Open Journal of Modelling and Simulation, 2015, 3, 63-69
Published Online July 2015 in SciRes. http://www.scirp.org/journal/ojmsi
http://dx.doi.org/10.4236/ojmsi.2015.33007

Some New Results on the Number of Paths

Beih S. El-Desouky[1], Abdelfattah Mustafa[1], E. M. Mahmoud[2]

[1]Mathematics Department, Faculty of Science, Mansoura University, Mansoura, Egypt
[2]Mathematics Department, Faculty of Science, Aswan University, Aswan, Egypt
Email: b_desouky@yahoo.com, abdelfatah_mustafa@yahoo.com

Received 23 April 2015; accepted 1 June 2015; published 5 June 2015

Abstract

Khidr and El-Desouky [1] derived a symmetric sum involving the Stirling numbers of the first kind through the process of counting the number of paths along a rectangular array $n\times m$ denoted by A_{nm}. We investigate the generating function for the general case and hence some special cases as well. The probability function of the number of paths along A_{nm} is obtained. Moreover, the moment generating function of the random variable *X* and hence the mean and variance are obtained. Finally, some applications are introduced.

Keywords

Stirling Numbers, Generating Function, Moment Generating Function, Comtet Numbers, Maple Program

1. Introduction

Let $\{a_i\}_{i=1}^{n}$ be a sequence of natural numbers $0\le a_i\le n, i=1,\cdots,n$, and A_{nm} be an $n\times m$ array associated with this sequence, whose entries $\alpha_{ij}=0,1$ such that

$$\sum_{j=1}^{m}\alpha_{ij}=a_i,\ i=1,2,\cdots,n,\ j=1,\cdots,m.$$

The path of order *k* along A_{nm} is defined to be a sequence of entries $\alpha_{1j_1},\cdots,\alpha_{nj_n}$ as follows

$$\sum_{i=1}^{n}\alpha_{ij_i}=k,\ k=0,1,\cdots,n,\ j_i=1,\cdots,m,\ i=1,\cdots,n.$$

The number of paths of order *k* will be denoted by

How to cite this paper: El-Desouky, B.S., Mustafa, A. and Mahmoud, E.M. (2015) Some New Results on the Number of Paths. *Open Journal of Modelling and Simulation*, **3**, 63-69. http://dx.doi.org/10.4236/ojmsi.2015.33007

$$g^k(a_1,\cdots,a_n;m),\ k=0,1,\cdots n.$$

By neglecting the last row in A_{nm} and then reconsidering it, we get the recurrence

$$g^k(a_1,\cdots,a_n;m)=a_n g^{k-1}(a_1,\cdots,a_{n-1};m)+(m-a_n)g^k(a_1,\cdots,a_{n-1};m). \tag{1}$$

When $a_i=a$, a is a constant, $i=1,\cdots,n,$ then

$$g^k(a,\cdots,a;m)\cdots g^k(\bar{a},m)=\binom{n}{k}a^k(m-a)^{n-k}, \tag{2}$$

and

$$\sum_{k=0}^{n} g^k(\bar{a},m)=m^n.$$

Khidr and El-Desouky [1] proved that, when $m=n$

$$g^k(a_1,a_2,\cdots,a_n;n)=(-1)^k\sum_{l=k}^{n}\binom{l}{k}s_{\bar{a}}(n,n-l)n^{n-l}, \tag{3}$$

where $s_{\bar{a}}(n,i)$ are the generalized Stirling numbers of the first kind associated with the sequence of real numbers $\bar{a}:=(a_1,a_2,\cdots,a_n)$, defined by [1]-[6],

$$(x-a_1)(x-a_2)\cdots(x-a_n)=\sum_{i=0}^{n}s_{\bar{a}}(n,i)x^i. \tag{4}$$

These numbers satisfy the recurrence relation

$$s_{\bar{a}}(n,i)=s_{\bar{a}}(n-1,i-1)-a_n s_{\bar{a}}(n-1,i),\ s_{\bar{a}}(n,i)=0\ \text{ for } i>n. \tag{5}$$

And

$$\sum_{k=0}^{n} g^k(\bar{a},n)=n^n.$$

Moreover, they introduced a special case of (3), when $a_i=i,\ i=1,2,\cdots,n$, then the number of paths of order k, $g^k(1,\cdots,n;n)$ is denoted by $g_n^k(n)$; and proved that

$$g_n^k(n)=(-1)^k\sum_{l=k}^{n}\binom{l}{k}s(n+1,n+1-l)n^{n-l}, \tag{6}$$

where $s(n,l)$ are the Stirling numbers of the first kind defined by, see [2] [3]

$$x(x-1)\cdots(x-n+1)=\sum_{l=0}^{n}s(n,l)x^n.$$

Also the generating function for $g_n^k(n)$ is given by

$$G(t)=\prod_{i=1}^{n}\left[n-j(1-t)\right]. \tag{7}$$

In this article, in Section 2, we derive a generalization of some results given in [1], for the number of paths of order k, $g^k(a_1,\cdots,a_n;m)$, when $m\neq n$. The generating function of $g^k(a_1,\cdots,a_n;m)$ is given. In Section 3, we find the probability distribution for $g^k(a_1,\cdots,a_n;m)$ and study some of their properties. The moment generating function, skewness and kurtosis for $g^k(a_1,\cdots,a_n;m)$ are investigated. Moreover special case and numerical results are given in Section 4.

2. Main Results

Theorem 1. The number of paths of order k is given by

$$g^{k}\left(a_1,a_2,\cdots,a_n;m\right)=\left(-1\right)^{k}\sum_{i=k}^{n}\binom{i}{k}s_{\bar{a}}\left(n,n-i\right)m^{n-i}. \tag{8}$$

Proof. Using (5) in (8), we get

$$\begin{aligned}
&g^{k}\left(a_1,a_2,\cdots,a_n;m\right)\\
&=\left(-1\right)^{k}\sum_{i=k}^{n}\binom{i}{k}\left\{s_{\bar{a}}\left(n-1,n-i-1\right)-a_n s_{\bar{a}}\left(n-1,n-i\right)\right\}m^{n-i}\\
&=\left(-1\right)^{k}\sum_{i=k}^{n-1}\binom{i}{k}s_{\bar{a}}\left(n-1,n-i-1\right)m^{n-i}-\left(-1\right)^{k}\sum_{i=k}^{n}a_n\left\{\binom{i-1}{k-1}+\binom{i-1}{k}\right\}s_{\bar{a}}\left(n-1,n-i\right)m^{n-i}\\
&=\left(-1\right)^{k}\sum_{i=k}^{n-1}\binom{i}{k}s_{\bar{a}}\left(n-1,n-i-1\right)m^{n-i}-\Bigg\{\left(-1\right)^{k}a_n\sum_{i=k+1}^{n}\binom{i-1}{k}\\
&\quad\times s_{\bar{a}}\left(n-1,n-i\right)m^{n-i}+\left(-1\right)^{k}a_n\sum_{i=k}^{n}\binom{i-1}{k-1}s_{\bar{a}}\left(n-1,n-i\right)m^{n-i}\Bigg\}\\
&=\left(-1\right)^{k}\sum_{i=k}^{n-1}\binom{i}{k}s_{\bar{a}}\left(n-1,n-i-1\right)m^{n-i}-\left(-1\right)^{k}a_n\sum_{i=k}^{n-1}\binom{i}{k}s_{\bar{a}}\left(n-1,n-i-1\right)\\
&\quad\times m^{n-i-1}-\left(-1\right)^{k}a_n\sum_{i=k-1}^{n-1}\binom{i}{k-1}s_{\bar{a}}\left(n-1,n-i-1\right)m^{n-i-1}\\
&=\left(m-a_n\right)g^{k}\left(a_1,\cdots,a_{n-1};m\right)+a_n g^{k-1}\left(a_1,\cdots,a_{n-1};m\right).
\end{aligned}$$

This by virtue of (1) completes the proof of (8).

Theorem 2. The generating function of the number of paths of order k is given by

$$G^{(n)}\left(n;m\right)=\prod_{i=1}^{n}\left(m-a_i\left(1-t\right)\right). \tag{9}$$

Proof. Let the generating function of the number of paths of order k be denoted by

$$G^{(n)}\left(n;m\right)=\sum_{k=0}^{n}g^{k}\left(a_1,\cdots,a_n;m\right)t^{k}. \tag{10}$$

Using (1), we obtain

$$\begin{aligned}
G^{(n)}\left(n;m\right)&=\sum_{k=1}^{n}a_n g^{k-1}\left(n-1;m\right)t^{k}+\left(m-a_n\right)\sum_{k=0}^{n}g^{k}\left(n-1;m\right)t^{k},\\
&=a_n t G^{(n-1)}\left(n-1;m\right)+\left(m-a_n\right)G^{(n-1)}\left(n-2;m\right)\\
&=\left(m-a_n+a_n t\right)G^{(n-1)}\left(n-1;m\right),
\end{aligned}$$

and hence we get

$$G^{(n)}\left(n;m\right)=\prod_{i=1}^{n}\left(m-a_i\left(1-t\right)\right)G^{(0)}\left(0;m\right),$$

where $G^{(0)}\left(0;m\right)=g^{0}\left(0;m\right)=1$. This completes the proof.

From (9), we get

$$G^{(n)}\left(n;m\right)=a_1\left(t-\frac{a_1-m}{a_1}\right)a_2\left(t-\frac{a_2-m}{a_2}\right)\cdots a_n\left(t-\frac{a_n-m}{a_n}\right)=\prod_{i=1}^{n}a_i\sum_{k=0}^{n}s_{\bar{\alpha}}\left(n,k\right)t^{k},$$

where $\alpha_i=-\left(\frac{m-a_i}{a_i}\right)$, $\bar{\alpha}:=\left(\alpha_1,\alpha_2,\cdots,\alpha_n\right)$ and hence we have

$$g^k\left(a_1,\cdots,a_n;m\right)=\prod_{i=1}^{n}a_i s_{\bar{\alpha}}\left(n,k\right), \tag{11}$$

where $\alpha_i=-\binom{m-a_i}{a_i}, i=1,\cdots,n.$

For the special case $a_i=i, m=n$, we get

$$g_n^k\left(n\right)=n!s_{\bar{\alpha}}\left(n,k\right), \tag{12}$$

where $\alpha_i=-\binom{n-i}{i}, i=1,\cdots,n.$

From (6) and (12), we have the identity

$$\left(-1\right)^k\sum_{m=k}^{n}\binom{m}{k}s\left(n+1,n+1-m\right)n^{n-m}=n!s_{\bar{\alpha}}\left(n,k\right), \tag{13}$$

where $\alpha_i=-\binom{n-i}{i}, i=1,\cdots,n.$

3. Some Applications

Let *X*, be the number of paths along A_{nm}, then by virtue of (8) we have

$$\begin{aligned}P\left(X=k\right)&=\left(-1\right)^k\sum_{i=k}^{n}\binom{i}{k}s_{\bar{\alpha}}\left(n,n-i\right)\frac{m^{n-i}}{m^n}\\&=\left(-1\right)^k\sum_{i=k}^{n}\binom{i}{k}\frac{s_{\bar{\alpha}}\left(n,n-i\right)}{m^i},\quad k=0,1,\cdots,n.\end{aligned} \tag{14}$$

On the other hand the moment generating function of the random variable *X* denoted by $M_X\left(t\right)$, is given by the following theorem.

Theorem 3. The moment generating function of *X*, is given by

$$M_X\left(t\right)=\sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{\left(-1\right)^i i!}{n!m^i}S\left(n,i\right)s_{\bar{\alpha}}\left(n,n-i\right)t^n. \tag{15}$$

Proof. We begin by the definition of the moment generating function as follows.

$$\begin{aligned}M_X\left(t\right)&=\sum_{k=0}^{n}e^{kt}f\left(k\right)=\sum_{k=0}^{n}e^{kt}\left(-1\right)^k\sum_{i=k}^{n}\binom{i}{k}\frac{s_{\bar{\alpha}}\left(n,n-i\right)}{m^i}\\&=\sum_{i=0}^{n}\sum_{k=0}^{i}\left(-1\right)^k\binom{i}{k}e^{kt}\frac{s_{\bar{\alpha}}\left(n,n-i\right)}{m^i}=\sum_{i=0}^{n}\left(1-e^t\right)^i\frac{s_{\bar{\alpha}}\left(n,n-i\right)}{m^i}\\&=\sum_{i=0}^{n}\sum_{n=i}\frac{\left(-1\right)^i i!}{n!m^i}S\left(n,i\right)s_{\bar{\alpha}}\left(n,n-i\right)t^n\\&=\sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{\left(-1\right)^i i!}{n!m^i}S\left(n,i\right)s_{\bar{\alpha}}\left(n,n-i\right)t^n,\end{aligned}$$

This completes the proof.

Corollary 1. The jth moments of *X* is

$$E\left[X^j\right]=\sum_{i=0}^{n}\frac{\left(-1\right)^i i!}{m^i}S\left(j,i\right)s_{\bar{\alpha}}\left(n,n-i\right). \tag{16}$$

Proof. The *j*th moments can be obtained from the moment generating function, $M_X\left(t\right)$ where

$$E\left[X^j\right]=\left[\frac{\mathrm{d}^n}{\mathrm{d}t^n}M_X(t)\right]_{t=0}.$$

This completes the proof.

Then from (16), we can calculate the mean and variance for the random variable *X* as follows.

$$\mu=E(X)=\sum_{i=0}^{n}\frac{(-1)^i i!}{m^i}S(1,i)s_{\bar{a}}(n,n-i)=-\frac{s_{\bar{a}}(n,n-1)}{m}. \tag{17}$$

$$\begin{aligned}E\left(X^2\right)&=\sum_{i=0}^{n}\frac{(-1)^i i!}{m^i}S(2,i)s_{\bar{a}}(n,n-i)\\&=-\frac{S(2,1)s_{\bar{a}}(n,n-1)}{m}+\frac{2!S(2,2)s_{\bar{a}}(n,n-2)}{m^2}\\&=-\frac{s_{\bar{a}}(n,n-1)}{m}+\frac{2s_{\bar{a}}(n,n-2)}{m^2},\end{aligned} \tag{18}$$

hence the variance is given by

$$\begin{aligned}Var(X)=E\left[X^2\right]-\mu^2&=-\frac{s_{\bar{a}}(n,n-1)}{m}+\frac{2s_{\bar{a}}(n,n-2)}{m^2}-\left(-\frac{s_{\bar{a}}(n,n-1)}{m}\right)^2\\&=-\frac{s_{\bar{a}}(n,n-1)}{m^2}\left[s_{\bar{a}}(n,n-1)+m\right]+\frac{2s_{\bar{a}}(n,n-2)}{m^2}.\end{aligned} \tag{19}$$

Corollary 2. The Skewness and kurtosis for the random variable *X* are given by

$$\text{Skewness}=\frac{\mu_3}{\mu_2^{3/2}},\ \text{kurtosis}=\frac{\mu_4}{\mu_2^2} \tag{20}$$

where

$$\mu_2=-\frac{s_{\bar{a}}(n,n-1)}{m}+\frac{2s_{\bar{a}}(n,n-2)}{m^2}-\frac{s_{\bar{a}}^2(n,n-1)}{m^2},$$

$$\begin{aligned}\mu_3=&-\frac{s_{\bar{a}}(n,n-1)}{m}-\frac{3s_{\bar{a}}^2(n,n-1)}{m^2}-\frac{2s_{\bar{a}}^3(n,n-1)}{m^3}+\frac{6s_{\bar{a}}(n,n-2)}{m^2}\\&+\frac{6s_{\bar{a}}(n,n-2)s_{\bar{a}}(n,n-1)}{m^3}-\frac{6s_{\bar{a}}(n,n-3)}{m^3},\end{aligned}$$

$$\begin{aligned}\mu_4=&-\frac{s_{\bar{a}}(n,n-1)}{m}-\frac{4s_{\bar{a}}^2(n,n-1)}{m^2}-\frac{6s_{\bar{a}}^3(n,n-1)}{m^3}-\frac{3s_{\bar{a}}^4(n,n-1)}{m^4}+\frac{14s_{\bar{a}}(n,n-2)}{m^2}\\&+\frac{24s_{\bar{a}}(n,n-2)s_{\bar{a}}(n,n-1)}{m^3}+\frac{12s_{\bar{a}}(n,n-2)s_{\bar{a}}^2(n,n-1)}{m^4}-\frac{36s_{\bar{a}}(n,n-3)}{m^3}\\&-\frac{24s_{\bar{a}}(n,n-3)s_{\bar{a}}(n,n-1)}{m^4}+\frac{24s_{\bar{a}}(n,n-4)}{m^4}.\end{aligned}$$

Proof. We can find the jth moments about the mean by using

$$\mu_j=E\left[\left(X-E[X]\right)^j\right]=\sum_{k=0}^{j}\binom{j}{k}(-1)^k\left(E[X]\right)^k E\left[X^{j-k}\right]. \tag{21}$$

From (16) and (21), we can find the moments μ_2,μ_3,μ_4 about mean which can be used to calculate the skweness and kurtosis.

Special Case:

If $a_i=i, m=n$, from (14), we have

$$P(X=k)=(-1)^k\sum_{i=k}^{n}\binom{i}{k}\frac{s(n+1,n+1-i)}{n^i},$$

and from (16) the jth moments has the form

$$E\left[X^j\right]=\sum_{i=0}^{n}\frac{(-1)^i\, i!S(j,i)s(n+1,n+1-i)}{n^i},$$

and the mean is given by

$$\mu=-\frac{s(n+1,n)}{n}=\frac{n+1}{2}.$$

$$E\left[X^2\right]=-\frac{s(n+1,n)}{n}+\frac{2s(n+1,n-1)}{n^2},$$

the variance can be obtained as follows.

$$\begin{aligned}Var(X)=E\left[X^2\right]-\mu^2&=-\frac{s(n+1,n)}{n}+\frac{2s(n+1,n-1)}{n^2}-\left(-\frac{s(n+1,n)}{n}\right)^2\\&=-\frac{s(n+1,n)}{n^2}\left[s(n+1,n)+n\right]+\frac{2s(n+1,n-1)}{n^2}\\&=\frac{(n-1)(n+1)}{6n}.\end{aligned}$$

where we used $s(n,n-1)=-\binom{n}{2}, s(n,n-2)=\frac{1}{4}(3n-1)\binom{n}{3}$, see [3].

4. Numerical Results

Setting $n=5, m=6, \alpha_i=i, i=1,2,\cdots,n$. Therefore the numerical values of $s_{\bar{\alpha}}(n,k)$, are reduced to $s(n,k)$, see [4] [5].

From Equation (14), we can find the probability distribution of the number of paths *X* along $A_{5\times 6}$ as follows

X	0	1	2	3	4	5	Total
p (*x*)	0.09259	0.34259	0.38889	0.15741	0.01852	0	1

From (16), we can compute the 4th moments as follows.

j	1	2	3	4
$E\left[X^j\right]$	1.66667	3.61111	8.88889	24.0556

The 4th moments about mean can be obtained as

j	1	2	3	4
μ_j	0	0.83333	0.09259	1.83333

The values of mean and variance can be obtained from (17) and (19) as follows.

$$\mu=1.66667, Var(X)=0.83333.$$

The skewness and kurtosis, respectively can be obtained from (20) as follows.

skewness = 0.12172, kurtosis = 2.64.

References

[1] Khidr, A.M. and El-Desouky, B.S. (1984) A Symmetric Sum Involving the Stirling Numbers of the First Kind. *European Journal of Combinatorics*, **5**, 51-54. http://dx.doi.org/10.1016/S0195-6698(84)80018-9

[2] Comtet, L. (1972) Nombres de Stirling generaux et fonctions symetriques. *Comptes Rendus de l'Académie des Sciences Paris* (*Series A*), **275**, 747-750.

[3] Comtet, L. (1974) Advanced Combinatorics: The Art of Finite and Infinite Expansions. D. Reidel Publishing Company, Dordrecht, Holand.

[4] El-Desouky, B.S. (1994) Multiparameter Non-Central Stirling Numbers. *The Fibonacci Quarterly*, **32**, 218-225.

[5] El-Desouky, B.S. and Cakić, N.P. (2011) Generalized Higher Order Stirling Numbers. *Mathematical and Computer Modelling*, **54**, 2848-2857. http://dx.doi.org/10.1016/j.mcm.2011.07.005

[6] Cakić, N.P., El-Desouky, B.S. and Milovanović, G.V. (2013) Explicit Formulas and Combinatorial Identities for Generalized Stirling Numbers. *Mediterranean Journal of Mathematics*, **10**, 57-72. http://dx.doi.org/10.1007/s00009-011-0169-x