bs} \cdot t_{\text{left}(\text{PCf})} \cdot \overline{m2_s} \tag{9}$$

The standard deviation of the Expected Value for each combination is:

$$\sigma_{\text{Exp Val C Rem}_{(\text{PCf}),s}} = P_A \cdot P_{Bs} \cdot \sigma 2_s \cdot t_{\text{left}(\text{PCf})} \tag{10}$$

While the Theoretical Revenues for the Remaining Capacity are:

$$\text{The Rev C Rem}_{(\text{PCf}),s} = t_{\text{left}(\text{PCf})} \cdot \overline{m2_s} \tag{11}$$

and the standard deviation of the Theoretical Revenues for each combination is:

$$\sigma_{\text{The Rev C Rem}_{(\text{PCf}),s}} = \sigma 2_s \cdot t_{\text{left}(\text{PCf})} \tag{12}$$

All the calculations concerning demand as well as the mean values of the prices of lease and standard deviations of prices are based on statistical computation of the real data coming from the international market and constitute the Parameter data set.

## 4.4. Output Function Formulation

All of these evaluated data are presented in the form of a unified decision tree (**Figure 10**), which is the output of the model.

The amount of Total Expected Revenues that will estimate the optimal policy for the firm is the sum of all revenues. This sum consists of the Real Revenues, plus the Expected Value resulting from the leasing of the Empty Capacity, plus the Expected Value resulting from the leasing of the Remaining Capacity, including their standard deviations, which are the Additional Expected Future Profits.

The Real Revenues are:

$$\text{Real Rev}_{(\text{PCf})} = \sum_{k=a}^{\zeta} \left(C_k\right)_{(\text{PCf})}, \quad k \in \{\alpha, \beta, \gamma, \delta, \varepsilon, \zeta\} \tag{13}$$

where $\left(C_k\right)_{(\text{PCf})}$ is the corresponding cost of lease, for the *k*-th customer, of the *f*-th combination, requested by the satellite operator, and belongs to the Input Parameters. An indicative cost, for each service, which applies to the international market, has been calculated by the gathered data. These values provide information to the incumbent operator for the current trends of satellite services billing and can be modified accordingly.

The Total Expected Revenues give a range of values, defining the best and worst case scenario for the revenues of the satellite operator.

$$\text{Total Expected Revenues} = \text{Tot Exp Val}_{(\text{PCf})} \pm \sigma_{\text{Tot Exp Val}_{(\text{PCf})}} \tag{14}$$

where:

$$\begin{aligned} \text{Tot Exp Val}_{(\text{PCf})} &= \text{Real Rev}_{(\text{PCf})} + \text{Exp Val C Rem}_{(\text{PCf}),s} + \text{Exp Val C Empty}_{(\text{PCf})} \\ &= \sum_{k=a}^{\zeta} \left(C_k\right)_{(\text{PCf})} + P_A \cdot \sum_{j=1}^{60} P_{C_n} \cdot \text{C Empty}_{(\text{PCf}),j} \cdot \overline{m1_n} + P_A \cdot P_{Bs} \cdot t_{\text{left}_{(\text{PCf})}} \cdot \overline{m2_s}. \end{aligned} \tag{15}$$

and

$$\sigma_{\text{Tot Exp Val}_{(\text{PCf})}} = \sigma_{\text{Exp Val C Empty}_{(\text{PCf})}} + \sigma_{\text{Exp Val CRem}_{(\text{PCf}),s}} \tag{16}$$

The real amount of money, resulting from each brand of the decision tree that will possibly result to the satellite operator is:

$$\text{Total Revenues} = \text{Tot The Rev}_{(\text{PCf})} \pm \sigma_{\text{Tot The Rev}_{(\text{PCf})}} \tag{17}$$

where:

$$\begin{aligned} \text{Tot The Rev}_{(\text{PCf})} &= \text{Real Rev}_{(\text{PCf})} + \text{The Rev C Rem}_{(\text{PCf}),s} + \text{The Rev C Empty}_{(\text{PCf})} \\ &= \sum_{k=a}^{\zeta} \left(C_k\right)_{(\text{PCf})} + \sum_{j=1}^{60} \text{C Empty}_{(\text{PCf}),j} \cdot \overline{m1_n} + t_{\text{left}_{(\text{PCf})}} \cdot \overline{m2_s}. \end{aligned} \tag{18}$$

and

$$\sigma_{\text{Tot The Rev}_{(\text{PCf})}} = \sigma_{\text{The Rev C Empty}_{(\text{PCf})}} + \sigma_{\text{The Rev C Rem}_{(\text{PCf}),s}} \tag{19}$$

This gives the opportunity to the satellite provider to determine which combination is the most profitable, for the present and in the future. The decision is made using the amount of Total Expected Revenues. This is not the real revenue that can be acquired, but it is an intermediate amount, which takes into account the corresponding probabilities, and is used for comparison purposes of the scenarios. The Total Expected Revenues is a decision quantity that incorporates profits and associated risk. This amount gives an estimate of the extent/worthness of the risk.

The decision making process starts from the identification of the highest Real Revenues, then the possible additional revenues are considered with their corresponding standard deviation that measures risk. This result to a range of Total Expected Revenues with central value: $\text{Tot Exp Val}_{(\text{PCf})}$, upper limit: $\text{Tot Exp Val}_{(\text{PCf})} + \sigma_{\text{Tot Exp Val}_{(\text{PCf})}}$ and lower limit: $\text{Tot Exp Val}_{(\text{PCf})} - \sigma_{\text{Tot Exp Val}_{(\text{PCf})}}$.

The final decision depends on the extent of risk that the firm is willing to take and on the particular policy that wants to apply. For a risk-loving decision maker the policy generating the highest Total Expected Revenues (upper limit) is chosen. A risk-neutral decision maker will take the policy with the central value, whereas for a risk-averse decision maker the policy generating the lowest of Total Expected Revenues (lower limit) will be chosen. The real amount of income that will result from each decision is the amount of Total Revenues of the corresponding scenario.

## 5. Sensitivity Analysis

Sensitivity analysis is necessary in validating the efficiency of a model. Due to the stochastic nature of the input parameters, we calculate the variance to the output of the model, caused by small variation of the input. The statistical processing of the pricing data gathered from the international market provided us with pricing and probability parameters. Since these variables may not be very accurate, we study the effect caused by small errors on their values.

These parameters include three categories:

- pricing and demand (probability) data concerning the Remaining Capacity $\left(P_{Bs},\overline{m2_s},\sigma 2_s\right)$ categorized on each type of service,
- pricing and demand (probability) data concerning the Empty Capacity $\left(P_{C_n},\overline{m1_n},\sigma 1_n\right)$ categorized on each selected range of capacity and
- the Probability of a new incoming customer asking the Satellite provider for satellite services $(P_A)$.

The sensitivity analysis for these input parameters was performed by creating a small perturbation for each one of them. The value of each parameter was varied by ±2% and the corresponding change in the output, which is the Total Expected Revenues (TER), was measured. The analysis was performed in two different ways: first by theoretical calculation of the shadow prices of the input parameters and second using a heuristic technique, implemented by immediate application of the input-change to the model and observation of the output.

### 5.1. Mathematical Calculation

The shadow prices of the price parameters $\overline{m1_n}$, $\overline{m2_s}$ where computed as the derivative of the output function TER, in the traditional sense of calculus: $S_k^{T(k)}=\dfrac{\mathrm{d}T(k)}{\mathrm{d}k}$, where $T(k)$ is the objective function and $k$ is the parameter that we would like to examine.

The Sensitivity of the Total Expected Revenues with respect to the parameter $\overline{m2_s}$ is:

$$S_{\overline{m2_s}}^{\text{TER}}=\frac{\mathrm{dTER}\left(\overline{m2_s}\right)}{\mathrm{d}\overline{m2_s}}=P_{Bs}\cdot t_{\text{left}_{(\text{PCf})}}\Rightarrow \mathrm{dTER}\left(\overline{m2_s}\right)=P_{Bs}\cdot t_{\text{left}_{(\text{PCf})}}\cdot \mathrm{d}\overline{m2_s} \tag{20}$$

The Sensitivity of the Total Expected Revenues with respect to the parameter $\overline{\text{m}1_n}$ is:

$$S_{\overline{m1_s}}^{\text{TER}}=\frac{\mathrm{dTER}\left(\overline{m1_s}\right)}{\mathrm{d}\overline{m1_s}}=P_A\cdot P_{C_n}\cdot\sum_{j=1}^{60}\text{C Empty}_{(\text{PCf}),j}\Rightarrow \mathrm{dTER}\left(\overline{m1_s}\right)=P_A\cdot P_{C_n}\cdot\sum_{j=1}^{60}\text{C Empty}_{(\text{PCf}),j}\cdot\mathrm{d}\overline{m1_s} \tag{21}$$

We proceed calculating the Bode Sensitivity function for the demand parameters $P_A$, $P_{C_n}$ and $P_{Bs}$. The Bode Sensitivity of the function $T(k)$ with respect to the parameter $k$ is calculated as: $S_k^{T(k)}=\dfrac{\mathrm{d}\ln T(k)}{\mathrm{d}\ln k}=\dfrac{\mathrm{d}T(k)}{\mathrm{d}k}\cdot\dfrac{k}{T(k)}$. This normalized sensitivity is more appropriate for the estimations concerning probabilities parameter, expressing the variation $\mathrm{d}T(k)$ subject to the standard value of $T$, in relation to the variation $\mathrm{d}k$ subject to the value of the parameter $k$.

The Sensitivity of the Total Expected Revenues with respect to the parameter $P_A$ is:

$$\begin{aligned}S_{P_A}^{\text{TER}}&=\frac{\mathrm{dTER}(P_A)}{\mathrm{d}P_A}=\sum_{j=1}^{60}P_{C_n}\cdot\text{C Empty}_{(\text{PCf}),j}\cdot\overline{m1_n}+P_{Bs}\cdot t_{\text{left}_{(\text{PCf})}}\cdot\overline{m2_s}\\ &\Rightarrow \mathrm{dTER}(P_A)=\sum_{j=1}^{60}P_{C_n}\cdot\text{C Empty}_{(\text{PCf}),j}\cdot\overline{m1_n}+P_{Bs}\cdot t_{\text{left}_{(\text{PCf})}}\cdot\overline{m2_s}\cdot\mathrm{d}P_A.\end{aligned} \tag{22}$$

The Sensitivity of the Total Expected Revenues with respect to the parameter $P_{C_n}$ is:

$$S_{P_{C_n}}^{\mathrm{TER}} = \frac{\mathrm{dTER}\left(P_{C_n}\right)}{\mathrm{d}P_{C_n}} = P_A \cdot \overline{m1_n} \cdot \sum_{j=1}^{60} \mathrm{C\ Empty}_{(\mathrm{PCf}),j} \Rightarrow \mathrm{dTER}\left(P_{C_n}\right) = P_A \cdot \overline{m1_n} \cdot \sum_{j=1}^{60} \mathrm{C\ Empty}_{(\mathrm{PCf}),j} \cdot \mathrm{d}P_{C_n} \quad (23)$$

The Sensitivity of the Total Expected Revenues with respect to the parameter $P_{Bs}$ is:

$$S_{P_{Bs}}^{\mathrm{TER}} = \frac{\mathrm{dTER}\left(P_{Bs}\right)}{\mathrm{d}P_{Bs}} = P_A \cdot t_{\mathrm{left}_{(\mathrm{PCf})}} \cdot \overline{m2_s} \Rightarrow \mathrm{dTER}\left(P_{Bs}\right) = P_A \cdot t_{\mathrm{left}_{(\mathrm{PCf})}} \cdot \overline{m2_s} \cdot \mathrm{d}P_{Bs} \quad (24)$$

All parameters ranges, involved in the calculation of the sensitivity analysis are presented in **Table 4**, resulting by the statistical analysis. Let us note that we consider a time period of 30 months allocated to the Remaining Capacity and 30 months allocated to the Empty Capacity. We calculated the summation of the Empty Capacity for each combination of customers $(\mathrm{PCf})$, over the total time period of 60 months, presume that for the months allocated to the Empty Capacity, only the 1/8 of the total available capacity of the transponder will be unoccupied. This assumption is compatible with the desirable requirement of choosing combinations of customers, with maximum occupation of the transponder, for the maximum time period.

**Table 5**, summarizes the change to the output (minimum-maximum value) of the model caused by 2% change of the price and demand parameters considered. The analysis showed very small sensitivity to the demand parameter (probabilities $P_A$, $P_{C_n}$ and $P_{Bs}$). On the other hand this mathematical calculation determined a wide ranging on the pricing parameters, with higher ranging to the parameter $\overline{m1_n}$. In order to derive limits of bounds of accuracy to the parameters above, we proceed with the heuristic approach.

## 5.2. Heuristic Calculation

The Sensitivity analysis is now performed by immediate application of the variance of each input parameter, by ±2%, to the input of the model, and measuring of the output. A case study of six incoming customers, arriving to the satellite operator has been evaluated.

Two cases were considered. The best case scenario, which is the scenario generating the larger amount of expected revenues. This was identified by choosing for the calculations of the Remaining Capacity those that result from the Media Company Service, since it is the one with the highest pricing. The second case considered was the worst case scenario. This case accordingly resulted from the choice of Broadcast Service for the calculations of the expected revenues of the Remaining Capacity, which is the one with the lowest pricing. For each of these cases the calculations were extended to considering the upper limit and the lower limit of the Total Expected Revenues. A sensitivity analysis was performed for these 4 cases.

Using this small variation for each of the input parameters and for each case, we calculated 20 points of the output of the decision model, corresponding to different percentages of change of the input parameters between the ranges of ±2% of the central value.

For each change of input, the change of the output was calculated. This change has the form of the percentage of difference of the value of the output calculated for each of the 20 points of change to the input, minus the value of the output at the central point (11th point, with zero alteration), normalized to this central value.

**Table 4.** Parameters values ranges.

| Variable | $P_A$ | $P_{C_n}$ | $P_{Bs}$ | $\overline{m1_n}$ (K€) | $\overline{m2_s}$ (K€) |
|---|---|---|---|---|---|
| min | 0.62 | 0.0164 | 0.015 | 4.148 | 125.100 |
| max | | 0.270 | 0.277 | 6.114 | 295.834 |

**Table 5.** Sensitivity analysis results.

| | $\mathrm{dTER}\left(\overline{m2_s}\right)$ | $\mathrm{dTER}\left(\overline{m1_n}\right)$ | $\mathrm{dTER}\left(P_A\right)$ | $\mathrm{dTER}\left(P_{C_n}\right)$ | $\mathrm{dTER}\left(P_{Bs}\right)$ |
|---|---|---|---|---|---|
| min | 0.06% | 0.02% | 0.0002% | 0.0001% | 0.0089% |
| max | 16.63% | 45.28% | 0.09% | 0.02% | 0.16% |

The results of the sensitivity analysis for the best case scenario are shown in **Figures 11-14**. For each of the 20 different input sets we calculated the output on the four most profitable combinations of customers. Scenario1, is the scenario with the higher amount of Total Expected Revenues, Scenario 2 is the one with the second higher amount of Total Expected Revenues and so on.

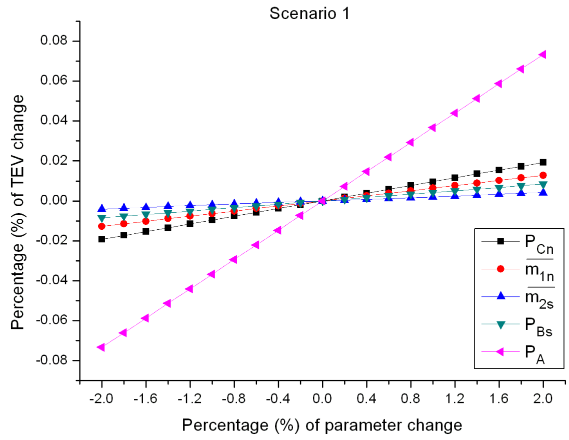


**Figure 11.** Sensitivity analysis for Scenario 1.

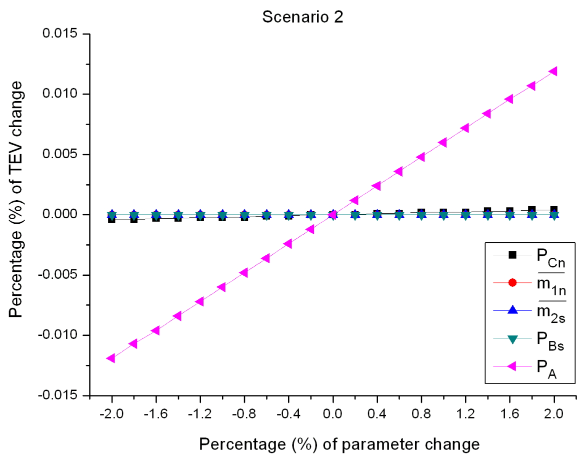


**Figure 12.** Sensitivity analysis for Scenario 2.

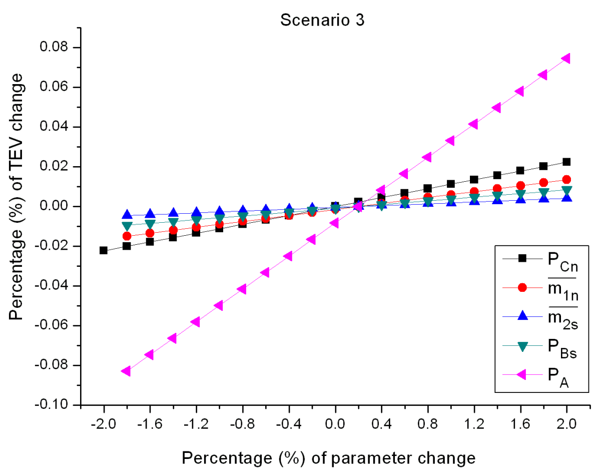


**Figure 13.** Sensitivity analysis for Scenario 3.

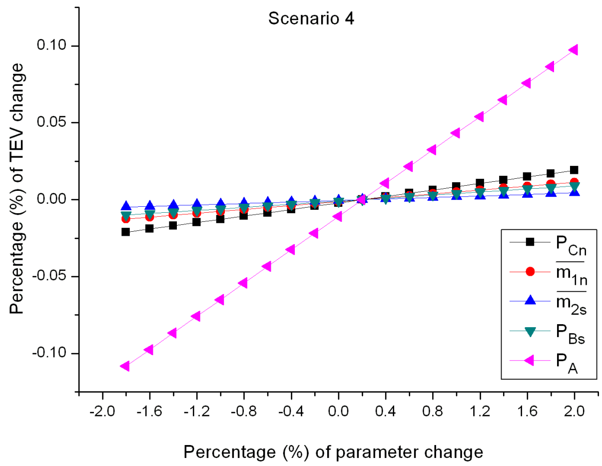


**Figure 14.** Sensitivity analysis for Scenario 4.

In Scenario 2 (**Figure 12**), we observe a zero sensitivity of parameters $P_{Bs}$, $\overline{m2_s}$, $\overline{m1_n}$. This is happening because the amounts of the Remaining Capacity, the $t_{\text{left}_{(\text{PCf})}}$ and the Empty Capacity for this combination of customers are zero.

The calculations were made for all 46 input parameters, categorized as shown in **Table 4** and **Table 5**, extended to all of the 4 most profitable combinations, for both the best and worst case scenario. The most characteristic ones were presented. The results turned out to be similar.

The analysis showed very low sensitivity of the output to changes in the input parameters, even to the pricing parameters. The outcome using Heuristic calculations for the Sensitivity analysis were consistent with the outcome from the mathematical calculations. A small change to the pricing and to the demand parameters will not significantly change the output of the model. This is a very desirable feature, resulting from the good balancing of the proposed model. Even if the parameter data are not very accurate, the decision will not be greatly affected.

## 6. Conclusions

In this paper, we consider the techno-economic valuation of satellite services. The incremental growth of the satellite market nowadays, makes important the study of the economic feasibility of a satellite operator considering technological aspects of the application.

A dynamic mathematical model addressing the decision needs of an operator that provides satellite services is created. This decision making tool considers different demands of customers that arrive to the satellite operator.

Demand and pricing data have been gathered and statistically processed, from the international market. We present a model that compares different scenarios of combinations of customers with different demands, asking to hire satellite capacity. The model evaluates all probable revenues, along with their associated risks that could result from each decision branch. The tool incorporates all the valuable information that will help the satellite operator to determine the most profitable leasing scenario and allows alternative courses of enterprising steps depending on the company policy.

Sensitivity analysis has been included and showed a very small impact of the uncertainty of the input demand and pricing parameters to the final decision. This analysis could also be extended on simultaneous changes of several combinations of input parameters. This work addresses the real need of optimal satellite business planning. Other analysis was mainly referred to the economic evaluation to the physical layer of satellite planning.

The benefits of the model and of the analysis presented here for any satellite operator are clear. The same benefits may apply to related areas of activity where leasing of specific volumes to customers is the essence of the business enterprise.

Finally, we are currently extending our work, using the dynamic programming formulation, in discrete time and with stochastic elements.

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## References


[1] Haase, E., Christensen, C.B. and Ten, C.H. (2012) Global Commercial Space Industry Indicators and Trends. *Acta Astronautica*, **50**, 747-757. http://dx.doi.org/10.1016/S0094-5765(02)00005-X

[2] Marchese, M. and Jamalipour, A. (2005) Key Technologies and Applications of Present and Future Satellite Communications. *IEEE Wireless Communications*, **12**, 8-9. http://dx.doi.org/10.1109/MWC.2005.1522097

[3] NSR (2010) Global Assessment of Satellite Supply & Demand. 8th Edition. http://www.nsr.com/research-reports/

[4] Wood, D. and Weigel, A. (2012) A Framework for Evaluating National Space Activity. *Acta Astronautica*, **73**, 221-236. http://dx.doi.org/10.1016/j.actaastro.2011.11.013

[5] Henri, Y. and Nozdrin, V. (2012) Economic Methods of Improving Efficient Use of the Orbit/Spectrum Resource by Satellite Systems. *Space Policy*, **28**, 185-191. http://dx.doi.org/10.1016/j.spacepol.2012.07.001

[6] Saleh, J.H. and Padilla, J.P.T. (2007) Beyond Cost Models: Communications Satellite Revenue Models. Integrating Cost Considerations into a Value-Centric Mindset. *International Journal of Satellite Communications and Networking*, **25**, 69-92. http://dx.doi.org/10.1002/sat.863

[7] Bertsekas, D.P. (2007) Dynamic Programming and Optimal Control. 3rd Edition, Athena Scientific, Belmont.

[8] Ross, S.M. (1983) Introduction to Stochastic Dynamic Programming. Academic Press, New York.

[9] Maral, G. and Bousquet, M. (1998) Satellite Communications Systems: Systems, Techniques and Technology. 3rd Edition, John Wiley & Sons Ltd., New York.

[10] Sarri, E. (2007) Modelling and Techno-Economic Evaluation of Telecommunication—Satellite Services Using Optimization Techniques. Ph.D. Thesis, National Technical University of Athens, Athens.

[11] Pace, P. and Sun, Z. (2007) Demand Sensitive Model for Tuning Price over Satellite Digital Multimedia Broadcast System. *IEEE Transactions on Broadcasting*, **53**, 329-337. http://dx.doi.org/10.1109/TBC.2006.889684

[12] Courcoubetis, C., Kelly, F.P., Siris, V.A. and Weber, R. (2000) A Study of Simple Usage-Based Charging Schemes for Broadband Networks. *Telecommunications Systems*, **15**, 323-343.

[13] Sat, H. Greek Satellite Operator. http://www.hellas-sat.net

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# Fingerprint Database Optimization Using Watershed Transformation Algorithm

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## Abstract

**Fingerprints are a unique feature for identification and verification of humans. The need to optimise several databases for storing the images of fingerprints is a major concerning issue. Several segmentation algorithms have been used in the time past but there are still several challenges facing some current segmentation algorithms like computational efficiency. Another challenge is that segmentation procedure can be impractically slow, or requires extremely large amounts of memory. This paper addresses the challenges by employing watershed flooding algorithm on the fingerprint images so as to optimize the sizes of the databases. A pre-processing plug-in that implements this segmentation process is developed using Java. We showed its effectiveness by testing it on fingerprint image dataset and the entropy showed that the segmented images sizes were reduced.**

## Keywords

**Watershed, Transformation, Fingerprint, Segmentation, Entropy**

## 1. Introduction

Image Segmentation is a fundamental step in analysing and understanding images. It is a process of partitioning the image into multiple segments [1] [2]. It is the first important step in many image processing applications like image analysis, image description and recognition, image visualization and object based image compression. Image segmentation means assigning a label to each pixel in the image such that pixels with same labels share common visual characteristics.

For more than a century, fingerprints were considered to be the identifying mark for the human beings. Fingerprint is a protected human organ and an effective biometric approach to human or personal identification. It acts like living passwords for humans as its texture is stable throughout the human life. Fingerprints are an im-

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pression left by the friction ridges of human finger.

For several reasons, we need to store these fingerprints in a database and among them one of the main reasons is they are used for analysis of forensic evidence worldwide. For storing several fingerprint impressions a huge database is needed, where the size of the database is also a matter of consideration. A huge database needs a huge amount of memory space. If we can reduce the size of the data then we can store more number of data in the same memory space.

Watershed transformation can be applied to human fingerprints segmentation by taking the idea from friction ridges of human finger and also with an effective storage capacity for the segmented images. Watershed algorithm depends on ridges to perform a proper segmentation, a property that is often fulfilled in contour detection where the boundaries of the objects are expressed as ridges.

In grey scale mathematical morphology the watershed transform, which was originally proposed by [3] and later improved by [4], was the method of choice for image segmentation [5]. When simulating the watershed transform for image segmentation, two approaches may be used: either one first finds basins, then watersheds by taking a set complement; or one computes a complete partition of the image into basins, and subsequently finds the watersheds by boundary detection.

The basic idea behind watershed algorithm by immersion comes from Geography. It requires that one think of an image as a surface; that bright areas are "high" surfaces and dark areas are "low" surfaces. With surfaces, it is natural to think in terms of catchment basins and watershed lines. Basins, also called catchment basins (**Figure 1**), will fill up with water starting at these local minima, and, at points where water coming from different basins would meet, dams are built.

When the water level has reached the highest peak in the landscape, the process is stopped. As a result, the landscape is partitioned into regions or basins separated by dams, called watershed lines or watersheds.

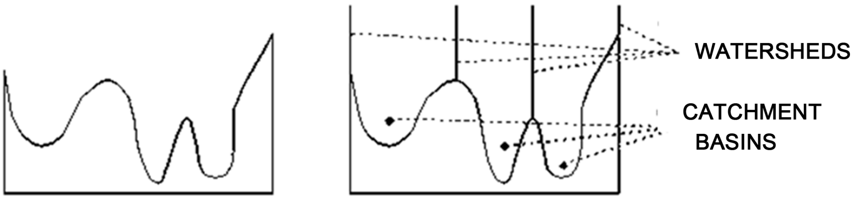
Finally, this paper is organized in sections. Section 2 explained the related work. Watershed algorithmic definitions were discussed in Section 3. While the implementation process in Section 4. Lastly, summary and conclusion is in Section 5.

## 2. Related Work

Although the research by [5] is the most related to our research, comprehensive reviews of early segmentation techniques can be found in [7] and [8]. Some classes of segmentation methods are considered below: Gray level thresholding, and region growing/merging techniques.

Gray level thresholding is a generalization of binary thresholding [9]. Binary thresholding works by determining the gray level value that separates pixels in the foreground from pixels in the background, and generating a "threshold image" where pixels are assigned one of two possible values corresponding to "foreground" and "background" depending on whether their gray level is above or below the selected threshold.

The work of Beveridge *et al*. [10] offers a good example of a procedure that integrates both gray level thresholding and region merging. In their paper, an input image (which can be either grayscale or colour) is divided into sectors of fixed size and fixed location. An intensity histogram is calculated for each sector (and on colour images, for each colour channel), and used to produce a local segmentation. For every sector, information from its neighbors is used to detect clusters for which there may not be enough local support due to the artificially induced partition of the image. After the local segmentations are complete, the sector boundaries are removed by merging together similar regions in neighboring sectors.




(a) (b)

**Figure 1.** One dimensional example of watershed segmentation. (a) Gray level profile of Image data; (b) Watershed segmentation—local minima of gray level (altitude) yield catchment basins; local maxima define the watershed lines (Source: [6]).

The above measure is computed for both the complete regions, and a band that is within a fixed, small distance on both sides of the boundary. Two regions are merged if the merge score is below a specified threshold for both the global and local measure.

The last step in the segmentation is region merging; this step uses a merge score composed of a pairwise comparison of several region features. Since the algorithm can only merge regions, the thresholds used during the local, threshold based segmentation stage are selected so that they'll yield a significantly over-segmented image; the merging step is then relied upon to turn the un-segmented image into a reasonable segmentation. Results presented in [10] show that this algorithm produces good segmentations in parts of the image that are reasonably homogeneous, and over-segmented regions when there is texture, significant intensity gradients, or objects with non-uniform coloring. The algorithm is not without problems, as there are several thresholds that must be chosen carefully depending on the image, and the region boundaries themselves have slight artifacts introduced by the sector-based initial segmentation. Even so, the algorithm illustrates what can be achieved with thresholding merging schemes.

The works in the literature showed that many current algorithms are able to produce reasonable results on images of moderate complexity; several of these algorithms are efficient enough that they can be used as a pre-processing stage for higher level vision tasks such as recognition and tracking.

However, there are still several challenges facing some current segmentation algorithms. Computational efficiency is still a concern issue when the processing of large affinity matrices is a part of the segmentation process, ultimately, a segmentation procedure can become impractically slow, or require extremely large amounts of memory. These have limited the size of the images that can be processed using many recent algorithms. However, we expect that the constant increase in computational power and storage capacity of modern computers should progressively reduce these limitations. The definition of a good similarity measure for general images remains an open issue. There is a general consensus that a robust image segmentation algorithm should combine multiple image cues and estimate similarity based on this combination, but so far there are few algorithms that use more than a single cue as a similarity measure, and only recently has a significant effort been dedicated to designing similarity measures based on the statistics of natural images, and human-generated segmentations.

The design of a good similarity measure is tied to the robustness of the segmentation algorithm in dealing with surface markings, lighting artifacts, and image texture. Evaluating the output of segmentation algorithms is still problematic. Therefore, watershed algorithm is adopted in this paper.

## 3. Watershed Algorithmic Definition

The diagrammatic descriptions of watershed lines and catchment basins have been presented in **Figure 1**. An algorithmic definition of the watershed transform by simulated immersion was given by [2]. Consider a digital grey value image, $f: D \to \mathbb{N}$ with minimum value $h_{\min}$ and maximum value $h_{\max}$ of $f$. We plunge this surface into a lake with a constant vertical speed, with water entering through the holes and flood the surface. Define a recursion with the grey level $h$ increasing from $h_{\min}$ to $h_{\max}$, in which the basins associated with the minima of $f$ are successively expanded. Let $X_h$ denote the union of the set of basins computed at level $h$. A connected component of the threshold set $T_{h+1}$ at level $h+1$ can be either a new minimum, or an extension of a basin in $X_h$: in the latter case one computes the geodesic influence zone of $X_h$ within $T_{h+1}$, resulting in an update $X_{h+1}$.

Let $\mathrm{MIN}_h$ denote the union of all regional minima at altitude $h$.

**Definition (Watershed Transform):** Let $f \in C(D)$ have a minima $\{m_k\}_{k\in I}$ for some index set $I$. The catchment basin $\mathrm{CB}(m_i)$ of a minimum $m_i$ is defines as the set of points $x \in D$ which are topographically closer to $m_i$ than to any other regional minimum $m_j$.

$$\mathrm{CB}(m_i) = \left\{x \in D \middle| \forall j \in I/\{i\} : f(m_i) + T_{f(x,m_i)} < f(m_j) + T_{f(x,m_j)}\right\}$$

The watershed of $f$ is the set of points which do not belong to any catchment basins:

$$W\mathrm{shed}(f) = D \cap \left(\bigcup_{i\in I} \mathrm{CB}(m_i)\right)^c$$

Let $W$ be some label, $W \notin I$. The watershed transform of $f$ is a mapping $\lambda: D \to I \cup \{W\}$, such that

$\lambda(p)=i$, if $p \in \mathrm{CB}(m_i)$, and $\lambda(p)=W$ if $p \in W\mathrm{shed}(f)$.

So the watershed transform of $f$ assigns labels to the points of $D$, such that i) different catchment basins are uniquely labeled, and ii) a special label $W$ is assigned to all points of the watershed of $f$.

We assume, from [11], a geodesic distance $d_A(a,b)$ between $a$ and $b$ within $A$ for $A \in \varepsilon$, with $\varepsilon = \mathbb{R}^d$ where $d_A(a,b)$ is the minimum path length among all paths within $A$ from $a$ to $b$ (in the continuous case, read "infimum" instead of "minimum"). If $B$ is a subset of $A$, define $d_A(a,B) = \mathrm{MIN}_{b\in B}(d_A(a,b))$. Let $B \in A$ be partitioned in $k$ connected components $B_i$, $i = 1,\cdots,k$. The *geodesic influence zone* of the set $B_i$ within $A$ is defined as:

$$iz_A(B_i) = \left\{ p \in A \middle| \forall j \in \{1,\cdots,k\} \setminus \{i\} : d_A(p,B_i) < d_A(p,B_j) \right\}$$

Let $B \in A$, The set $\mathrm{IZ}_A(B)$ is the union of the geodesic influence zones of the connected components of $B$ [12], *i.e.*,

$$\mathrm{IZ}_A(B) = \bigcup_{i=1}^{k} iz_A(B_i)$$

The definition of $\mathrm{IZ}_A(B)$ is used in the construction of a watershed algorithm by immersion.

**Definition (watershed by immersion):**

$$\text{Define the following recursion}: \begin{cases} X_{h+1} = \left\{ p \in D \middle| f(p) = h_{\min} \right\} = T_{h_{\min}} \\ X_{h+1} = \mathrm{MIN}_h \cup \mathrm{IZ}_{T_{h+1}}(X_h) \qquad h \in [h_{\min}, h_{\max}) \end{cases}$$

The watershed $W\mathrm{shed}(f)$ of $f$ is the complement of $X_{h_{\max}}$ in $D$:

$$W\mathrm{shed}(f) = D / X_{h_{\max}}.$$

## 4. The Implementation Process

A pre-processing plug-in that implements this segmentation process is developed using Java PL. This plug-in would be compatible with an application called Image J [12], a general purpose image-processing and image-analysis package. Image J is of choice because it has a public domain licence, it runs on several operating system platforms. This application would apply the watershed flooding algorithm which can be interrupted to a user-specified level. Each particle should have a local maximum (or local minimum when the objects are dark) to define a catchment basin.

The Watershed algorithm by immersion is presented in **Table 1**.

The SKIZ (skeleton by influence zones) function is the complement of the set $\mathrm{IZ}_A(B)$ (defined in Section 3) within $A$, and is defined as:

$$\mathrm{SKIZ}_A(B) = A \setminus \mathrm{IZ}_A(B)$$

So the SKIZ consists of all points which are equidistant (in the sense of the geodesic distance) to at least two nearest connected components. For a binary image $f$ with domain $A$, the SKIZ can be defined by identifying $B$ with the set of foreground pixels [11].

This approach is being implemented on different fingers, where the segmented results are very clear. **Figures 2-6** presents the application of the approach on the little finger, the ring finger, the middle finger, grooming finger and thumb finger respectively. In the figures presented, (a) is the original image, (b) is the grey scale image of the original one, (c) shows the overlaid basins of the grey scale image, (d) shows the catchment basins, (e) shows the composite image and finally one gets the segmented image (f) after applying watershed algorithm.

From **Table 2**, the entropy for original images with their segmented results has been measured.

Entropy is defined as:

$$H(x) = -\sum_x p(x)\log_2\left[p(x)\right]$$

where $p(x)$ is the probability that $X$ is in the state $x$ and $p\log_2 p$ is defined as 0 if $p = 0$. The joint en-

**Table 1.** Watershed algorithm by immersion.

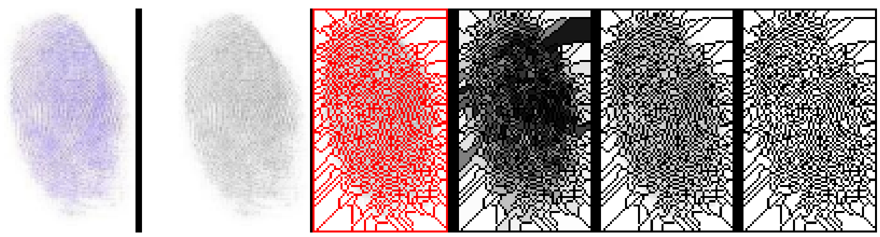
| Vincent and Soille watershed algorithm by immersion [2] |
|---|
| **procedure** Watershed by Immersion |
| INPUT: digital grey scale image G = (D, E, *im*) |
| OUTPUT: labeled watershed image *lab* on *D*. |
| #define INIT −1 (*initial value of lab image*) |
| #define MASK −2 (*initial value at each level*) |
| #define WSHED 0 (*label of the watershed pixels*) |
| #define FICTITIOUS (−1, −1) (*fictitious pixel $\notin D$ *) |
| curlab ← 0 (**curlab* is the current label*) |
| *fifo_init* (*queue*) |
| **for all** $p \in D$ **do** |
| *lab* [p]← INIT; *dist* [p]← 0 (*$dist$ is a work image of distances*) |
| **end for** |
| SORT pixels in increasing order of grey values (minimum *hmin*, maximum *hmax*) |
| (*Start the Flooding*) |
| **for** *h* = *hmin* **to** *hmax* **do** (*Geodesic SKIZ of level *h* −1 inside level h*) |
| **for all** $p \in D$ with *im* [p] = **do** (*mask all pixels at level h*) |
| (*these are directly accessible because of the sorting step*) |
| *lab* [p] ← mask |
| **if** $p$ has a neighbor *q* **with** (*lab* [q] > 0 or *lab* [q] = WSHED) **then** |
| (*Initialize queue with neighbours at level h of current basins or watersheds*) |
| *dist* [p] ← 1; *fifo_add* (p, queue) |
| **end if** |
| **end for** |
| curdist ← 1; *fifo_add* (*FICTITIOUS*, *queue*) |
| **loop** (*extend basins*) |
| *p* ← *fifo_remove* (*queue*) |
| **if** *p* = *FICTITIOUS* **then** |
| **if** *fifo_empty* (*queue*) **then** |
| BREAK |
| **else** |
| *fifo_add* (*FICTITIOUS*, *queue*); *curdist* ← *curdist* + 1; |
| *p* ← *fifo_remove* (*queue*) |
| **end if** |
| **end if** |
| **for all** $q \in N_G(p)$ **do** (*labeling p by inspecting neighbors*) |
| **if** *dist*[*q*] < *curdist* **and** (*lab*[*q*] > 0 or lab[q] = WSHED) **then** |
| (*q belongs to an existing basin or to watersheds *) |
| **if** *lab*[*q*] > 0 **then** |
| **if***lab*[*p*] = MASK **or** *lab*[*p*] = WSHED **then** |
| *lab*[*p*] ← *lab*[*q*] |
| **else if** *lab*[*p*] ≠ *lab*[*q*] **then** |
| *lab*[*p*] ← WSHED |
| **end if** |
| **else if** *lab*[*p*] = MASK **then** |
| *lab*[*p*] WSHED |
| **end if** |
| **else if** *lab*[*q*] = MASK **and** *dsit*[*q*] = 0 then (*q is plateau pixel*) |
| *dsit*[*q*] ← *curdist* + 1; *fifo_add* (*p*, *queue*) |
| **end if** |
| **end for** |

**Continued**

```
end loop
(*detect and process new minima at level h*)
for all p ∈ D with im[p] = h do
dist [p] ← 0 (*reset distance to zero*)
if lab [p] = MASK then (*p is inside a new minimum*)
curlab ← curlab + 1; (*create new label*)
fifo_add (p, queue); lab[p] ← curlab
while not fifo_empty (queue) do
q ← fifo_remove (queue)
for all r ∈ N_G (q) do                (*inspect neighbours of q*)
if lab [r] = MASK then
fifo_add (r, queue); lab [r] ← cur lab
end if
end for
end while
end if
end for
end for
(*End Flooding*)
```

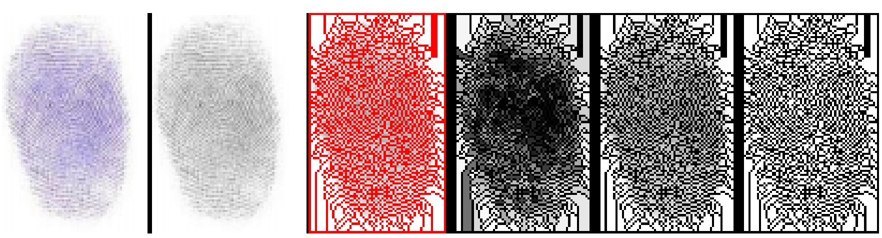
**Table 2.** The entropy for original images with segmented results.

| Name of Finger | Entropy | | |
|---|---|---|---|
| | Entropy of Original Image | Entropy of Segmented Image | Percentage of Reduction |
| Little Finger | 5.491650659963482 | 0.9847813571240879 | 82.07% |
| Ring Finger | 6.147308152748742 | 0.9960550245994975 | 83.79% |
| Middle Finger | 5.924018496000674 | 0.987188916509631 | 83.33% |
| Grooming Finger | 5.739202551639768 | 0.9866855429259958 | 82.81% |
| Thumb Finger | 5.879363918691973 | 0.9948162715104257 | 83.08% |



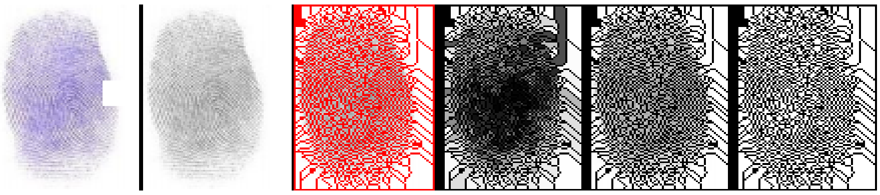
(a) (b) (c) (d) (e) (f)

**Figure 2.** Application of this approach on the little finger.



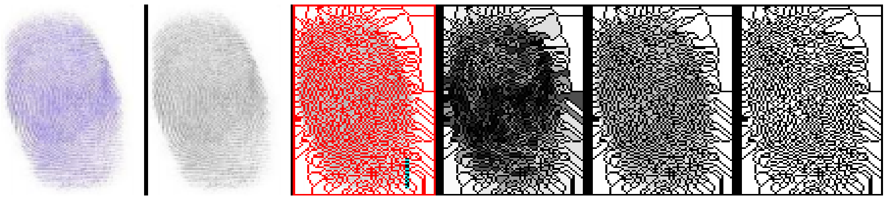
(a) (b) (c) (d) (e) (f)

**Figure 3.** Application of this approach on the ring finger.



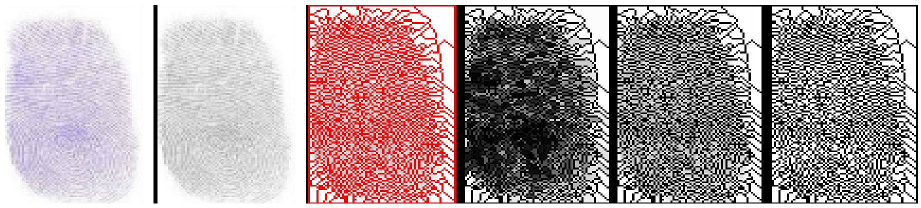
(a) (b) (c) (d) (e) (f)

**Figure 4.** Application of this approach on the middle finger.



(a) (b) (c) (d) (e) (f)

**Figure 5.** Application of this approach on the grooming finger.



(a) (b) (c) (d) (e) (f)

**Figure 6.** Application of the approach on the thumb finger.

tropy of variable $x_1, \cdots, x_n$ is defined by:

$$H\left(x_1, \cdots, x_n\right) = -\sum_{x_1} p \cdots \sum_{x_n} p\left(x_1, \cdots, x_n\right) \log_2\left[p\left(x_1, \cdots, x_n\right)\right]$$

The performance of segmentation algorithm can be measured with the help of entropy and as in term of visual quality of the original image and the resulted image. The image entropy can provide a good level of information to describe a given image.

Low entropy images have very little contrast and large runs of pixels with the same values. An image that is perfectly flat will have entropy of zero. Consequently, they can be compressed to a relatively small size. On the other hand, high entropy images such as an image of heavily cratered areas on the moon have a great deal of contrast from one pixel to the next and consequently cannot be compressed as much as low entropy images.

### Comparison with Related Works

As depicted earlier, the research by [5] is a good threshold to adjudge the efficiency of our algorithm. In the end, the case study images selected is random and entropies also become random. However, the images selected by [5] have almost similar entropies to our images and so the percentage of reduction of their original images to segmented images is presented in **Table 3**.

The difference comparison is shown in **Table 4**.

As shown, we can see that the implementation of watershed algorithm in our scheme offers better computation (and hence, better storage reduction) capacity.

## 5. Summary and Conclusions

This approach has provided an easy method of the segmentation of the human fingerprints based on watershed

**Table 3.** The entropy for original images with segmented results [5].

| Name of Finger | Entropy | | |
|---|---|---|---|
| | Entropy of Original Image | Entropy of Segmented Image | Percentage of Reduction |
| Little Finger | 5.4766 | 1.0281 | [(5.4766 – 1.0281)/5.4766 ∗ 100] = 81.23% |
| Ring Finger | 5.5974 | 0.9537 | [(5.5974 – 0.9537)/5.5974 ∗ 100] = **82.96%** |
| Middle Finger | 5.6865 | 0.9820 | [(5.6865 – 0.9820)/5.6865 ∗ 100] = **82.73%** |
| Grooming Finger | 5.2182 | 0.9620 | [(5.2182 – 0.9620)/5.2182 ∗ 100] = **81.56%** |
| Thumb Finger | 5.4733 | 0.9969 | [(5.4733 – 0.9969)/5.4733 ∗ 100] = **81.78%** |

**Table 4.** Comparing our scheme and [5].

| Entropy | | |
|---|---|---|
| Percentage of Reduction, [5] | Percentage of Reduction, Our Scheme | Difference in Percentages |
| 81.23% | 82.07% | 0.84 |
| 82.96% | 83.79% | 0.83 |
| 82.73% | 83.33% | 0.60 |
| 81.56% | 82.81% | 1.25 |
| 81.78% | 83.08% | 1.30 |

transformation. The concept of watershed algorithm has been used for segmentation purpose. In practical applications, fingerprints are unique feature for identification and verification of humans, and as well as we need to maintain several databases for storing the images of fingerprints and the sizes of the databases are a major concerned issue. We can store the segmented images of fingerprints instead of the original images to reduce the size of the databases. The final result of segmentation depends upon the quality of scanner and the inkpad which we use.

Thus method is therefore recommended in achieving the aim of optimizing the size of image database.

# References


[1] Das, S. Lecture Notes. IIT Madras, India. http://vplab.iitm.ac.in/courses/CV_DIP/PDF/lect-Segmen.pdf

[2] Vincent, L. and Soille, P. (1991) Watersheds in Digital Spaces: An Efficient Algorithm Based on Immersion Simulations. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, **13**, 583-598. http://dx.doi.org/10.1109/34.87344

[3] Digabel, H. and Lantuejoul, C. (1978) Iterative Algorithms. *Proceedings of the* 2nd *European Symposium Quantitative Analysis of Microstructures in Material Science, Biology and Medicine*, 85-89.

[4] Beucher, S. and Meyer, F. (1993) The Morphological Approach to Segmentation: The Watershed Transformation, Mathematical Morphology in Image Processing. Marcel Dekker Inc., New York, 433-481.

[5] Acharjya, P.P. and Ghoshal, D. (2012) An Effective Human Fingerprint Segmentation Method Using Watershed Algorithm. *International Journal of Computer Applications*, **53**. http://dx.doi.org/10.5120/8482-2422

[6] Ulbsibiu, R. (2014) Watershed Segmentation. http://remus.ulbsibiu.ro/teaching/courses/docs/acs/Watershed%20Segmentation.doc

[7] Haralick, R.M. and Shapiro, L.G. (1985) Image Segmentation Techniques. *Computer Vision, Graphics and Image Processing*, **29**, 100-132. http://dx.doi.org/10.1016/s0734-189x(85)90153-7

[8] Pal, N.R. and Pal, S.K. (1993) A Review on Image Segmentation Techniques. *Pattern Recognition*, **26**, 1277-1294. http://dx.doi.org/10.1007/978-3-662-21817-4_10

[9] Berthold, K. and Horn, P. (1986) Robot Vision. MIT Press, Cambridge.

[10] Ross Beveridge, J., Griffith, J., Kohler, R.R., Hanson, A.R. and Rise-Man, E.M. (1989) Segmenting Images Using Localized Histograms and Region Merging. *International Journal of Computer Vision*, **2**, 311-347.

http://dx.doi.org/10.1007/bf00158168

[11] Roerdink, J.B.T.M. and Meijster, A. (2001) The Watershed Transform: Definitions, Algorithms and Parallelization Strategies. *Fundamenta Informaticae*, **41**, 187-228.

[12] http://imagej.nih.gov/ij/

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# Compromise Allocation for Combined Ratio Estimates of Population Means of a Multivariate Stratified Population Using Double Sampling in Presence of Non-Response

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## Abstract

**This paper is an attempt to work out a compromise allocation to construct combined ratio estimates under multivariate double sampling design in presence of non-response when the population mean of the auxiliary variable is unknown. The problem has been formulated as a multi-objective integer non-linear programming problem. Two solution procedures are developed using goal programming and fuzzy programming techniques. A numerical example is also worked out to illustrate the computational details. A comparison of the two methods is also carried out.**

## Keywords

**Multivariate Stratified Sampling, Compromise Allocation, Non-Response, Double Sampling Goal Programming, Fuzzy Programming**

## 1. Introduction

Often in sample surveys the main variable is highly correlated to another variable called an auxiliary variable and the data on auxiliary variable are either available or can be easily obtained. In this situation to obtain the estimate of the parameters regarding the main variable the auxiliary information can be used to enhance the precision of the estimate. Ratio and Regression Methods and double sampling technique are some examples. When data are collected on the sampled units of the main variable due to one or the other reason, data for all the se-

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lected units cannot be obtained. This result is an incomplete and less informative sample. This phenomenon is termed as "non response". [1] is the first one to consider this problem. Furthermore, when auxiliary parameters are unknown, they can be estimated from a preliminary large sample. Then a second sample is obtained in which the main and auxiliary, both the variables are measured. Often a second sample is a subsample of the first. In such cases only the main variable is to be measured in the second sample. This technique is called "Double Sampling" or "Two Phase Sampling", [2]-[11] are some who used the auxiliary information in sample surveys. [10] has worked on the problem in which ratio estimator has been considered for population mean under double sampling in presence of non-response for a univariate population.

In the present paper, we considered combined ratio estimators of the population means of a multivariate stratified population using double sampling in presence of non-response. Compromise allocations at first and second phase of double sampling are obtained by formulating the problems as multi-objective integer non-linear programming problems. Solution procedures are developed by using goal programming and fuzzy programming techniques. A numerical example is also worked out to illustrate the computational details. A comparison of the two methods is also carried out.

When auxiliary information is available, the use of Ratio method of estimation is well known in univariate stratified sampling. Formulae are also available to work out optimum allocations to various strata [12]. In multivariate case finding an allocation that gives optimum results for all the characteristics is not possible due to the conflicting nature of the characteristics. Compromise allocation is used in such situations. Furthermore, if the problem of non-response is also there, the situation becomes more complicated. The paper is structured as below:

In Section 2 of the manuscript combined ratio estimates for the population means of the "*p*" characteristics in presence of non-response using double sampling are constructed. Section 3 formulates the problem of obtaining compromise allocations for phase-I and phase-II of the double sampling as an integer nonlinear programming problem (INLPP). Sections 4 and 5 show that how these INLPP's can be transformed to apply the Goal Programming Technique (GPT) and the Fuzzy Programming Technique (FPT) to solve the transformed problems. Section 6 provides an application of the techniques through a numerical data. In the last Section 8 gives the conclusion and the future work trend for interested readers.

## 2. The Combined Ratio Estimate in Multivariate Stratified Double Sampling Design in Presence of Non-Response

Consider a multivariate stratified population of size $N$ with $L$ non-overlapping strata of sizes $N_1, N_2, \cdots, N_L$ with $\sum_{h=1}^{L} N_h = N$. Let $p$ characteristics be defined on each unit of the population. If $N_1, N_2, \cdots, N_L$ are not known in advance then the strata weights $W_h = N_h / N, h = 1, 2, \cdots, L$ also remain unknown. In such situation double sampling technique may be used to estimate the unknown strata weights. For this a large preliminary simple random sample of size $n'$ is obtained at the first phase of the double sampling, treating the population as unstratified. The number of sampled units $n'_h; h = 1, 2, \cdots, L$ falling in each stratum is recorded. The quantity $w_h = n'_h / n'$ will give an unbiased estimate of $W_h$. Simple random subsamples, without replacement of sizes $n_h = \mu_h n'_h; h = 1, 2, \cdots, L; 0 < \mu_h \le 1$ are then drawn out of $n'_h$ from each stratum for values of $\mu_h$ chosen in advance.

For the $jth$ characteristics and the $hth$ stratum denote by

$y_{jhi}$ the value of the $ith$ population (sample) unit of the main variable.

$x_{jhi}$ the value of the $ith$ population (sample) units of the auxiliary variable.

$\bar{Y}_{jh} = \frac{1}{N_h} \sum_{i=1}^{N_h} y_{jhi}$ and $\bar{y}_{jh} = \frac{1}{n_h} \sum_{i=1}^{n_h} y_{jhi}$ the stratum mean and the sample mean respectively for the main variable.

$\bar{X}_{jh} = \frac{1}{N_h} \sum_{i=1}^{N_h} X_{jhi}$ and $\bar{x}_{jh} = \frac{1}{n_h} \sum_{i=1}^{n_h} y_{jhi}$ denote the same values for the auxiliary variable.

In double sampling for stratification the combined ratio estimate of the population mean of the $jth$ characteristics is given as

$$\bar{y}_{j(CRDS)} = \sum_{h=1}^{L} w_h \bar{r}_{jh}; \quad j = 1, 2, \cdots, p \tag{1}$$

where "*CR*" and "*DS*" stand for "combined ratio" and "double sampling" respectively.

Further,

$$\bar{r}_{jh} = \hat{R}_{jst}\bar{X}_j, \qquad \hat{R}_{jst} = \frac{\bar{y}_{jst}}{\bar{x}_{jst}}$$

$$\bar{y}_{js} = \sum_{h=1}^{L} w_h \bar{y}_{jh}$$

$$\bar{x}_{js} = \sum_{h=1}^{L} w_h \bar{x}_{jh}$$

The sampling variance of $\bar{y}_{j(CRDS)}$ is

$$V\left(\bar{y}_{j(CRDS)}\right) = \left(\frac{1}{n'} - \frac{1}{N}\right) S_{yj}^2 + \frac{1}{n'}\sum_{h=1}^{L} w_h \left(\frac{1}{\mu_h} - 1\right) S_{rjh}^2 \tag{2}$$

$S_{rjh}^2$ in (2) is defined as

$$S_{rjh}^2 = S_{yjh}^2 + R_j^2 S_{xjh}^2 - 2R_j S_{x_j y_j h} \tag{3}$$

where $R_j$ are true population ratios given as

$$R_j = \frac{\bar{Y}_j}{\bar{X}_j}$$

$S_{yjh}^2, S_{xjh}^2$ are the stratum variances of the $jth$ characteristics in the $hth$ stratum for main variable and auxiliary variables respectively and $S_{x_j y_j h}$ are the stratum co-variances of the $jth$ characteristics in the $hth$ stratum for $j = 1, 2, \cdots, p; h = 1, 2, \cdots, L$.

In the presence of non-response, let out of the $n_h$ units $n_{h1}$ units respond at the first call and $n_{h2} = n_h - n_{h1}$ units constitute the non-respondents group. Using [1], a subsample of size $m_{h2} = k_h^* n_{h2}; 0 < k_h^* \le 1$ out of $n_{h2}$ is drawn and interviewed with extra efforts. Where $k_h^*; h = 1, 2, \cdots, L$ are fixed in advance.

An combined ratio estimate $\bar{y}_{j(CRDS)}^*$ of $\bar{Y}_j$ may be given as

$$\bar{y}_{j(CRDS)}^* = \sum_{h=1}^{L} w_h \bar{r}_{jh}^* \tag{4}$$

where

$$\bar{r}_{jh}^* = \frac{n_{h1}\bar{r}_{jh1} + n_{h2}\bar{r}_{jm_{h2}}}{n_h}$$

$\bar{r}_{jh1}$, $\bar{r}_{jm_{h2}}$ are the sample mean of the ratio estimates for respondents (based on $n_{h1}$ units) and non-respondents group (based on $m_{h2}$ units) respectively.

Using the results presented in [12]—Sections 5A.2, 12.9 and 13.6 we get $V\left(\bar{y}_{j(CRDS)}^*\right)$ in presence of non-response as

$$\begin{aligned} V\left(\bar{y}_{j(CRDS)}^*\right) &= V\left(\bar{y}_{j(CRDS)}\right) + \frac{1}{n'}\sum_{h=1}^{L} w_{h2}\left(\frac{1-k_h^*}{k_h^*\mu_h}\right) S_{2rjh}^2 \\ &= \left(\frac{1}{n'} - \frac{1}{N}\right) S_{yj}^2 + \frac{1}{n'}\sum_{h=1}^{L} w_h \left(\frac{1}{\mu_h} - 1\right) S_{rjh}^2 + \frac{1}{n'}\sum_{h=1}^{L} w_{h2}\left(\frac{1-k_h^*}{k_h^*\mu_h}\right) S_{2rjh}^2 \end{aligned} \tag{5}$$

where,

$$S_{2rjh}^2 = S_{2yjh}^2 + R_j^2 S_{2xjh}^2 - 2R_j S_{2x_j y_j h}$$

$S_{2yjh}^2, S_{2xjh}^2$ are the stratum variances of the $jth$ characteristics of the non-respondents in the $hth$ stratum for main variable and auxiliary variable respectively. $S_{2x_jy_jh}$ is the stratum co-variances of the $jth$ characteristics of the non-respondents in the $hth$ stratum [11].

The total cost of the survey may be given

$$C = c_0 n' + \sum_{h=1}^{L} c_{h1} n_h + \sum_{h=1}^{L} c_{h11} n_{h1} + \sum_{h=1}^{L} c_{h12} m_{h2} \tag{6}$$

where,

$c_0$ is the per unit cost of getting information from the preliminary sample $n'$.

$c_{h1}$ is the per unit cost of of making the first attempt (Phase I).

$c_{h11}$ is the per unit cost of processing and analyzing the result of all the $p$ characteristics on the $n_{h1}$ respondents units in the $hth$ stratum at Phase I.

$c_{h12}$ is the per unit cost of measuring and processing the result of all the $p$ characteristics on the $m_{h2}$ subsampled units from non-respondents group in the $hth$ stratum at Phase II.

Since $n_{h1}$ is not known until the first attempt is made, the quantity $w_{h1} n_h$ may be used as its expected value. The total expected cost $\hat{C}$ of the survey is then given as

$$\hat{C} = c_0 n' + \sum_{h=1}^{L} \left(c_{h1} + c_{h11} w_{h1}\right) n_h + \sum_{h=1}^{L} c_{h12} m_{h2}. \tag{7}$$

## 3. Formulation of the Problem

In Phase I, we obtain the sample size $n_h$ in each stratum by minimizing variance given in (5) for fixed cost given in (7). At Phase II subsample size from non-respondents group has been obtained by minimizing the sampling variance in (5) for given cost in (7).

### 3.1. Formulation of the Problem at Phase I

Expression (5) can be expressed as

$$\left(\bar{y}^*_{j(CRDS)}\right) = V_j = \sum_{h=1}^{L} \frac{\alpha_{jh}}{n_h}; \quad j = 1, 2, \cdots, p \tag{8}$$

where the terms independent of $n_h$ are ignored

$$\text{and } \alpha_{jh} = \frac{1}{n'}\left( w_h n'_h S_{jh}^2 + w_{h2}\left(\frac{\left(1-k_h^*\right)}{k_h^*}\right) n'_h S_{jh2}^2 \right) \tag{9}$$

The cost constraint (7) becomes

$$\sum_{h=1}^{L} \left(c_{h1} + w_{h1} c_{h11}\right) n_h \le \hat{C}_0 \tag{10}$$

where

$$\hat{C}_0 = \hat{C} - c_0 n' - \sum_{h=1}^{L} c_{h12} m_{h2}$$

Thus the multi-objective formulation of the problem at Phase I becomes

$$\left.\begin{aligned} &\text{Minimize } V_j = \sum_{h=1}^{L} \frac{\alpha_{jh}}{n_h}; \; j = 1, 2, \cdots, p \text{ simulteniously} \\ &\text{Subject to } \sum_{h=1}^{L} \left(c_{h1} + w_{h1} c_{h11}\right) n_h \le \hat{C}_0 \\ &\qquad 2 \le n_h \le n'_h \\ &\text{and } n_h \text{ integers}; h = 1, 2, \cdots, L \end{aligned}\right\} \tag{11}$$

(see [11]).

### 3.2. Formulation of the Problem for Phase II

Ignoring the term independent from $m_{h2}$ in (5), substituting $k_h^* = m_{h2}/n_{h2}$ and $v_h = \frac{n_h}{n'_h}$, for $j = 1, 2, \cdots, p$, expression (5) can be written as

$$V'_j = \sum_{h=1}^{L} \frac{\beta_{jh}}{m_{h2}}; \quad j = 1, 2, \cdots, p \tag{12}$$

where

$$\beta_{jh} = \frac{1}{n'} \sum_{h=1}^{L} \frac{w_{h2} n_{h2} n'_h S_{jh2}^2}{n_h} \tag{13}$$

The cost constraint becomes

$$\sum_{h=1}^{L} c_{h12} m_{h2} \le \hat{C}'_0$$

where

$$\hat{C}'_0 = \hat{C} - c_0 n' - \sum_{h=1}^{L} \left( c_{h1} + w_{h1} c_{h11} \right) n_h \tag{14}$$

Then the multi-objective formulation of the problem at Phase II becomes

$$\left.\begin{array}{c} \text{Minimize } V'_j = \sum_{h=1}^{L} \frac{\beta_{jh}}{m_{h2}};\ j = 1, 2, \cdots, p; \text{ simultaneously} \\ \text{Subject to } \sum_{h=1}^{L} c_{h12} m_{h2} \le \hat{C}'_0 \\ 2 \le m_{h2} \le n_{h2} \\ \text{and } m_{h2} \text{ integers; } h = 1, 2, \cdots, L \end{array}\right\}. \tag{15}$$

## 4. Formulation as a Goal Programming Problem

### 4.1. Phase I

Let $V_j^*$ be the optimal value of $V_j$ under optimum allocation for the $jth$ characteristics obtained by solving the following integer non-linear programming for all the $j = 1, 2, \cdots, p$ characteristics separately.

$$\left.\begin{array}{c} \text{Minimize } V_j = \sum_{h=1}^{L} \frac{\alpha_{jh}}{n_h} \\ \text{Subject to } \sum_{h=1}^{L} \left( c_{h1} + w_{h1} c_{h11} \right) n_h \le \hat{C}_0 \\ 2 \le n_h \le n'_h \\ \text{and } n_h \text{ integers; } h = 1, 2, \cdots, L \end{array}\right\} \tag{16}$$

Further let

$$\tilde{V}_j = \tilde{V}_j \left( n_1, n_2, \cdots, n_L \right) = \sum_{h=1}^{L} \frac{\alpha_{jh}}{n_h} \tag{17}$$

denote the variance under the compromise allocation, where $n_h; h = 1, 2, \cdots, L$ are to be worked out.

Obviously $\tilde{V}_j \ge V_j^*$ and $\tilde{V}_j - V_j^* \ge 0; j = 1, 2, \cdots, p$ will give the increase in the variances due to not using the individual optimum allocation for $jth$ characteristics.

Let $d_j \ge 0$ denote the tolerance limit specified for $\left( \tilde{V}_j \ge V_j^* \right); j = 1, 2, \cdots, p$.

We have $\tilde{V}_j - V_j^* \le d_j; j = 1, 2, \cdots, p$

$$\text{or} \quad \tilde{V}_j - d_j \le V_j^*;\ j = 1, 2, \cdots, p$$

$$\text{or} \quad \sum_{h=1}^{L} \frac{\alpha_{jh}}{n_h} - d_j \le V_j^*;\ j = 1, 2, \cdots, p \tag{18}$$

A suitable compromise criterion to work out a compromise allocation at phase-I will then be to minimize the sum of deviations $d_j$. Therefore the Goal Programming problem at phase-I may be given as

$$\left.\begin{gathered}
\text{Minimize} \sum_{j=1}^{p} d_j \\
\text{Subject to} \sum_{h=1}^{L} \frac{\alpha_{jh}}{n_h} - d_j \le V_j^* \\
\sum_{h=1}^{L} \left(c_{h1} + w_{h1} c_{h11}\right) n_h \le \hat{C}_0 \\
2 \le n_h \le n_h' \\
d_j \ge 0 \\
\text{and } n_h \text{ integers} \\
h = 1, 2, \cdots, L;\ j = 1, 2, \cdots, p
\end{gathered}\right\} \tag{19}$$

(See [13]). Where $d_j \ge 0;\ j = 1, 2, \cdots, p$ are the goal variables.
The goal is now to minimize the sum of deviations from the respective optimum variances.

### 4.2. Phase II

Similarly, at phase II Goal Programming formulation of the problem (15) will be

$$\left.\begin{gathered}
\text{Minimize} \sum_{j=1}^{p} d_j' \\
\text{Subject to} \sum_{h=1}^{L} \frac{\beta_{jh}}{m_{h2}} - d_j' \le V_j^{*\prime} \\
\sum_{h=1}^{L} c_{h12} m_{h2} \le \hat{C}_0' \\
2 \le m_{h2} \le n_{h2} \\
d_j' \ge 0 \\
\text{and } m_{h2} \text{ integers};\ h = 1, 2, \cdots, L;\ j = 1, 2, \cdots, p
\end{gathered}\right\}. \tag{20}$$

## 5. Formulation as a Fuzzy Programming Problem

### 5.1. Phase I

To obtain Fuzzy solution we first compute maximum value $U_k$ and minimum value $L_k$ for each characteristic. where

$$U_k = \max V_j\left(\left(n_{h,j}^*\right)\right) L_k = \min V_j\left(\left(n_{h,j}^*\right)\right);\ j, k = 1, 2, \cdots, p \tag{21}$$

where $\left(n_{h,j}^*\right)$ denote the optimum allocation for the $jth$ characteristics and the maximum and minimum are for all $V_j$, among their values for a particular $j = k$.

The difference of the maximum value $U_k$ and minimum values $L_k$ are denoted by $d_k = U_k - L_k; k = 1, 2, \cdots, p$.

The Fuzzy Programming Problem (FPP) corresponding to the (11) at phase I is given by the following NLPP

$$\left.\begin{aligned} &\text{Minimize}\,\delta \\ &\text{Subject to } \sum_{h=1}^{L}\frac{\alpha_{jh}}{n_h}-\delta\left(d_k\right)\le V_j^{*} \\ &\sum_{h=1}^{L}\left(c_{h1}+w_{h1}c_{h11}\right)n_h\le \hat{C}_0 \\ &2\le n_h\le n'_h \\ &d_k\ge 0 \\ &\text{and } n_h \text{ integers} \\ &h=1,2,\cdots,L;\ j,k=1,2,\cdots,p \end{aligned}\right\} \tag{22}$$

where $\delta \ge 0$ is the decision variable representing the worst deviation level.

## 5.2. Phase II

Similarly, the Fuzzy Programming Problem corresponding to the (15) at phase II is given by the following NLPP

$$\left.\begin{aligned} &\text{Minimize}\,\delta' \\ &\text{Subject to } \sum_{h=1}^{L}\frac{\beta_{jh}}{m_{h2}}-\delta'\left(d'_k\right)\le V_j^{*\prime} \\ &\sum_{h=1}^{L}c_{h12}m_{h2}\le \hat{C}'_0 \\ &2\le m_{h2}\le n_{h2} \\ &d'_k\ge 0 \\ &\text{and } m_{h2} \text{ integers} \\ &h=1,2,\cdots,L;\ j,k=1,2,\cdots,p \end{aligned}\right\} \tag{23}$$

where $\delta' \ge 0$ is the decision variable representing the worst deviation level.

The NLPPs may be solved by using the optimization software [14]. For further information about LINGO one may visit the site: http://www.lindo.com.

# 6. A Numerical Example

The data in **Table 1** use are from [15]. A population of size $N = 3850$ is divided into four strata. Two characteristic $Y_1$ and $Y_2$ are defined on each unit of the population. The values of $X_1$ and $X_2$ are used as the auxiliary information corresponding on the main variable $Y_1$ and $Y_2$. The authors have assumed the values for $R_1 = 1.48$ and $R_2 = 0.843$. **Table 2** shows the other data. Each stratum is divided into respondents and non-respondents as shown in **Table 2**.

It is assumed that $v_h$ and $k_h^*$ are known and the preliminary sample size $n' = 1000$.

In the last column of **Table 2**, $l = 1$ is for respondents group and $l = 2$ is for non-respondents group.

The total cost for the survey is taken as $C$ 3000 units. Out of which 750 units are for the preliminary sample $n'$, 1900 units are for phase-I and 350 units are for phase-II.

**Table 1.** Data for four strata and two characteristics.

| | | | | | | | $j=1$ | | | $j=2$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $h$ | $w_h$ | $v_h$ | $k_h^*$ | $c_{h1}$ | $c_{h11}$ | $c_{h12}$ | $S^2_{y_{1h}}$ | $S^2_{x_{1h}}$ | $S^2_{x_1y_{1h}}$ | $S^2_{y_{2h}}$ | $S^2_{x_{2h}}$ | $S^2_{x_2y_{2h}}$ |
| 1 | 0.32 | 0.4 | 0.5 | 1 | 2 | 3 | 784 | 242 | 341 | 1444 | 481 | 628 |
| 2 | 0.21 | 0.5 | 0.6 | 1 | 3 | 4 | 576 | 192 | 250 | 676 | 255 | 294 |
| 3 | 0.27 | 0.6 | 0.7 | 1 | 4 | 5 | 1024 | 341 | 445 | 1936 | 645 | 842 |
| 4 | 0.20 | 0.65 | 0.75 | 1 | 5 | 6 | 2916 | 972 | 1268 | 6084 | 2028 | 2645 |

**Table 2.** Data for groups of respondents and non-respondents.

| $h$ | Group | $S^2_{y_{1h}}$ | $S^2_{x_{1h}}$ | $S^2_{x_1 y_{1h}}$ | $S^2_{y_{2h}}$ | $S^2_{x_{2h}}$ | $S^2_{x_2 y_{2h}}$ | $w_{hl}$ $l = 1,2$ |
|---|---|---|---|---|---|---|---|---|
| 1 | Respondent | 361.06 | 103.16 | 157.04 | 767.82 | 255.76 | 333.93 | $w_{11} = 0.70$ |
| | Non-respondent | 310.55 | 88.73 | 135.07 | 454.76 | 151.48 | 197.93 | $w_{12} = 0.30$ |
| 2 | Respondent | 373.79 | 124.60 | 162.24 | 449.92 | 169.72 | 195.67 | $w_{21} = 0.80$ |
| | Non-respondent | 326.29 | 108.76 | 141.62 | 353.81 | 133.46 | 153.88 | $w_{22} = 0.20$ |
| 3 | Respondent | 930.15 | 309.75 | 404.22 | 1272.88 | 424.07 | 553.60 | $w_{31} = 0.75$ |
| | Non-respondent | 560.28 | 186.85 | 243.48 | 1165.98 | 388.46 | 507.10 | $w_{32} = 0.25$ |
| 4 | Respondent | 2355.98 | 785.33 | 1024.48 | 2690.53 | 896.84 | 1169.70 | $w_{41} = 0.72$ |
| | Non-respondent | 1013.08 | 337.69 | 440.53 | 2403.55 | 801.18 | 1044.93 | $w_{42} = 0.28$ |

Using estimated values of strata weights the values of $n'_h = w_h n';\ h = 1,2,\cdots,L$ are obtained as

$$n'_1 = 320,\ \ n'_2 = 210,\ \ n'_3 = 270,\ \ n'_4 = 200 \ \text{ with } \sum_{h=1}^{4} n'_h = 1000.$$

## 6.1. Computation of Compromise Allocation Using Goal Programming Technique (GPT)

### 6.1.1. Individual Optimum Allocation (Phase I)

Using data from **Table 1** and **Table 2**, we compute the individual optimum allocation for each characteristic by using NLPP (11) will be the solution to:

For $j = 1$

$$\text{Minimize } V_1 = \frac{39.16074}{n_1} + \frac{13.82271}{n_2} + \frac{55.71960}{n_3} + \frac{62.80995}{n_4}$$

Subject to

$$2.4n_1 + 3.4n_2 + 4n_3 + 4.6n_4 \le 1900$$
$$2 \le n_1 \le 320$$
$$2 \le n_2 \le 210$$
$$2 \le n_3 \le 270$$
$$2 \le n_4 \le 200$$

and $n_h$ are integers $h = 1,2,\cdots,L$

Using optimization software LINGO we get the optimal solution as

$$\left(n^*_{h,1}\right) = \left(159,80,146,144\right),\quad V^*_1 = 1.236899$$

For $j = 2$

$$\text{Minimize } V_2 = \frac{109.6218}{n_1} + \frac{21.00712}{n_2} + \frac{85.26067}{n_3} + \frac{151.4584}{n_4}$$

Subject to

$$2.4n_1 + 3.4n_2 + 4n_3 + 4.6n_4 \le 1900$$
$$2 \le n_1 \le 320$$
$$2 \le n_2 \le 210$$
$$2 \le n_3 \le 270$$
$$2 \le n_4 \le 200$$

and $n_h$ are integers $h = 1,2,\cdots,L$

Using optimization software LINGO we get the optimal solution as

$$\left(n_{h,2}^{*}\right)=\left(183,69,126,157\right),\quad V_2^{*}=2.544852.$$

### 6.1.2. Compromise Solution Using Goal Programming (Phase I)

Using data from **Table 1** and **Table 2** the Goal Programming Problem (19) can be formulated as

$$\text{Minimize} d_1+d_2$$

$$\text{Subject to}$$

$$\frac{39.16074}{n_1}+\frac{13.82271}{n_2}+\frac{55.71960}{n_3}+\frac{62.80995}{n_4}-d_1\le 1.236899$$

$$\frac{109.6218}{n_1}+\frac{21.00712}{n_2}+\frac{85.26067}{n_3}+\frac{151.4584}{n_4}-d_2\le 2.544852$$

$$2.4n_1+3.4n_2+4n_3+4.6n_4\le 1900$$

$$2\le n_1\le 320$$

$$2\le n_2\le 210$$

$$2\le n_3\le 270$$

$$2\le n_4\le 200$$

$$d_1\ge 0, d_2\ge 0$$

$$\text{and } n_h \text{ are integers } h=1,2,\cdots,L$$

Using optimization software LINGO we get the optimal solution as

$$n_1^{*}=175,\quad n_2^{*}=72,\quad n_3^{*}=134,\quad n_4^{*}=152,$$

with $d_1^{*}=0.79$ and $d_2^{*}=0.603434$ and the optimum value of the objective function $d_1^{*}+d_2^{*}=0.0139343$.

### 6.1.3. Individual Optimum Allocation (Phase II)

As in Section 6.1.1 for the given data the individual optimum allocations for each the two characteristics using NLPP (15) are:

For $j=1$

$$\left(m_{h,1}^{*}\right)=\left(21,12,19,24\right)\quad V_1^{*\prime}=0.7599099$$

For $j=2$

$$\left(m_{h,2}^{*}\right)=\left(21,9,19,26\right)\quad V_2^{*\prime}=1.742418.$$

### 6.1.4. Compromise Solution Using Goal Programming (Phase II)

For the given data, as in Section 6.1.2 Goal Programming Problem (20) gives the following optimal solution

$$m_{12}^{*}=22,\quad m_{22}^{*}=10,\quad m_{32}^{*}=20,\quad m_{42}^{*}=24,$$

with $d_1^{*\prime}=0.002751545$ and $d_2^{*\prime}=0.002866150$ and the optimum value of the objective function

$d_1^{*\prime}+d_2^{*\prime}=0.005617695.$

## 6.2. Computations of Compromise Solution Using Fuzzy Programming Technique (FPT)

### 6.2.1. Compromise Solution Using Fuzzy Programming (Phase I)

To obtain fuzzy solution we first obtained the maximum value and minimum value as given in (21) for each characteristic by using individual optimum allocation worked out in Section 6.1.1

$$U_1=1.256605\qquad L_1=1.236899$$

$$U_2=2.58685\qquad L_2=2.544852$$

$$\text{and} \quad d_1 = 0.019706 \qquad d_2 = 0.042954.$$

After computing the optimum allocation and optimum variances for two characteristics the compromise optimal solution for the above problem can be obtained by solving the given Fuzzy Programming Problem (FPP) of (23)

$$\text{Minimize } \delta$$
$$\text{Subject to}$$
$$\frac{39.16074}{n_1}+\frac{13.82271}{n_2}+\frac{55.71960}{n_3}+\frac{62.80995}{n_4}-\delta(0.019706)\le 1.236899$$
$$\frac{109.6218}{n_1}+\frac{21.00712}{n_2}+\frac{85.26067}{n_3}+\frac{151.4584}{n_4}-\delta(0.042954)\le 2.544852$$
$$2.4n_1+3.4n_2+4n_3+4.6n_4\le 1900$$
$$2\le n_1\le 320$$
$$2\le n_2\le 210$$
$$2\le n_3\le 270$$
$$2\le n_4\le 200$$
$$\delta\ge 0$$
$$\text{and} n_h \text{ are integers } h=1,2,\cdots,L$$

Using optimization software LINGO we get the optimal solution as

$$n_1^* = 171, \quad n_2^* = 75, \quad n_3^* = 135, \quad n_4^* = 151 \text{ with } \delta = 0.2486958.$$

#### 6.2.2. Compromise Solution Using Fuzzy Programming (Phase II)

Similarly, using data from **Table 1** and **Table 2** the Fuzzy Programming Problem (23) gives the following optimal solution

$$m_{12}^* = 22, \quad m_{22}^* = 10, \quad m_{32}^* = 20, \quad m_{42}^* = 24 \text{ with } \delta' = 0.2423372.$$

## 7. Summary of the Results

In the following results obtained using Goal Programming Technique and Fuzzy Programming Technique are summarized.

## 8. Conclusions

**Table 3** and **Table 4** show the values of the variance of the combined ratio estimates of the population means at Phase-I and Phase-II respectively, for the two characteristics. The figures show that both the approaches the Goal Programming Approach and the Fuzzy Programming Approach give almost same results. However, at Phase-I the Goal Programming Approach is slightly more precise in terms of the trace value (See [16]).

The Goal Programming and Fuzzy Programming technique and some other techniques like Dynamic Programming and Separable Programming can be used to solve a wide variety of mathematical programming problems. These techniques may be of great help in solving multivariate sampling problem also. Like determining

**Table 3.** Compromise solution at Phase I.

| Techniques | Allocations | | | | Variances | | Trace | Cost incurred |
|---|---|---|---|---|---|---|---|---|
| | $n_1$ | $n_2$ | $n_3$ | $n_4$ | $V_1$ | $V_2$ | $V_1+V_2$ | $C$ |
| GPT | 175 | 72 | 134 | 152 | 1.244799 | 2.550886 | 3.795685 | 1900 |
| FPT | 171 | 75 | 135 | 151 | 1.242011 | 2.555754 | 3.797765 | 1900 |

**Table 4.** Compromise solution at Phase II.

| Techniques | Allocations | | | | Variances | | Trace | Cost incurred |
|---|---|---|---|---|---|---|---|---|
| | $m_{12}$ | $m_{22}$ | $m_{32}$ | $m_{42}$ | $V_1'$ | $V_2'$ | $V_1'+V_2'$ | |
| GPT | 22 | 10 | 20 | 24 | 0.7626614 | 1.745284 | 2.507945 | 350 |
| FPT | 22 | 10 | 20 | 24 | 0.7626614 | 1.745284 | 2.507945 | 350 |

the number of strata, strata boundaries and compromise allocations in multivariate stratified sampling. Little work has been done to solve the above mentioned optimization problems in real life situations. For example when the estimates of the population parameters used in formulating the problems are themselves treated as random variables with assumed or known distributions. In such cases the formulated problems becomes a multivariate stochastic programming. Further, apart from a linear cost function, nonlinear functions may be used that may include travel cost, labour cost, rewards to the respondent and incentives to the investigators etc. Interested researchers may expose these situations.

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## References

[1] Hansen, M.H. and Hurwitz, W.N. (1946) The Problem of Non-Response in Sample Surveys. *Journal of the American Statistical Association*, **41**, 517-529. http://dx.doi.org/10.1080/01621459.1946.10501894

[2] Rao, P.S.R.S. (1986) Ratio Estimation with Sub-Sampling the Non-Respondents. *Survey Methodology*, **12**, 217-230.

[3] Rao, P.S.R.S. (1987) Ratio and Regression Estimates with Sub-Sampling the Non-Respondents. *Special Contributed Session of the International Statistical Association Meeting*, 2-16 September 1987, Tokyo.

[4] Khare, B.B. and Srivastva, S. (1993) Estimation of Population Mean Using Auxiliary Character in Presence of Non-Response. *National Academy Science Letters*, **16**, 111-114.

[5] Khare, B.B. and Srivastva, S. (1997) Transformed Ratio Type Estimators for the Population Mean in the Presence of Non-Response. *Communications in Statistics—Theory and Methods*, **26**, 1779-1791. http://dx.doi.org/10.1080/03610929708832012

[6] Raiffa, H. and Schlaifer, R. (1961) Applied Statistical Decision Theory. Graduate School of Business Administration, Harvard University, Boston.

[7] Ericson, W.A. (1965) Optimum Stratified Sampling Using Prior Information. *Journal of the American Statistical Association*, **60**, 750-771. http://dx.doi.org/10.1080/01621459.1965.10480825

[8] Ahsan, M.J. and Khan, S.U. (1982) Optimum Allocation in Multivariate Stratified Random Sampling with Overhead Cost. *Metrika*, **29**, 71-78. http://dx.doi.org/10.1007/BF01893366

[9] Dayal, S. (1985) Allocation in Sample Using Values of Auxiliary Characteristics. *Journal of Statistical Planning and Inference*, **11**, 321-328. http://dx.doi.org/10.1016/0378-3758(85)90037-0

[10] Khan, M.G.M., Maiti, T. and Ahsan, M.J. (2010) An Optimal Multivariate Stratified Sampling Design Using Auxiliary Information: An Integer Solution Using Goal Programming Approach. *Journal of Official Statistics*, **26**, 695-708.

[11] Varshney, R., Najmussehar and Ahsan, M.J. (2011) An Optimum Multivariate Stratified Double Sampling Design in Non-Response. *Optimization Letters*, **6**, 993-1008.

[12] Cochran, W.G. (1977) Sampling Techniques. 3rd Edition, John Wiley& Sons, New York.

[13] Schniederjans, M.J. (1995) Goal Programming: Methodology and Applications. Kluwer, Dordrecht. http://dx.doi.org/10.1007/978-1-4615-2229-4

[14] Lingo User's Guide (2013) Lingo-User's Guide. LINDO SYSTEM INC., Chicago.

[15] Haseen, S., Iftekhar, S., Ahsan, M.J. and Bari, A. (2012) A Fuzzy Approach for Solving Double Sampling Design in Presence of Non-Response. *International Journal of Engineering Science and Technology*, **4**, 2542-2551.

[16] Sukhatme, P.V., Sukhatme, B.V., Sukhatme, S. and Asok, C. (1984) Sampling Theory of Surveys with Applications. 3rd Edition, Iowa State University Press, Iowa and Indian Society of Agricultural Statistics, New Delhi.