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Calogero Model with Different Masses

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ABSTRACT

We study a multispecies one-dimensional Calogero model with two- and three-body interactions. Here, we factorize the ground state ψ_0 out of the Hamiltonian H in order to get the new operator $\tilde{H}$ which preserves some spaces of polynomials $\mathcal{P}_n$ in the case of equal masses, *i.e.* $m_i = m$ (the usual Calogero model) and in the case with different masses. The spectrum of these both cases is found easily.

Keywords: Multispecies One-Dimensional Calogero; Equal Masses; Different Masses; Usual Calogero Model

1. Introduction

The ordinary Cologero [1,2] model describes N indistinguishable particles on the line which interact through an inverse-square two-body interaction. The model is completely integrable in both the classical and quantum case [3]. The spectrum is known and the wave functions are given implicitly. In the present paper, which is in a sense a continuation of the investigation of the ordinary model [4], we use an algebraic method to find some of the salient features of the multispecies Calogero model on the line with two- and three-body interactions. After performing a certain transformation of the operator H, we get a new Hamiltonian $\tilde{H}$ for which we find its spectrum in the both cases with equal masses and different masses.

2. Calogero Model with Different Masses

In this section, we reconsider the "multispecies" Calogero model considered in [5]. The Hamiltonian reads

$$H = -\frac{1}{2}\sum_{i=1}^{N}\frac{1}{m_i}\frac{\partial^2}{\partial x_i^2} + \frac{\omega^2}{2}\sum_{i=1}^{N} m x_i^2 + \frac{1}{4}\sum_{i\neq j}\frac{\upsilon_{ij}\left(\upsilon_{ij}-1\right)}{\left(x_i - x_j\right)^2}\left(\frac{1}{m_i}+\frac{1}{m_j}\right) + \frac{1}{2}\sum_{i,j,k\neq}\frac{\upsilon_{ij}\upsilon_{jk}}{m_j\left(x_j - x_i\right)\left(x_j - x_k\right)}\left(\frac{1}{m_j}+\frac{1}{m_k}\right), \tag{1}$$

where $\upsilon_{ij} = \upsilon_{ji}$.

We factorize the full ground state

$$\psi_o = \Delta\phi, \tag{2}$$

with

$$\Delta = \prod_{i<j}\left(x_i - x_j\right)^{\upsilon_{ij}}, \tag{3}$$

$$\phi = \exp\left(-\frac{\omega}{2}\sum_{j=1}^{N} m_j x_j^2\right). \tag{4}$$

When factorizing the factor ψ_0 out of H, we got the new operator

$$\tilde{H} = \psi_0^{-1} H \psi_0 = -\frac{1}{2}\sum_{i=1}^{N}\frac{1}{m_i}\frac{\partial^2}{\partial x_i^2} + \omega D - \frac{1}{2}\sum_{i\neq j}^{N}\frac{\upsilon_{ij}}{x_i - x_j}\left(\frac{1}{m_i}\frac{\partial}{\partial x_i} - \frac{1}{m_j}\frac{\partial}{\partial x_j}\right) + \omega\left(N + \frac{1}{2}\sum_{i\neq j}^{N}\upsilon_{ij}\right) \tag{5}$$

$$D = \sum_{i=1}^{N} x_i \frac{\partial}{\partial x_i}. \tag{6}$$

The operator $\tilde{H}$ preserves some spaces of polynomials that we would like to study and compare with the invariant spaces $\mathcal{P}_n$ available in the case of equal masses, *i.e.* $m_i = m$ (the usual Calogero model). We first proceed with the $N = 2$ *i.e.* two body case. Then it is easy to check that the following vector spaces are preserved by $\tilde{H}$:

$$P_0 = \text{span}\{1\},\ P_1 = \text{span}\{X\},$$
$$P_2 = \text{span}\left\{X^2, \left(x_1 - x_2\right)^2\right\}, \tag{7}$$
$$P_3 = \text{span}\left\{X^3, X\left(x_1 - x_2\right)^2\right\}, \cdots$$

with

$$X = \frac{\sum_{i=1}^{N} m_i x_i}{\sum_{i=1}^{N} m_i} \equiv \frac{1}{M}\sum_{i=1}^{N} m_i x_i = \frac{1}{m_1 + m_2}\left(m_1 x_1 + m_2 x_2\right). \tag{8}$$

It should be stressed that the combination $x_1 - x_2$ has to be eliminated from P_1 because it is not preserved by the part

$$\begin{aligned} H_0 &= -\frac{1}{2}\sum_{i\neq j}\frac{\upsilon_{ij}}{x_i - x_j}\left(\frac{1}{m_i}\frac{\partial}{\partial x_i} - \frac{1}{m_j}\frac{\partial}{\partial x_j}\right) \\ &= -\frac{1}{2}\left[\frac{\upsilon_{12}}{x_1 - x_2}\left(\frac{1}{m_1}\frac{\partial}{\partial x_1} - \frac{1}{m_2}\frac{\partial}{\partial x_2}\right)\right. \\ &\qquad \left. + \frac{\upsilon_{21}}{x_2 - x_1}\left(\frac{1}{m_2}\frac{\partial}{\partial x_2} - \frac{1}{m_1}\frac{\partial}{\partial x_1}\right)\right] \\ &= -\frac{1}{2}\left[2\frac{\upsilon_{12}}{x_1 - x_2}\left(\frac{1}{m_1}\frac{\partial}{\partial x_1} - \frac{1}{m_2}\frac{\partial}{\partial x_2}\right)\right] \\ &= -\frac{\upsilon_{12}}{x_1 - x_2}\left(\frac{1}{m_1}\frac{\partial}{\partial x_1} - \frac{1}{m_2}\frac{\partial}{\partial x_2}\right). \end{aligned} \tag{9}$$

of the operator $\tilde{H}$. As a consequence the monomial $\left(x_1 - x_2\right)^3$ has to be discarded from P_3 since $\tilde{H}\left(x_1 - x_2\right)^3$ *i.e*, the following part of the operator $\tilde{H}$

$$\begin{aligned} &-\frac{1}{2}\sum_{i=1}^{2}\frac{1}{m_i}\frac{\partial^2}{\partial x_i^2}\left(x_1 - x_2\right)^3 \\ &= -\frac{1}{2}\left(\frac{1}{m_1}\frac{\partial^2}{\partial x_1^2} + \frac{1}{m_2}\frac{\partial^2}{\partial x_2^2}\right)\left(x_1 - x_2\right)^3, \\ &= \frac{3}{m_1}\left(x_1 - x_2\right) + \frac{3}{m_2}\left(x_1 - x_2\right). \end{aligned} \tag{10}$$

would naturally involve a term of the form $\left(x_1 - x_2\right)$ in the first order monomial which is excluded by the above argument (*i.e.* P_1). Proceeding along the same lines we conclude that the set of spaces P_n can be rephased in terms of the vector spaces $\mathcal{P}_n$ defined in [6] *i.e.*

$$\begin{aligned} &\mathcal{P}_n = \text{span}\left\{\sigma_1^{n_1}\tau_2^{n_1}, n_1 + n_2 \le n\right\},\ \mathcal{P}_0 = \text{span}\{1\}, \\ &\mathcal{P}_1 = \text{span}\{1, \sigma_1, \tau_2\},\ \mathcal{P}_2 = \text{span}\left\{1, \sigma_1, \tau_2, \sigma_1^2, \sigma_1\tau_2, \tau_2^2\right\}, \\ &\mathcal{P}_3 = \text{span}\left\{1, \sigma_1, \tau_2, \sigma_1^2, \sigma_1\tau_2, \tau_2^2, \sigma_1^3, \sigma_1^2\tau_2, \sigma_1\tau_2^2, \tau_2^3\right\}\cdots \end{aligned} \tag{11}$$

with σ_1 is the center-of-mass coordinate

and
$$\tau_2 \div \left(x_1 - x_2\right)^2 \tag{12}$$

in this respect, the operator H (and then also $\tilde{H}$) is integrable and solvable for $N = 2$.

Notice that the space $\mathcal{P}_n$ is equivalent to the ones considered by [6], apart from the fact that the variable X (the analogue of σ_1) is defined with the masses.

Let us now investigate the case $N = 3$. Again we can show that the following vector spaces are preserved by the operator $\tilde{H}$,

$$\begin{aligned} &P_0 = \text{span}\{1\},\ P_1 = \text{span}\{X\}, \\ &P_2 = \text{span}\left\{X^2, Q\right\},\ P_3 = \text{span}\left\{X^3, XQ\right\}, \cdots \end{aligned} \tag{13}$$

where

$$\begin{aligned} &X \equiv \frac{1}{m_1 + m_2 + m_3}\left(m_1 x_1 + m_2 x_2 + m_3 x_3\right), \\ &Q \equiv m_1 m_2\left(x_1 - x_2\right)^2 + m_1 m_3\left(x_1 - x_3\right)^2 + m_2 m_3\left(x_2 - x_3\right)^2. \end{aligned} \tag{14}$$

Note that Q above is the generalization of the variable τ_2 of [6]. However, it turns out to be impossible to construct a translation invariant-cubic polynomial of the form

$$\begin{aligned} \tau_3 &= y_1 y_2 y_3 \\ &= \left(x_1 - \frac{1}{3}\left(x_1 + x_2 + x_3\right)\right)\left(x_2 - \frac{1}{3}\left(x_1 + x_2 + x_3\right)\right) \\ &\quad \cdot\left(x_3 - \frac{1}{3}\left(x_1 + x_2 + x_3\right)\right) \\ &= \left(\frac{2}{3}x_1 - \frac{1}{3}\left(x_2 + x_3\right)\right)\left(\frac{2}{3}x_2 - \frac{1}{3}\left(x_1 + x_3\right)\right) \\ &\quad \cdot\left(\frac{2}{3}x_3 - \frac{1}{3}\left(x_1 + x_2\right)\right) \\ &\div \left(2x_1 - x_2 - x_3\right)\left(2x_2 - x_1 - x_3\right)\left(2x_3 - x_1 - x_2\right) \end{aligned} \tag{15}$$

which is preserved by the operator $\tilde{H}$ if the masses m_i are generic (*i.e* $m_1 \neq m_2 \neq m_3$, etc). As a consequence, the dimension of the vector spaces of monomials preserved by $\tilde{H}\Big|_{m_i \neq}$ is lower than the vector spaces preserved by $\tilde{H}\Big|_{m_i =}$ and the number of algebraic eigenvalues is lower than the usual Calogero case. In the next, this can be demonstrated easily in the particular case $N = 2$.

2.1. Eigenvalues for the Case with Equal Masses

We use the operator

$$D = \sigma_1\frac{\partial}{\partial\sigma_1} + \sum_{i=2}^{2} i\tau_i\frac{\partial}{\partial\tau_i} = \sigma_1\frac{\partial}{\partial\sigma_1} + 2\tau_2\frac{\partial}{\partial\tau_2}. \tag{16}$$

i) $$D\mathcal{P}_0 = \left(\sigma_1 \frac{\partial}{\partial \sigma_1} + 2\tau_2 \frac{\partial}{\partial \tau_2}\right)(1) = 0. \tag{17}$$

ii) $$D\mathcal{P}_1 = \left(\sigma_1 \frac{\partial}{\partial \sigma_1} + 2\tau_2 \frac{\partial}{\partial \tau_2}\right)(\sigma_1) = 1\sigma_1,$$
$$D\mathcal{P}_1 = \left(\sigma_1 \frac{\partial}{\partial \sigma_1} + 2\tau_2 \frac{\partial}{\partial \tau_2}\right)(\tau_2) = 2\tau_2. \tag{18}$$

iii) $$D\mathcal{P}_2 = \left(\sigma_1 \frac{\partial}{\partial \sigma_1} + 2\tau_2 \frac{\partial}{\partial \tau_2}\right)(\sigma_1^2) = 2\sigma_1^2,$$
$$D\mathcal{P}_2 = \left(\sigma_1 \frac{\partial}{\partial \sigma_1} + 2\tau_2 \frac{\partial}{\partial \tau_2}\right)(\sigma_1 \tau_2) = 3\sigma_1\tau_2, \tag{19}$$
$$D\mathcal{P}_2 = \left(\sigma_1 \frac{\partial}{\partial \sigma_1} + 2\tau_2 \frac{\partial}{\partial \tau_2}\right)(\tau_2^2) = 4\tau_2^2.$$

The spectrum for the above case is
0;
1, 2;
2, 3, 4.

2.2. Eigenvalues for the Case with Different Masses

In this case we apply the some procedure used in the previous case (*i.e.* we consider also $N = 2$) but the operator D has the following form

$$D = \sum_{i=1}^{2} x_i \frac{\partial}{\partial x_i} = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}. \tag{20}$$

i) $$DP_0 = \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\right)(1) = 0. \tag{21}$$

ii) $$DP_1 = \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\right)(X)$$
$$= \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\right)\left[\frac{1}{m_1 + m_2}(m_1 x_1 + m_2 x_2)\right] \tag{22}$$
$$= \frac{1}{m_1 + m_2}(m_1 x_1 + m_2 x_2) = 1X.$$

iii) $$DP_2 = \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\right)(X^2)$$
$$= \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\right)\left[\frac{1}{m_1 + m_2}(m_1 x_1 + m_2 x_2)\right]^2,$$
$$2X^2,$$
$$DP_2 = \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\right)(x_1 - x_2)^2 = 2(x_1 - x_2)^2. \tag{23}$$

The spectrum for the above case is
0;
1;
2, 2.

More generally, the vector spaces preserved by $\tilde{H}$ are of the form

$$P_n = \mathrm{spa}\left\{X^n, X^{n-2}Q, X^{n-4}Q^2, \cdots\right\} \tag{24}$$

and any eigenvector of $\tilde{H}$ can be written according to

$$\psi_{n_1 n_2}(x) = X^{n_1} Q^{n_2} + \text{"lowest degree"} \tag{25}$$

while the corresponding eigenvalues are given by

$$\omega D\psi_{n_1 n_2}(x) = \omega(n_1 + 2n_2)\psi_{n_1 n_2}(x) \tag{26}$$

so that the spectrum of H consists of integers of the form

$$E_{n_1 n_2} = \omega(n_1 + 2n_2) \tag{27}$$

as generic in [5]. In this way, we have redemonstrated the result of these authors by following the algebraic technique of operators preserving spaces of monomials as suggested by [6].

We have attempted to construct invariant spaces of polynomials involving the monomials

$$R_p = m_1 m_2 (x_1 - x_2)^p + m_1 m_3 (x_1 - x_3)^p + m_2 m_3 (x_2 - x_3)^p \tag{28}$$

with $p \geq 3$. These polynomials are indeed such that $H_0 R_p$ is a polynomial but the new polynomials $\tilde{H}R_p$ are not in general expressible as polynomials of the two variables X and Q ($i.e.\, Q \equiv R_2$). More generally, the polynomials for N body can be written as follows

$$\begin{aligned} R_p &= A(x_1 - x_2)^p + B(x_1 - x_3)^p + C(x_2 - x_3)^p \\ &\quad + \cdots + W(x_1 - x_N)^p + \cdots + Z(x_{N-1} - x_N)^p \\ &= m_1 m_2 (x_1 - x_2)^p + m_1 m_3 (x_1 - x_3)^p + m_2 m_3 (x_2 - x_3)^p \\ &\quad + \cdots + m_1 m_N (x_1 - x_N)^p + \cdots + m_{N-1} m_N (x_{N-1} - x_N)^p. \end{aligned} \tag{29}$$

3. Conclusion

Here we have constructed the operator $\tilde{H}$ which preserves some spaces of polynomials and compared with the invariant spaces available in the usual Calogero model ($m_i = m$ *i.e.* the masses are equal). We have determined the real spectrum for the case with different masses and for the case for equal masses where $N = 2$ *i.e.* two body case. This extended Calogero model exhibits some remarkable properties which are absent in the case of usual Calogero model. For example, the number of eigenvalues in the case with different masses is lower than one of eigenvalues of the usual Calogero model.

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