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Deriving the Kutta-Joukowsky Equation and Some of Its Generalizations Using Momentum Balances

David H. Wood
Department of Mechanical and Manufacturing Engineering, Schulich School of Engineering, University of Calgary, Calgary, Canada
E-mail: dhwood@ucalgary.ca
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Abstract

Momentum balances are used to derive the Kutta-Joukowsky equation for an infinite cascade of aerofoils and an isolated aerofoil. These derivations are simpler than those based on the Blasius theorem or more complex unsteady control volumes, and show the close relationship between a single aerofoil and an infinite cascade. The modification of lift due to the presence of another lifting body is similarly derived for a wing in ground effect, a biplane, and tandem aerofoils. The results are identical to those derived from the vector form of the Kutta-Joukowsky equation.

Keywords: Lift, Vorticity, Kutta-Joukowsky Equation, Aerofoils, Cascades, Biplane, Ground Effect, Tandem Aerofoils

1. Introduction

The Kutta-Joukowsky (KJ) equation can be viewed as the answer to the question: what is the simplest possible singularity representation of a lifting body in an inviscid fluid flow? It is fundamental to aerofoil theory and subsequent developments in turbomachinery, wind turbines, and propellers. The purpose of this note is to provide a derivation of the equation using simple techniques of conservation of momentum and the Reynolds transport theorem, along with a basic knowledge of singularities and circulation. The demonstration includes isolated bodies, infinite cascades with application to rotating fluid machines, and pairs of identical or mirror-image bodies modeling wings in ground effect and biplanes.

In the educational literature there are three common developments of the KJ equation for an isolated aerofoil:

1) the demonstration of its validity for a specific body, often a rotating circular cylinder, followed by an unproved statement of its generality, e.g. White [1],

2) the derivation using the Blasius theorem combined with residue calculus and complex variables, e.g. Panton [2], and

3) the moving and expanding control volume method of Batchelor [3], which requires a thorough knowledge of the unsteady Bernoulli equation and careful consideration of the decay of induced velocities at large distances.

2. Cascades and Isolated Aerofoils

The following demonstrations are considerably simpler than 2 or 3. Consider an infinite cascade of identical bodies—usually aerofoils—spaced distance s apart along the y-axis in **Figure 1**. Only four bodies are shown. The undisturbed velocity of the incompressible fluid is U_0. For simplicity, one body is located at the origin surrounded by a rectangular control volume (CV) with horizontal faces at $y = \pm s/2$. The vertical faces are equidistant from the y-axis: the actual distance is not important. The faces are labeled in clockwise order from the upstream one. Symmetry requires that for faces 2 and 4:

- the pressures are equal at the same x,
- there is no net efflux of x- or y-direction momentum, and
- the contribution to, the circulation around the contour, will cancel.

Γ is positive in the clockwise direction. Only the flow through faces 1 and 3 contributes to the momentum balance. The x-velocity at any point in the flow is $U_0 + u$ where the latter is due to the singularities, as is the vertical velocity, v.

Applying the Reynolds transport theorem to the CV gives for the vertical force on the body, F_y:

$$-F_y = \rho\int_{-s/2}^{s/2}(U_0+u_3)v_3\mathrm{d}y - \rho\int_{-s/2}^{s/2}(U_0+u_1)v_1\mathrm{d}y \quad (1)$$

where the subscripts on u and v denote the face, or

$$F_y = \rho U_0\Gamma + \rho\int_{-s/2}^{s/2}(u_1v_1 - u_3v_3)\mathrm{d}y \quad (2)$$

Similarly, the x-direction force, F_x, is found from

$$-F_x + \int_{-s/2}^{s/2}(P_1-P_3)\mathrm{d}y = \rho\int_{-s/2}^{s/2}(U_0+u_3)^2\,\mathrm{d}y - \rho\int_{-s/2}^{s/2}(U_0+u_1)^2\,\mathrm{d}y \quad (3)$$

where P is the pressure which can be removed by assuming that the Bernoulli constant, C, is the same for all streamlines in the flow[1]:

$$P = C - \frac{1}{2}\rho\left\{(U_0+u)^2+v^2\right\} \quad (4)$$

Equation (3) is rewritten as

$$F_x = \frac{1}{2}\rho\int_{-s/2}^{s/2}\left\{u_1^2-u_3^2+2U_0(u_1-u_3)+v_3^2-v_1^2\right\}\mathrm{d}y \quad (5)$$

The first term in Equation (2) makes it necessary to represent a lifting body by a vortex of strength Γ. This representation is now shown to be sufficient as (2) and (5) are fully satisfied. If all the bodies in the cascade are replaced by vortices of strength Γ, u is an even function of y and v is an odd function. Thus uv is odd and the integral in (2) identically zero. Equi-spacing of the CV faces 1 and 3 about the y-axis requires $u_1(y)=u_3(y)$ and $v_1(y)=-v_3(y)$ so the integrand in (5) is zero for any y. Thus Equations (3) and (5) reduce to

$$F_x = 0,\quad F_y = \rho U_0\Gamma \quad (6a)$$

which is the simplest form of the KJ equation. Note that the forces are independent of the spacing s. The vector form is

$$\mathbf{F} = \rho\mathbf{U}\times\mathbf{\Gamma} \quad (6b)$$

for a straight line vortex with no internal structure, e.g. Section 11.4 of Saffman [4]. It will be shown that results of the momentum balances can be interpreted in terms of the general from (6b) by appropriately altering the magnitude of the vector velocity, **U**, from U_0.

Figure 1 for a cascade can be replaced by **Figure 2** for an isolated body. This CV extends to ±∞ and it is assumed that no x- or y-direction momentum enters or leaves the horizontal faces. The contribution to the circulation on faces 1 and 3 induced by all the vortices representing the bodies in **Figure 1** is the same as that induced by the single vortex over the infinite faces in **Figure 2**. Equations (2) and (5) are unaltered by the change in CV except that $\pm s/2$ are replaced by ±∞ and the argument leading to Equation (6) is the same. This establishes the KJ equation for an isolated body.

[1]This is rarely the case for cascades that model fluid machines; large flow deflections can result in much larger (or smaller) exiting y-direction velocity than the entering one. Since the x-direction velocity is constrained by conservation of mass, the pressure and the Bernoulli constant will change. It may be useful to distinguish between cascades of blades with these changes and cascades of aerofoils, where they do not.

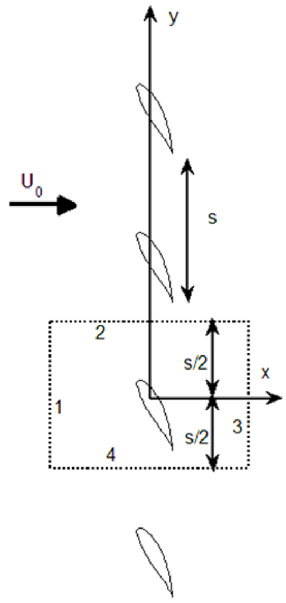

Figure 1. Control volume for cascade of equi-spaced identical bodies.

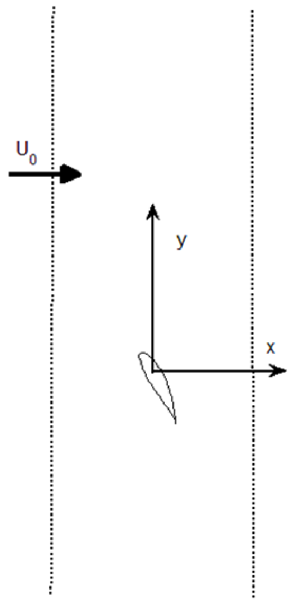

Figure 2. Control volume for an isolated body.

3. Aerofoil in Ground Effect and Biplanes

A single lifting body and an infinite cascade of identical bodies are the simplest arrangements in which to establish the KJ equation because there is no induced velocity on any of the bodies. For a finite "stack" of lifting bodies, the analysis becomes considerably more complex, e.g. Crowdy [5], and momentum balances quickly lose their attraction. However, for two lifting bodies, there is benefit in extending the present analysis. The geometry and control volume are shown in **Figure 3** for two cases of vertical separation: in the first the body at $-h$ is a mirror image of that at h and so has opposite circulation. This is common model for a lifting body in ground effect, GE. In the second, the bodies are identical, modeling a biplane, symbolised as B. This case is treated in Chapter 13 of Glauert [6] who gives the lift in terms of elliptic functions.

The rectangular control volume shown in **Figure 3** is used for both GE and B. It has height Y, and half width $X/2$ and it will be necessary to examine the effect of letting both X and Y tend to infinity. At a distance from the bodies large compared to h, the biplane acts as a single vortex of strength 2Γ, and the GE bodies as having no circulation. Thus the interaction between the two bodies must be only an exchange of lift for the biplane and a mutual increase or decrease in the magnitude of lift for GE.

For the CV in **Figure 3**, Equation (2) becomes

$$F_y = \rho U_0 \Gamma + \rho \int_0^Y \left(u_1 v_1 - u_3 v_3\right) \mathrm{d}y$$

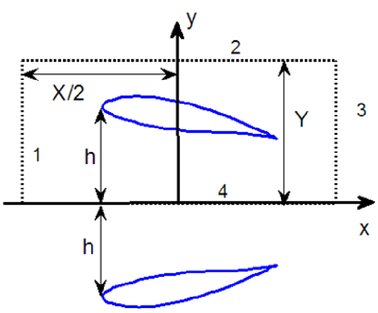

Figure 3. Aerofoil in ground effect.

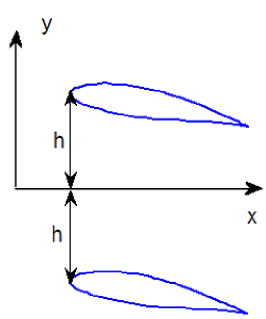

Figure 4. Aerofoils comprising a biplane. Control volume as in Figure 3.

$$+\frac{1}{2}\rho \int_{-X/2}^{X/2} \left(u_2^2 - u_4^2 - v_2^2 + v_4^2\right) \mathrm{d}x \tag{7}$$

and the only immediate simplification is that $v_4 = 0$ for GE and $u_4 = 0$ for B. Similarly, (5) becomes

$$\begin{aligned} F_x &= \frac{1}{2}\rho \int_0^Y \left\{u_1^2 - u_3^2 + 2U_0\left(u_1 - u_3\right) - v_1^2 + v_3^2\right\} \mathrm{d}y \\ &\quad + \rho \int_{-X/2}^{X/2} \left\{U_0\left(v_4 - v_2\right) + u_4 v_4 - u_2 v_2\right\} \mathrm{d}x \end{aligned} \tag{8}$$

where $u_1(y) = u_3(y)$, and v_2 and v_4 must be even in x. It is now shown that the first integral in (7) becomes negligible as X, $Y \to \infty$ and the integrand of second reduces to $-u_4^2$ for GE and v_4^2 for B as X, $Y \to \infty$. The unchanged first term on the right of (7) requires the continued use of vortices to represent the bodies. It is trivially easy to show that the velocities at any point (x, y) in the flow are given by

$$\begin{aligned} u &= \frac{\Gamma}{2\pi}\left(\frac{y-h}{r_1^2} \pm \frac{y+h}{r_2^2}\right) \\ v &= -\frac{\Gamma x}{2\pi}\left(\frac{1}{r_1^2} \pm \frac{1}{r_2^2}\right) \end{aligned} \tag{9}$$

with $r_1^2 = x^2 + (y-h)^2$, $r_2^2 = x^2 + (y+h)^2$. The + sign is for B and the – for GE. All the integrals in (7) and (8) can be evaluated exactly. For example,

$$\int_{-X/2}^{X/2} v_2^2 \mathrm{d}x = \frac{\Gamma^2}{4\pi^2} \int_{-X/2}^{X/2} \left[\frac{x^2}{\left\{x^2 + (Y-h)^2\right\}^2} + \frac{x^2}{\left\{x^2 + (Y+h)^2\right\}^2} \pm 2\frac{x^2}{\left\{x^2 + (Y-h)^2\right\}\left\{x^2 + (Y+h)^2\right\}}\right] \mathrm{d}x \tag{10}$$

Obviously the integral becomes negligible as X, $Y \to \infty$ for GE and it is easy to show that it does also for B. This is because, for example,

$$\begin{aligned} &\int_{-X/2}^{X/2} \frac{x^2}{\left\{x^2 + (Y-h)^2\right\}\left\{x^2 + (Y+h)^2\right\}} \mathrm{d}x \\ &= \frac{1}{4Yh}\left[\int_{-X/2}^{X/2} \frac{x^2}{x^2 + (Y-h)^2} \mathrm{d}x - \int_{-X/2}^{X/2} \frac{x^2}{x^2 + (Y+h)^2} \mathrm{d}x\right] \end{aligned} \tag{11}$$

and

$$\begin{aligned} &\int_{-X/2}^{X/2} \frac{x^2}{\left\{x^2 + (Y-h)^2\right\}^2} \mathrm{d}x \\ &= \left.\frac{a\tan\left(x/|Y-h|\right)}{2|Y-h|} - \frac{x}{2\left\{x^2 + (Y-h)^2\right\}}\right|_{-X/2}^{X/2} \end{aligned} \tag{12}$$

(12) tends to zero as X, $Y \to \infty$. Using (7) and results like (10) to (12) as X, $Y \to \infty$ shows that

$$F_{x,GE} = F_{x,B} = 0 \tag{13a}$$

$$\begin{aligned} F_{y,GE} &= \rho U_0 \Gamma - \tfrac{1}{2}\rho \int_{-\infty}^{\infty} u_{4,GE}^2 \mathrm{d}x \\ F_{y,B} &= \rho U_0 \Gamma + \tfrac{1}{2}\rho \int_{-\infty}^{\infty} v_{4,B}^2 \mathrm{d}x \end{aligned} \tag{13b}$$

Along $y = 0$, Equation (9) simplifies to

$$u_{4,GE} = \frac{-\Gamma h}{\pi\left(x^2 + h^2\right)}$$

$$v_{4,B} = \frac{\Gamma x}{\pi\left(x^2 + h^2\right)}$$

Thus

$$F_{y,GE} = \rho U_0 \Gamma - \rho \Gamma^2 / \left(4\pi h\right) \tag{15}$$

and

$$F_{y,B} = \rho U_0 \Gamma + \rho \Gamma^2 / \left(4\pi h\right) \tag{16}$$

Equation (15) is well known: it is, for example, Equation (16) of Katz & Plotkin [7] derived from a lumped-vortex model, and is equivalent to their (6.113) obtained from the Blasius theorem. (15) can be interpreted as the modification of (6a) due to the induced velocity of the image vortex, $-\Gamma/\left(4\pi h\right)$ on the "real" vortex in the direction opposite to U_0. This causes a lift reduction of $\rho\Gamma^2/\left(4\pi h\right)$ according to (6b).

Equations (4) and (13b) show that the second term in (15) is due to the non-zero gauge pressure acting on the ground plane. The total force (per unit length) on the fluid $\rho U_0 \Gamma$ is shared between the lifting body and the surface pressure. Other analyses of ground effect that include information about the body geometry usually show an increase in lift as the ground is approached but only when h is comparable to the chord length c, e.g. Thwaites [8, p. 527ff] and Katz & Plotkin [7, Section 12.3]. Assuming the usual relation between Γ. and aerofoil lift coefficient, C_l, gives the ratio of the GE lift to the aerofoil lift from (15) as

$$1 - cC_l / \left(8\pi h\right) \tag{17}$$

At $C_l \approx 1.0$ nand $h/c \approx 0.5$, the reduction is only 8% and can be easily overwhelmed by other effects.

As far as the author can tell, Equation (16) for the lift on the upper body of a biplane, is new but is easily established from (6b). The lower vortex induces a velocity of $\Gamma/\left(4\pi h\right)$ on the upper vortex in the direction of U_0 which increases the lift by $\rho\Gamma^2/\left(4\pi h\right)$ As with the formulation of Crowdy [5], the lift is increased and that on the lower body reduced by the same amount. Equation (16) shows the difference in lift increases with Γ, as found by Crowdy [5], but, in contrast, the difference here is zero when Γ = 0.

4. Tandem Aerofoils

If two identical lifting bodies are placed at $\pm d$ on the x-axis then the analysis of the last Section is easily modified to show that proximity does not alter the lift but causes a force

$$F_x = -\rho\Gamma^2 / \left(4\pi d\right) \tag{18}$$

on the forward aerofoil and an equal and opposite force on the rear one. The rear vortex induces a vertical velocity of $\Gamma/\left(4\pi d\right)$ on the front vortex which now produces a thrust (opposite of drag) of $\rho\Gamma^2/\left(4\pi d\right)$). The argument is readily reversed to show that the rear vortex experiences an equal and opposite force.

5. Summary and Conclusions

Momentum balances provide a straightforward proof of the usual form of the Kutta-Joukowsky Equation, (6a), for the fluid forces acting on isolated bodies and infinite cascades of equi-spaced identical bodies. The analysis implies—but *does not* assume—that each body is represented simply as a vortex. These two geometries are unique in that there is no induced velocity on any of the bodies. Deriving the forces using momentum balances shows the close link between cascade flow and that over a single aerofoil.

For pairs of identical or mirror image bodies the analysis becomes more complex and it is likely that momentum balances would become too cumbersome for larger numbers. The forces acting on aerofoils in ground effect, biplanes, and tandem aerofoils as determined from momentum balances are in agreement with the more general form of the Kutta-Joukowsky Equation, (6b), which includes the effect of the velocity induced by one body on the other. This idea is easily generalised, so that, for example, three aerofoils spaced equally apart by distance h in the vertical direction will experience the following vertical forces: $\rho U_0 \Gamma + \rho\Gamma^2/\left(12\pi h\right)$ on the upper, $\rho U_0 \Gamma$ on the middle, and $\rho U_0 \Gamma - \rho\Gamma^2/\left(12\pi h\right)$ on the lower.

In practice, the determination of the forces on multiple bodies can be more complex with differences in circulation for geometrically identical bodies, as opposed to the assumption of equal circulation made here. For example, Section 5.5 of Katz & Plotkin [7] shows that a lumped vortex model of tandem aerofoils requires the upstream aerofoil to have a greater circulation. Momentum balances will still give the forces if the circulations differ but they will not fix the magnitudes or the ratios of the circulations.

6. Acknowledgements

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