Open Journal of Ecology, 2015, 5, 87-109
Published Online April 2015 in SciRes. http://www.scirp.org/journal/oje
http://dx.doi.org/10.4236/oje.2015.54009

Conservation of Forestry Biomass with the Use of Alternative Resource

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Received 30 November 2014; accepted 25 March 2015; published 1 April 2015

Abstract

The effect of the alternative resource and time delay on conservation of forestry biomass is studied by considering a nonlinear mathematical model. In this paper, interaction between forestry biomass, industrialization pressure, toxicant pressure and technological effort is proposed and analysed. We find out the critical value of delay and observe that there is Hopf bifurcation. Using the normal form theory and the center manifold theorem, we determine the stability and direction of the bifurcating periodic solutions. Numerical simulations are given to illustrate the analytical results.

Keywords

Forestry Biomass, Industrialization, Alternative Resource, Toxicant, Technological Effort, Local Stability, Hopf Bifurcation

1. Introduction

Forest is an integral part of our biosphere. It used for fuel, furniture etc. and thus provides strong foundation for the development of any country. Forest assists in the global cycling of water, oxygen, carbon and nitrogen. In many developing countries, people burn wood to get energy for heating and cooking. Forest also provides food and shelter to many wild life species. Due to overpopulation, industrialization and associated pollution forests are depleted alarmingly. A typical example is the Doon Valley in the northern part of India where the forestry resources are being depleted by limestone quarries, wood and paper based industries, growth of human and livestock populations, expansion of forest land for agriculture and settlement etc., threatening the ecological stability of the entire region [1]. It is therefore required a suitable harvesting plan to keep ecological balance. For controlling depletion of forestry biomass, alternative resources like synthetic, liquid wood, plastic, wood composite lumber etc. can play an important part. The following examples also motivate us to consider biomass-

How to cite this paper: Agarwal, M. and Pathak, R. (2015) Conservation of Forestry Biomass with the Use of Alternative Resource. *Open Journal of Ecology*, **5**, 87-109. http://dx.doi.org/10.4236/oje.2015.54009

industry system with alternative resource.

1) To overcome the worldwide problem of conservation of forestry resources, synthetic is a good alternative of wood based product as it is cheap, and needs not much maintenance, and the one most important thing is that it looks fresher than wood based products.

2) Plastic and wood composite lumber are quickly becoming a common replacement for redwood, cedar, and treated lumber in such applications as decking, door and window frames, and exterior moldings. Redwood and cedar decking use virgin trees, maintaining our dependence on scarce wood resources. Plastic and wood composite lumber are worked similarly to real wood and do not require treatment, yet they hold up well to water, sun, insects, and salt air, typical enemies of wood [2].

[3] proposed and analyzed a mathematical model for the survival of a resource-dependent biological population (such as human beings) where both the population and its resource were affected by a toxicant emitted into the environment from external sources as well as formed by its precursors. [4] investigated a nonlinear mathematical model to study the depletion of forestry resources caused by population and population pressure augmented industrialization. It is shown that the equilibrium density of resource biomass decreases as the equilibrium densities of population and industrialization increase. It is found that even if the growth of population (whether intrinsic or by migration) is only partially dependent on resource, still the resource biomass is doomed to extinction due to large population pressure augmented industrialization. It is noted that for sustained industrialization, control measures on its growth are required to maintain the ecological stability. In [5], they proposed a nonlinear mathematical model and analyzed to study the survival of resource-dependent competing species. It is assumed that competing species and its resource are affected simultaneously by a toxicant emitted into the environment from external sources as well as formed by precursors of competing species. It is concluded from the analysis that as the cumulative rates of emission and formation of toxicants into the environment increase, the densities of both competing species and its resource decrease. [6] studied the effect of alternative resource (synthetic) on the conservation of forestry biomass which grew logistically decays due to presence of wood based industries.

In same year, [7] studied the effect of time delay on conservation of forestry biomass by proposing a nonlinear mathematical model. They assumed that the density of forestry biomass depleted due to the presence of human population and it was being conserved by applying some technological efforts. Further, [8] and [9] investigated and concluded a nonlinear mathematical model to study the depletion of forest resources caused by population and the corresponding population pressure.

As a consequence, we propose a model for the interaction of forestry biomass with industrialization pressure, toxicant pressure and applied technological effort. Further, the effect of alternative resource on the growth of forestry biomass is seen. The time delay is the inherent property of the dynamical systems and plays an important role in almost all branches of science and particularly in the biological sciences. In the further study of the model, we see the effect of time delay on the growth rate of forestry biomass. The rest of this paper is organized as follows: In Section 2, we analyze our model with regard to equilibria and their positive conditions. In Section 3, we investigate the stability of positive equilibrium and stability and direction of Hopf bifurcation. In Section 4, some numerical supports are carried out to justify the analytic results obtained in the manuscript. Section 5 deals with the conclusions of the paper.

2. Mathematical Model

We consider the following system of differential equations:

$$\begin{aligned} \frac{\mathrm{d}B}{\mathrm{d}t} &= rB\left(1-\frac{B}{K}\right)-\frac{\alpha_1 BI}{1+B}, \\ \frac{\mathrm{d}I}{\mathrm{d}t} &= r_1 I\left(1-\frac{I}{L}\right)+\frac{\alpha_2 BI}{1+B}, \end{aligned} \tag{1}$$

where $B(0)=B_0 \ge 0$, $I(0)=I_0 \ge 0$.

In model system (1), $B(t)$ and $I(t)$ are the concentration of forestry biomass and industries, respectively. r and K are intrinsic growth rate and carrying capacity of biomass and r_1, L are intrinsic growth and carrying capacity of industries, respectively. α_1 and α_2 represents the depletion rate of forest biomass and growth rate of industries in presence of forestry biomass. In the above system (1), growths of industries are based on forestry biomass. For controlling depletion of forestry biomass, alternative resources can play an important role. Using alternative resource (A_r), the model (1) can be formulated as

$$\begin{aligned}\frac{\mathrm{d}B}{\mathrm{d}t} &= rB\left(1-\frac{B}{K}\right)-\frac{\alpha_1 BIA_r}{1+B},\\ \frac{\mathrm{d}I}{\mathrm{d}t} &= r_1 I\left(1-\frac{I}{L}\right)+\frac{\alpha_2 BIA_r}{1+B}+(1-A_r)I,\end{aligned} \tag{2}$$

where $B(0)=B_0\ge 0$, $I(0)=I_0\ge 0$.

Here A_r is a time independent constant and its origin is the alternative resource. If $A_r=1$, the industries depend only on the forestry biomass and thus it is clear that the system (1) is special case of system (2). If $A_r=0$ then both the forestry biomass and industries grow without any interaction. In such case, the industries pressure on forestry biomass is completely removed and industries evolve in presence of alternative food only. But such decoupled system is out of our interest. For neglecting above both cases, A_r always lies between 0 and 1 in our system. Due to advancement in technology and industrialization at rapid pace, large amount of toxicants enter into both aquatic ant terrestrial environment and affect biomass. Let us assume that $T(t)$ is the concentration of toxicant in the environment at time t. Emission of toxicant into the environment from various external sources and industries is Q_0 and Q_1. The constant δ_0 is the natural washout rate coefficient of toxicant present in the environment, γ_1 and γ_2 are the depletion rate coefficients of toxicant concentration in the environment due to its uptake by the forestry biomass. After adding this in system (2), our extend model is as follows:

$$\begin{aligned}\frac{\mathrm{d}B}{\mathrm{d}t} &= rB\left(1-\frac{B}{K}\right)-\frac{\alpha_1 BIA_r}{1+B}-\gamma_1 BT,\\ \frac{\mathrm{d}I}{\mathrm{d}t} &= r_1 I\left(1-\frac{I}{L}\right)+\frac{\alpha_2 BIA_r}{1+B}+(1-A_r)I,\\ \frac{\mathrm{d}T}{\mathrm{d}t} &= Q(I)-\delta_0 T-\gamma_2 BT.\end{aligned} \tag{3}$$

Here $Q(I)=Q_0+Q_1 I$.

Where $B(0)=B_0\ge 0$, $I(0)=I_0\ge 0$, $T(0)=T_0\ge 0$.

The system (3) is further modified when the technological effort (T_c) is applied to conserve the biomass. Thus system (3) become as:

$$\begin{aligned}\frac{\mathrm{d}B}{\mathrm{d}t} &= rB\left(1-\frac{B}{K}\right)-\frac{\alpha_1 BIA_r}{1+B}-\gamma_1 BT+\phi_1 BT_c,\\ \frac{\mathrm{d}I}{\mathrm{d}t} &= r_1 I\left(1-\frac{I}{L}\right)+\frac{\alpha_2 BIA_r}{1+B}+(1-A_r)I,\\ \frac{\mathrm{d}T}{\mathrm{d}t} &= Q(I)-\delta_0 T-\gamma_2 BT,\\ \frac{\mathrm{d}T_c}{\mathrm{d}t} &= \phi\left(K-B(t-\tau)\right)-\phi_0 T_c\end{aligned} \tag{4}$$

where $B(\gamma)=B_0\ge 0$ for $\gamma\in[-\tau,0]$, $I(0)=I_0\ge 0$, $T(0)=T_0\ge 0$, $T_c(0)=T_{c0}\ge 0$.

Here T_c is the measure of effort due to technology applied for conservation of forestry and ϕ_1 is the growth rate coefficient of forestry biomass due to technological effort. The constant ϕ is the growth-rate coefficient of technological efforts and ϕ_0 is the natural depletion-rate coefficient of technological effort.

Lemma: The region of attraction for the model system (4) is given by the set:

$$\Omega=\left\{(B,I,T,T_c):0\le B\le B_m,0\le I\le I_m,0\le T\le T_m,0\le T_c\le T_{c_m}\right\}$$

where $B_m=\frac{K(r\phi_0+\phi_1\phi K)}{\phi_0 r}$, $I_m=\frac{L(r_1+1-A_r+\alpha_2 B_m A_r)}{r_1}$, $T_m=\frac{Q_0+Q_1 I_m}{\delta_0}$ and $T_{c_m}=\frac{\phi K}{\phi_0}$ and it attracts all solutions initiating in the interior of the positive octant.

Equilibrium analysis: It can be checked that system (4) has four nonnegative equilibria namely, $E_0\left(0,0,\frac{Q_0}{\delta_0},\frac{\phi K}{\phi_0}\right)$, $E_1\left(\bar{B},0,\bar{T},\bar{T}_c\right)$, $E_2\left(0,\tilde{I},\tilde{T},\tilde{T}_c\right)$ and $E_3\left(B^*,I^*,T^*,T_c^*\right)$. The existence of the equilibrium point E_0 is obvious hence omitted. We show the existence of the other equilibria as follows:

Existence of $E_1\left(\bar{B},0,\bar{T},\bar{T}_c\right)$.

Here $\bar{B}$, $\bar{T}$ and $\bar{T}_c$ are the positive solutions of the following algebraic equations:

$$r\left(1-\frac{\bar{B}}{K}\right)-\gamma_1\bar{T}+\phi_1\bar{T}_c=0, \tag{5}$$

$$Q_0-\delta_0\bar{T}-\gamma_2\bar{B}\bar{T}=0, \tag{6}$$

$$\phi\left(K-\bar{B}\right)-\phi_0\bar{T}_c=0. \tag{7}$$

From Equation (7), we get

$$\bar{T}_c=\frac{\phi\left(K-\bar{B}\right)}{\phi_0}.$$

Thus, E_1 exists if: $\bar{B}<K$ which is obvious.
From Equation (6), we get

$$\bar{T}=\frac{Q_0}{\delta_0+\gamma_2\bar{B}}.$$

Putting the value of $\bar{T}$ and $\bar{T}_c$ in Equation (5), we get

$$\bar{B}=\frac{\left(-r\phi_0\delta_0-\phi_1\phi\delta_0K+r\phi_0\gamma_2K+\phi_1\phi K^2\gamma_2\right)}{2\left(\phi_1\phi K\gamma_2-r\phi_0\gamma_2\right)}$$
$$\pm\frac{\sqrt{\left(r\phi_0\delta_0+\phi_1\phi\delta_0K-r\phi_0\gamma_2K-\phi_1\phi K^2\gamma_2\right)^2-4\left(r\phi_0\gamma_2-\phi_1\phi K\gamma_2\right)\left(-r\phi_0\delta_0K+Q_0\gamma_1\phi_0K-\phi_1\phi K^2\delta_9\right)}}{2\left(\phi_1\phi K\gamma_2-r\phi_0\gamma_2\right)}.$$

Thus, E_2 exists if condition $K\gamma_2>\delta_0$ holds.
Existence of $E_2\left(0,\tilde{I},\tilde{T},\tilde{T}_c\right)$.
Here $\tilde{I}$, $\tilde{T}$ and $\tilde{T}_c$ are the positive solutions of the following algebraic equations:

$$r_1\left(1-\frac{\tilde{I}}{L}\right)+1-A_r=0, \tag{8}$$

$$Q_0+Q_1\tilde{I}-\delta_0\tilde{T}=0, \tag{9}$$

$$\phi K-\phi_0\tilde{T}_c=0. \tag{10}$$

From Equation (10), we get

$$\tilde{T}_c=\frac{\phi K}{\phi_0}.$$

From Equation (8), we get

$$\tilde{I}=\frac{L\left(r_1+1-A_r\right)}{r_1}.$$

Putting the value of $\tilde{I}$ in Equation (9), we get

$$\tilde{T} = \frac{r_1 Q_0 + Q_1 L\left(r_1 + 1 - A_r\right)}{\delta_0}.$$

Existence of $E_3\left(B^*, I^*, T^*, T_c^*\right)$.

Here B^*, I^*, T^* and T_c^* are the positive solutions of the following algebraic equations:

$$r\left(1 - \frac{B^*}{K}\right) - \frac{\alpha_1 I^* A_r}{1 + B^*} - \gamma_1 T^* + \phi_1 T_c^* = 0, \tag{11}$$

$$r_1\left(1 - \frac{I^*}{L}\right) + \frac{\alpha_2 B^* A_r}{1 + B^*} + \left(1 - A_r\right) = 0, \tag{12}$$

$$Q_0 + Q_1 I^* - \delta_0 T^* - \gamma_2 B^* T^* = 0, \tag{13}$$

$$\phi\left(K - B^*\right) - \phi_0 T_c^* = 0. \tag{14}$$

From Equation (14), we get

$$T_c^* = \frac{\phi\left(K - B^*\right)}{\phi_0}.$$

After simple manipulation, we get from Equation (12) is

$$I^* = \frac{L\left\{r_1 + 1 - A_r + B^*\left(r_1 + 1 - A_r + \alpha_2 A_r\right)\right\}}{r_1\left(1 + B^*\right)}.$$

Putting the value of I^* in Equation (13), we get

$$T^* = \frac{Q_0 r_1\left(1 + B^*\right) + Q_1 L\left\{r_1 + 1 - A_r + B^*\left(r_1 + 1 - A_r + \alpha_2 A_r\right)\right\}}{r_1\left(1 + B^*\right)\left(\delta_0 + \gamma_2 B^*\right)}.$$

Putting the value of T^*, T_c^* and I^* in Equation (11), we get

$$F\left(B^*\right) = A_1^* B^{*4} + A_2^* B^{*3} + A_3^* B^{*2} + A_4^* B^* + A_5^* \tag{15}$$

where

$$A_1^* = \frac{\gamma_2 \phi_0 r\, r_1}{K} + \gamma_2 \phi_1 \phi r_1,$$

$$A_2^* = \frac{\left(\delta_0 + \gamma_2\right)\phi_0 r r_1}{K} + \frac{r r_1 \phi_0 \gamma_2}{K} - r r_1 \phi_0 \gamma_2 + \delta_0 \phi_1 \phi r_1 + 2\gamma_2 \phi_1 \phi r_1 - K\gamma_2 \phi_1 \phi r_1,$$

$$\begin{aligned} A_3^* = {} & r r_1 \phi_0 \delta_0\left(\frac{2}{K} + \frac{\gamma_2}{\delta_0 K} - 1 - \frac{\gamma_2}{\delta_0}\right) + \gamma_1 Q_0 r_1 \phi_0 + \gamma_1 \phi_0\left(r_1 + 1 - A_r + \alpha_2 A_r\right) + \delta_0 \phi_1 \phi r_1 - K\delta_0 \phi_1 \phi r_1 - K\gamma_2 \phi_1 \phi r_1 \\ & - r r_1 \gamma_2 \phi_0 + \phi_1 \phi r_1 \delta_0 + \phi_1 \phi r_1 \gamma_2 - \phi_1 \phi r_1 K \gamma_2 + \alpha_1 A_r L \phi_0 \gamma_2\left(r_1 + 1 - A_r + \alpha_2 A_r\right), \end{aligned}$$

$$\begin{aligned} A_4^* = {} & r r_1 \phi_0 \delta_0\left(\frac{1}{K} - 2\right) + \gamma_1\left(r_1 + 1 - A_r\right)\phi_0 - \phi_1 \phi r_1\left(K\delta_0 + \gamma_2\right) + \gamma_1 Q_0 r_1 \phi_0 + \gamma_1 \phi_0\left(r_1 + 1 - A_r + \alpha_2 A_r\right) \\ & + \phi_1 \phi \delta_0 r_1\left(1 - K - \frac{K\gamma_2}{\delta_0}\right) + \alpha_1 A_r L \phi_0 \gamma_2\left(r_1 + 1 - A_r\right) + \gamma_1 Q_0 r_1 \phi_0 + \alpha_1 A_r L \phi_0 \delta_0\left(1 + r_1 - A_r + \alpha_2 A_r\right), \end{aligned} \tag{16}$$

$$A_5^* = \gamma_2 Q_0 \phi_0 r_1 - r r_1 \phi_0 \delta_0 + \gamma_1 Q_0 r_1 \phi_0 + \gamma_1\left(r_1 + 1 - A_r\right)\phi_0 - \phi \phi_1 r_1 K \delta_0 + \alpha_1 A_r L \phi_0 \delta_0\left(r_1 + 1 - A_r\right).$$

$$F\left(0\right) = A_5^* > 0, \text{ if } \gamma_2 Q_0 > r\delta_0 ? \quad \gamma_1 Q_0 \phi_0 > \phi\phi_1 K \delta_0,$$

$$F\left(K\right) = A_1^* K^4 + A_2^* K^3 + A_3^* K^2 + A_4^* K + A_5^* < 0.$$

We note that $F\left(0\right) > 0$ and $F\left(K\right) < 0$, showing the existence of B^* in the interval $0 < B^* < K$. Now, the sufficient condition for B^* to be unique positive real is $F'\left(B^*\right)<0$ at E_3, where

$$F'\left(B^*\right)=4A_1^*B^{*3}+3A_2^*B^{*2}+2A_3^*B^*+A_4^*<0\,.$$

Remark 1. From Equation (15), it is easy to note that $\frac{\mathrm{d}B^*}{\mathrm{d}\phi}>0$ and $\frac{\mathrm{d}B^*}{\mathrm{d}A_r}>0$, which implies that the equilibrium density of forestry biomass increases as the growth rate coefficient of technological efforts and value of alternative resources increases.

2.1. Local Stability Analysis without Delay, (*i.e.* $\tau=0$)

To discuss the local stability of system (4), we compute the variational matrix of system (4). The entries of general variational matrix are given by differentiating the right side of system (4) with respect to B, I, T and T_c *i.e.*

$$V(E)=\begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & 0 & 0 \\ t_{31} & t_{32} & t_{33} & 0 \\ t_{41} & 0 & 0 & t_{44} \end{pmatrix}$$

where

$$t_{11}=r-\frac{2rB}{K}-\gamma_1T+\phi_1T_c-\frac{\alpha_1IA_r}{(1+B)^2},\quad t_{12}=-\frac{\alpha BA_r}{1+B},\quad t_{13}=-\gamma_1B,\quad t_{14}=\phi_1B,\quad t_{21}=\frac{\alpha_2IA_r}{(1+B)^2},$$

$$t_{22}=r_1-\frac{2r_1I}{L}+\frac{\alpha_2BA_r}{1+B}+1-A_r,\quad t_{31}=-\gamma_2T,\quad t_{32}=Q_1,\quad t_{33}=-\delta_0-\gamma_2B,\quad t_{41}=-\phi,\quad t_{44}=-\phi_0.$$

The variational matrix $V\left(E_0\right)$ at equilibrium point E_0 is given by

$$V\left(E_0\right)=\begin{pmatrix} r-\frac{\gamma_1Q_0}{\delta_0}+\frac{\phi_1\phi K}{\phi_0} & 0 & 0 & 0 \\ 0 & r_1+1-A_r & 0 & 0 \\ -\frac{\gamma_2Q_0}{\delta_0} & Q_1 & -\delta_0 & 0 \\ -\phi & 0 & 0 & -\phi_0 \end{pmatrix}.$$

The eigenvalues of matrix $V\left(E_0\right)$ in the direction of T and T_c are negative. So E_0 has stable manifold in $T-T_c$ plane and unstable manifold in I-direction. The equilibrium point E_0 is stable manifold in $B-T-T_c$ plane if $\delta_0\left(\phi_0r+\phi_1\phi K\right)>\gamma_1Q_0$ otherwise it becomes unstable in $B-T-T_c$ plane.

The variational matrix $V\left(E_1\right)$ at equilibrium point E_1 is given by

$$V\left(E_1\right)=\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & 0 & 0 & a_{44} \end{pmatrix}$$

where

$$a_{11}=r-\frac{2r\bar{B}}{K}-\gamma_1\bar{T}+\phi_1\bar{T}_c,\quad a_{12}=-\frac{\alpha\bar{B}A_r}{1+\bar{B}},\quad a_{13}=-\gamma_1\bar{B},\quad a_{14}=\phi_1\bar{B},$$

$$a_{22}=r_1+\frac{\alpha_2\bar{B}A_r}{1+\bar{B}}+1-A_r,\quad a_{31}=-\gamma_2\bar{T},\quad a_{32}=Q_1,\quad a_{33}=-\delta_0-\gamma_2\bar{B},\quad a_{41}=-\phi,\quad a_{44}=-\phi_0.$$

The characteristic equation corresponding to the variational matrix $V\left(E_1\right)$ is given by

$$\lambda^4+C_1\lambda^3+C_2\lambda^2+C_3\lambda+C_4=0\,, \tag{17}$$

where

$$C_1 = -a_{11} - a_{22} - a_{33} - a_{44},$$
$$C_2 = a_{33}a_{44} + a_{22}a_{44} + a_{11}a_{44} + a_{22}a_{33} + a_{11}a_{33} + a_{11}a_{22} - a_{13}a_{31} - a_{14}a_{41},$$
$$C_3 = -a_{22}a_{33}a_{44} - a_{11}a_{33}a_{44} - a_{11}a_{22}a_{44} - a_{11}a_{22}a_{33} + a_{13}a_{31}a_{44} + a_{22}a_{13}a_{31} + a_{14}a_{41}a_{33} + a_{14}a_{41}a_{22},$$
$$C_4 = a_{11}a_{22}a_{33}a_{44} - a_{13}a_{31}a_{22}a_{44} - a_{22}a_{33}a_{14}a_{41}.$$

According to Routh-Hurwitz criterion, equilibrium point E_1 is locally asymptotically stable provided the following conditions are satisfied

$$C_1 > 0, \quad C_2 > 0, \quad C_3 > 0, \quad C_4 > 0, \quad C_1C_2 - C_3 > 0, \quad C_3\left(C_1C_2 - C_3\right) - \left(C_1\right)^2 C_4 > 0.$$

The variational matrix $V(E_2)$ at equilibrium point E_2 is given by

$$V(E_2) = \begin{pmatrix} b_{11} & 0 & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 \\ b_{41} & 0 & 0 & b_{44} \end{pmatrix}$$

where

$$b_{11} = r - \gamma_1\left[\frac{Q_0}{\delta_0} + \frac{Q_1L\left(r_1 + 1 - A_r\right)}{r_1\delta_0}\right] + \frac{\phi_1\phi K}{\phi_0} - \frac{\alpha_1 A_r L\left(r_1 + 1 - A_r\right)}{r_1},$$
$$b_{21} = \alpha_2\tilde{I}A_r, \quad b_{22} = -\left(r_1 + 1 - A_r\right),$$
$$b_{31} = -\gamma_2\tilde{T}, \quad b_{32} = Q_1, \quad b_{33} = -\delta_0, \quad b_{41} = -\phi, \quad b_{44} = -\phi_0.$$

The variational matrix $V(E_2)$ has four eigenvalues. The sign of three eigenvalues b_{22}, b_{23} and b_{44} are negative so the stability of equilibrium point E_2 depends on sign of b_{11}. The equilibrium point E_2 is stable manifold in $B - I - T - T_c$ plane if $rr_1\delta_0 > \gamma_1\left\{Q_0 r_1 + Q_1 L\left(r_1 + 1 - A_r\right)\right\}$ otherwise it is unstable in $B - I - T - T_c$ plane.

The variational matrix $V(E_3)$ at equilibrium point E_3 is given by

$$V(E_3) = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & 0 & 0 \\ c_{31} & c_{32} & c_{33} & 0 \\ c_{41} & 0 & 0 & c_{44} \end{pmatrix}$$

where

$$c_{11} = r - \frac{2rB^*}{K} - \gamma_1 T^* + \phi_1 T_c^* - \frac{\alpha_1 I^* A_r}{\left(1 + B^*\right)^2}, \quad c_{12} = -\frac{\alpha B^* A_r}{1 + B^*}, \quad c_{13} = -\gamma_1 B^*, \quad c_{14} = \phi_1 B^*, \quad c_{21} = \frac{\alpha_2 I^* A_r}{\left(1 + B^*\right)^2},$$
$$c_{22} = r_1 - \frac{2r_1 I^*}{L} + \frac{\alpha_2 B^* A_r}{1 + B^*} + 1 - A_r, \quad c_{31} = -\gamma_2 T^*, \quad c_{32} = Q_1, \quad c_{33} = -\delta_0 - \gamma_2 B^*, \quad c_{41} = -\phi, \quad c_{44} = -\phi_0.$$

The characteristic equation corresponding to the variational matrix $V(E_3)$ is given by

$$\lambda^4 + G_1\lambda^3 + G_2\lambda^2 + G_3\lambda + G_4 = 0, \tag{18}$$

where

$$G_1 = -c_{11} - c_{22} - c_{33} - c_{44},$$
$$G_2 = c_{33}c_{44} + c_{22}c_{44} + c_{11}c_{44} + c_{22}c_{33} + c_{11}c_{33} + c_{11}c_{22} - c_{13}c_{31} - c_{14}c_{41} - c_{13}c_{31},$$
$$G_3 = -c_{22}c_{33}c_{44} - c_{11}c_{33}c_{44} - c_{11}c_{22}c_{44} - c_{11}c_{22}c_{33} + c_{12}c_{21}c_{44} + c_{12}c_{21}c_{33} - c_{13}c_{12}c_{32}$$
$$+ c_{13}c_{31}c_{44} + c_{22}c_{13}c_{31} + c_{14}c_{41}c_{33} + c_{14}c_{41}c_{22},$$
$$G_4 = c_{11}c_{22}c_{33}c_{44} - c_{12}c_{21}c_{33}c_{44} + c_{13}c_{21}c_{32}c_{44} - c_{13}c_{31}c_{22}c_{44} - c_{22}c_{33}c_{14}c_{41}.$$

According to Routh-Hurwitz criterion, equilibrium point E_3 is locally asymptotically stable provided the following conditions are satisfied

$$G_1 > 0,\quad G_2 > 0,\quad G_3 > 0,\quad G_4 > 0,\quad G_1G_2 - G_3 > 0,\quad G_3\left(G_1G_2 - G_3\right) - \left(G_1\right)^2 G_4 > 0\,.$$

2.2. Local Stability Analysis with Delay, (*i.e.* $\tau \neq 0$)

To discuss the stability behavior of equilibrium E_3 of system (4) with time delay, (*i.e.* $\tau \neq 0$), we linearize system (4) by using the following transformations:

$$B = B^* + b,\quad I = I^* + i,\quad T = T^* + a,\quad T_c = T_c^* + c$$

where b, i, a and c are small perturbations around the equilibrium E_3.

The linearized system of system (4) about E_3 is given by:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = M_1 u(t) + M_2 u(t-\tau) \tag{19}$$

where $u(t) = \left[b(t), i(t), a(t), c(t)\right]$,

$$M_1 = \begin{bmatrix} -B^*\left(\dfrac{r}{K} - \dfrac{A_r\alpha_1 I^*}{\left(1+B^*\right)^2}\right) & -\dfrac{\alpha_1 B^* A_r}{1+B^*} & -\gamma_1 B^* & \phi_1 B^* \\ \dfrac{\alpha_2 A_r I^*}{1+B^*} - \dfrac{\alpha_2 A_r B^* I^*}{\left(1+B^*\right)^2} & -\dfrac{r_1 I^*}{L} & 0 & 0 \\ -\gamma_2 T^* & Q_1 & -\left(\delta_0 + \gamma_2 B^*\right) & 0 \\ 0 & 0 & 0 & -\phi_0 \end{bmatrix}$$

and

$$M_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\phi & 0 & 0 & 0 \end{bmatrix}.$$

The characteristic equation for linearized system (19) is obtained as:

$$\psi^4 + p_1\psi^3 + p_2\psi^2 + p_3\psi + p\left(q_1\psi^2 + q_2\psi + q_3\right)\mathrm{e}^{-\psi\tau} = 0 \tag{20}$$

where

$$p_1 = \phi_0 + \delta_0 + \gamma_2 B^* + B^*\left(\frac{r}{K} - \frac{A_r\alpha_1 I^*}{\left(1+B^*\right)^2}\right) + \frac{r_1 I^*}{L},$$

$$p_2 = \phi_0\left(\delta_0 + \gamma_2 B^*\right) + \phi_0\left[\frac{r_1 I^*}{L} + B^*\left(\frac{r}{K} - \frac{A_r\alpha_1 I^*}{\left(1+B^*\right)^2}\right)\right] - \left[\frac{r_1 I^*}{L} + B^*\left(\frac{r}{K} - \frac{A_r\alpha_1 I^*}{\left(1+B^*\right)^2}\right)\right]\left(\delta_0 + \gamma_2 B^*\right)$$
$$+ \frac{r_1 I^*}{L} B^*\left(\frac{r}{K} - \frac{A_r\alpha_1 I^*}{\left(1+B^*\right)^2}\right) + \frac{\alpha_1\alpha_2 B^* I^* A_r^2}{\left(1+B^*\right)^2} - \gamma_2 T^*\gamma_1 B^*,$$

$$p_3 = \phi_0\left[B^*\left(\frac{r}{K} - \frac{A_r\alpha_1 I^*}{\left(1+B^*\right)^2}\right) + \frac{r_1 I^*}{L}\right]\left(\delta_0 + \gamma_2 B^*\right) + \frac{r_1 I^* B^*}{L}\left(\frac{r}{K} - \frac{A_r\alpha_1 I^*}{\left(1+B^*\right)^2}\right)\left(\phi_0 + \delta_0 + \gamma_2 B^*\right)$$
$$+ \frac{\alpha_1\alpha_2 B^* A_r^2 I^*}{\left(1+B^*\right)^2}\left(\phi_0 + \delta_0 + \gamma_2 B^*\right) + \frac{\gamma_1 B^*\alpha_2 A_r I^* Q_1}{1+B^*} - \gamma_1\gamma_2 B^* T^*\phi_0 - \frac{\gamma_1\gamma_2 r_1 B^* I^* T^*}{L},$$

$$p_4 = \phi_0\left[\left(\delta_0+\gamma_2 B^*\right)\left\{\frac{r_1 I^* B^*}{L}\left(\frac{r}{K}-\frac{A_r\alpha_1 I^*}{\left(1+B^*\right)^2}\right)+\frac{\alpha_1 B^* A_r}{1+B^*}\left(\frac{\alpha_2 A_r I^*}{1+B^*}-\frac{\alpha_2 A_r B^* I^*}{\left(1+B^*\right)^2}\right)\right\}\right.$$

$$\left.-\gamma_1 B^*\left\{\frac{r_1 I^*\gamma_2 T^*}{L}-Q_1\left(\frac{\alpha_2 A_r I^*}{1+B^*}-\frac{\alpha_2 A_r B^* I^*}{\left(1+B^*\right)^2}\right)\right\}\right],$$

$$q_1 = -\phi_1\phi B^*,\quad q_2 = -\phi_1\phi B^*\left(\frac{r_1 I^*}{L}+\delta_0+\gamma_2 B^*\right),\quad q_3 = -\frac{\phi_1\phi r_1 B^* I^*\left(\delta_0+\gamma_2 B^*\right)}{L}.$$

Let $\psi = i\omega$ be one such root. Substituting this in Equation (20) and equating real and imaginary parts, we get

$$\left(q_3+q_1\omega^2\right)\cos\omega\tau+q_2\omega\sin\delta\tau = p_2\omega^2-\omega^4-p_4, \tag{21}$$

$$q_2\omega\cos\delta\tau-\left(q_3+q_1\omega^2\right)\sin\delta\tau = p_1\omega^3+p_2\omega. \tag{22}$$

Squaring and adding Equations (21) and (22), we get

$$\omega^8+\bar{D}_1\omega^6+\bar{D}_2\omega^4+\bar{D}_3\omega^2+\bar{D}_4=0, \tag{23}$$

where

$$\bar{D}_1 = p_1^2-2p_2,\quad \bar{D}_2 = p_2^2+2p_4+2p_1p_3-q_1^2,\quad \bar{D}_3=-2p_2p_4+p_3^2+2q_1q_3-q_2^2,\quad \bar{D}_4 = p_4^2-q_3^2.$$

Substituting $\omega^2=\sigma$ Equation (23) becomes

$$F(\sigma)=\sigma^4+\bar{D}_1\sigma^3+\bar{D}_2\sigma^2+\bar{D}_3\sigma+\bar{D}_4=0, \tag{24}$$

$$F(0)=\bar{D}_4 = p_4^2-q_3^2.$$

We assume that:

$$(\mathrm{H}_1):\quad p_4<q_3.$$

We notice that F is continuous everywhere with $F(0)<0$ when condition (H_1) holds and $F(\infty)>0$. Therefore, the Equation (24) always has at least one positive root. Consequently, the stability criteria of the system for $\tau=0$ will not necessarily ensure the stability of the system for $\tau\neq 0$. We assume the Equation (24) has four positive roots denoted by $\sigma_1,\ \sigma_2,\ \sigma_3,\ \sigma_4$ denoted as:

$$\omega_1=\sqrt{\sigma_1},\quad \omega_2=\sqrt{\sigma_2},\quad \omega_3=\sqrt{\sigma_3},\quad \omega_4=\sqrt{\sigma_4}.$$

Again solving (21) and (22), we get a critical value of delay given as follows

$$\tau_k=\frac{1}{\omega}\cos^{-1}\left[\frac{q_2\omega\left(p_1\omega^3+p_3\omega\right)+\left(q_3-q_1\omega^2\right)\left(p_2\omega^2-\omega^4-p_4\right)}{q_2^2\omega^2+\left(q_3-q_1\omega^2\right)^2}\right]+\frac{2k\pi}{\omega},\quad k=0,1,2,\cdots.$$

2.2.1. Hopf Bifurcation

To investigate the behavior of the system (4) in the neighborhood of τ_0. We represent the following theorem.

Theorem:

We observe that the conditions for Hopf bifurcation are satisfied yielding the required periodic solution, that is,

$$\left[\frac{\mathrm{d}(\mathrm{Re}\,\psi)}{\mathrm{d}\tau}\right]_{\tau=\tau_0}\neq 0.$$

This signifies that there exists at least one eigenvalue with positive real part for $\tau>\tau_0$.

Proof:

Differentiating Equation (20) with respect to τ, we obtain

$$\left[4\psi^3+3p_1\psi^2+2p_2\psi+p_3+e^{-\psi\tau}\left(2q_1\psi+q_2\right)-\tau e^{-\psi\tau}\left(q_1\psi^2+q_2\psi+q_3\right)\right]\frac{d\psi}{d\tau}=\psi e^{-\psi\tau}\left(q_1\psi^2+q_2\psi+q_3\right). \quad (25)$$

Therefore

$$\left(\frac{d\psi}{d\tau}\right)^{-1}=\frac{4\psi^3+3p_1\psi^2+2p_2\psi+p_3+e^{-\psi\tau}\left(2q_1\psi+q_2\right)}{\psi e^{-\psi\tau}\left(q_1\psi^2+q_2\psi+q_3\right)}-\frac{\tau}{\psi}. \quad (26)$$

We can obtain here

$$\left[\frac{d\left(\mathrm{Re}\psi\right)}{d\tau}\right]_{\tau=\tau_0}\neq 0. \quad (27)$$

Verifying numerically it has been obtained that the transversality condition holds and hence Hopf bifurcation occurs at $\tau=\tau_0$.

2.2.2. Stability and Direction of Periodic Solutions

In this section, we will derive explicit formulae for determining the direction, stability and period of the bifurcating periodic solutions arises through Hopf bifurcation. The method we will follow is based on the normal form theory and center manifold theorem as given in [10]. Without loss of generality we denote any of the critical values of τ by τ_k at which Equation (20) has a pair of purely imaginary roots $\pm i\omega_0$ and system undergoes Hopf bifurcation. Hence for any root of Equation (20) of the form $\psi(\tau)=\nu(\tau)+i\omega(\tau)$, $\nu(\tau_k)=0$, $\omega(\tau_k)=\omega_0$ and $\left.\frac{d\nu}{d\tau}\right|_{\tau=\tau_k}\neq 0$. Let $\tau=\tau_k+\mu$, $\mu\in R$, so that $\mu=0$ is Hopf bifurcation value for the system.

Define the space of continuous real valued functions as $C=C\left([-1,0],R^4\right)$. Using the transformation $u_1(t)=B(t)-B^*$, $u_2(t)=I(t)-I^*$, $u_3(t)=T(t)-T^*$ and $u_4(t)=T_c(t)-T_c^*$ and $\chi_i(t)=u_i(\tau t)$ for $i=1,2,3,4$; the delay model system (4), then transform to FDE in C as,

$$\frac{d\chi}{dt}=L_\mu\chi_t+f\left(\mu,\chi_t\right) \quad (28)$$

where $\chi(t)=\left(\chi_1(t),\chi_2(t),\chi_3(t),\chi_4(t)\right)^{\mathrm{T}}\in R^4$, $\chi_t(\Theta)=\chi(t+\Theta)$, $\Theta\in[-1,0]$ and $L_\mu:C\to R^4$ and $f:C\times R\to R^4$ are given by

$$L_\mu\zeta=\left(\tau_k+\mu\right)\left[M_1\zeta(0)+M_2\zeta(-1)\right] \quad (29)$$

and

$$f(\mu,\zeta)=\left(\tau_k+\mu\right)\begin{pmatrix}Z_1\\ Z_2\\ Z_3\\ 0\end{pmatrix} \quad (30)$$

where

$$Z_1=\left(\frac{\alpha_1 A_r I^*}{\left(1+B^*\right)^2}-\frac{r}{K}\right)\zeta_1^2(0)+\left(\frac{\alpha_1 A_r B^*}{\left(1+B^*\right)^2}-\frac{\alpha_1 A_r}{1+B^*}\right)\zeta_1(0)\zeta_2(0)+\frac{\alpha_1 A_r}{\left(1+B^*\right)^2}\zeta_1^2(0)\zeta_2(0)-\gamma_1\zeta_1(0)\zeta_3(0)$$
$$+\phi_1\zeta_1(0)\zeta_4(0),$$

$$Z_2=-\frac{r_1\zeta_2^2(0)}{L}-\frac{\alpha_2 A_r I^*}{\left(1+B^*\right)^2}\zeta_1^2(0)+\left(\frac{\alpha_2 A_r}{1+B^*}-\frac{\alpha_2 A_r B^*}{\left(1+B^*\right)^2}\right)\zeta_1(0)\zeta_2(0)-\frac{\alpha_2 A_r\zeta_1^2(0)\zeta_2(0)}{\left(1+B^*\right)^2},$$

$$Z_3 = -\gamma_2 \zeta_1(0)\zeta_3(0),$$
$$Z_4 = 0.$$

For $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)^{\mathrm{T}} \in C$.

By the Reisz representation theorem there exists a function $\eta(\Theta, \mu)$ whose components are of bounded variation for $\Theta \in [-1, 0]$ such that

$$L_\mu \zeta = \int_0^{-1} \mathrm{d}\eta(\Theta, \mu)\zeta(\Theta). \tag{31}$$

In view of Equation (29) we can choose

$$\eta(\Theta, \mu) = (\tau_k + \mu)\left[M_1\delta(\Theta) - M_2\delta(\Theta + 1)\right], \tag{32}$$

where $\zeta \in C^1([-1, 0], R^4)$, define

$$A(\mu)\zeta = \begin{cases} \dfrac{\mathrm{d}\zeta(\Theta)}{\mathrm{d}\Theta}, & \Theta \in [-1, 0), \\ \int_{-1}^{0} \mathrm{d}\eta(\rho, \mu)\zeta(\rho) = L_\mu \zeta, & \Theta = 0, \end{cases} \tag{33}$$

and

$$R(\mu)\zeta = \begin{cases} 0, & \Theta \in [-1, 0), \\ f(\zeta, \mu), & \Theta = 0. \end{cases} \tag{34}$$

The system (28) is the equivalent to

$$\dot{\chi}_t = A(\mu)\chi_t + R(\mu)\chi_t, \tag{35}$$

where $\chi_t(\Theta) = \chi(t + \Theta)$ for $\Theta \in [-1, 0]$.

For $\psi \in C^1\left([-1, 0], (R^4)^*\right)$, define

$$A^*\xi(\rho) = \begin{cases} -\dfrac{\mathrm{d}\xi(\rho)}{\mathrm{d}s}, & \rho \in (0, 1], \\ \int_{-1}^{0} \mathrm{d}\eta^{\mathrm{T}}(t, 0)\xi(-t) & \rho = 0. \end{cases} \tag{36}$$

and a bilinear inner product

$$\langle \xi, \zeta \rangle = \bar{\xi}(0)\cdot\zeta(0) - \int_{\Theta=-1}^{0}\int_{\nu=0}^{\Theta} \bar{\xi}^{\mathrm{T}}(\nu - \Theta)\mathrm{d}\eta(\Theta)\zeta(\nu)\mathrm{d}\nu \tag{37}$$

where $\eta(\Theta) = \eta(\Theta, 0)$. Then $A(0)$ (from here onwards we shall refer $A(0)$ by A) and A^* are adjoint operators. We know that $\pm i\omega_0\tau_k$ are the eigenvalues of A. Thus, they are also eigenvalues of A^*. We need to compute eigenvectors of A and A^* corresponding to $+i\omega_0\tau_k$ and $-i\omega_0\tau_k$ respectively.

Suppose $q(\Theta) = (1, a_1, a_2, a_3)^{\mathrm{T}} \mathrm{e}^{i\omega_0\tau_k\Theta}$ be the eigenvector of A corresponding to eigenvalues $i\omega_0\tau_k$ then

$$Aq(\Theta) = i\omega_0\tau_k q(\Theta), \tag{38}$$

which for $\Theta = 0$, gives

$$\tau_k \begin{bmatrix} i\omega_0 + B^*\left(\dfrac{r}{K} - \dfrac{A_r\alpha_1 I^*}{(1+B^*)^2}\right) & \dfrac{\alpha_1 B^* A_r}{1+B^*} & \gamma_1 B^* & -\phi_1 B^* \\ -\dfrac{\alpha_2 A_r I^*}{1+B^*} + \dfrac{\alpha_2 A_r B^* I^*}{(1+B^*)^2} & i\omega_0 + \dfrac{r_1 I^*}{L} & 0 & 0 \\ \gamma_2 T^* & Q_1 & i\omega_0 + \delta_0 + \gamma_2 B^* & 0 \\ \phi \mathrm{e}^{-i\omega_0\tau_k} & 0 & 0 & i\omega_0 + \phi_0 \end{bmatrix} q(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{39}$$

Solving the system of Equation (39), we get

$$a_1=\frac{\alpha_2 A_r I^* L}{\left(1+B^*\right)^2\left(i\omega_0 L+r_1 I^*\right)},\quad a_2=-\frac{\gamma_2 T^*}{i\omega_0+\delta_0+\gamma_2 B^*}-\frac{\alpha_2 A_r L I^* Q_1}{\left(1+B^*\right)^2\left(i\omega_0 L+r_1 I^*\right)\left(i\omega_0+\delta_0+\gamma_2 B^*\right)}$$

and $a_3=-\dfrac{\phi \mathrm{e}^{-i\omega_0\tau_k}}{i\omega_0+\phi_0}$.

Similarly calculating $q^*(\rho)=\left(1,a_1^*,a_2^*,a_3^*\right)^{\mathrm{T}}\mathrm{e}^{i\omega_0\tau_k\rho}$ such that

$$A^*q^*(\rho)=-i\omega_0\tau_k q^*(\rho), \tag{40}$$

where

$$a_1^*=-\frac{\gamma_1 B^* Q_1}{\left(-i\omega_0+\dfrac{r_1 I^*}{L}\right)\left(i\omega_0-\delta_0-\gamma_2 B^*\right)}-\frac{\alpha_1 B^* A_r}{\left(1+B^*\right)\left(-i\omega_0+\dfrac{r_1 I^*}{L}\right)},\quad a_2^*=\frac{\gamma_1 B^*}{i\omega_0-\delta_0-\gamma_2 B^*}$$

$$a_3^*=-\frac{\phi_1 B^*}{\phi_0-i\omega_0}.$$

Now the normalization condition gives

$$\left\langle q^*(\rho)\cdot q(\Theta)\right\rangle=1,$$

$$\bar{q}^*(0)\cdot q(0)-\bar{D}\int_{\Theta=-1}^{0}\int_{\nu=0}^{\Theta}\bar{q}^*(0)\mathrm{e}^{-i\omega_0\tau_k(\nu-\Theta)}\mathrm{d}\eta(\Theta)\times q(0)\mathrm{e}^{i\omega_0\tau_k\nu}\mathrm{d}\nu=1$$

$$\bar{D}\left[1+a_1\bar{a}_1^*+a_2\bar{a}_2^*+a_3\bar{a}_3^*-\mathrm{e}^{i\omega_0\tau_k}\tau_k\phi\bar{a}_3^*\right]=1.$$

Thus, $\bar{D}$ is so chosen such that

$$\bar{D}=\frac{1}{1+a_1\bar{a}_1^*+a_2\bar{a}_2^*+a_3\bar{a}_3^*-\mathrm{e}^{i\omega_0\tau_k}\tau_k\phi\bar{a}_3^*}. \tag{41}$$

Proceeding same as [10] and using same notation, we compute the coordinates to describe the center manifold C_0 at $\mu=0$. Let χ_t be solution of Equation (35) when $\mu=0$. Define

$$Z(t)=\left\langle q^*,\chi_t\right\rangle,\quad W(t,\Theta)=\chi_t(\Theta)-2\mathrm{Re}\left\{Z(t)q(\Theta)\right\}. \tag{42}$$

On the center manifold C_0, we have

$$W(t,\Theta)=W\left(Z,\bar{Z},\Theta\right) \tag{43}$$

$$W\left(z,\bar{z},\Theta\right)=W_{20}(\Theta)\frac{z^2}{2}+W_{11}(\Theta)z\bar{z}+W_{02}(\Theta)\frac{\bar{z}^2}{2}+\cdots. \tag{44}$$

z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$. Note that W is real if χ_t is real. We only consider real solutions. For solution $\chi_t\in C_0$ of Equation (35). Since $\mu=0$, we have

$$\begin{aligned}\dot{z}&=i\omega_0\tau_k z+\bar{q}^*(0)\cdot f\left(0,W\left(z,\bar{z},0\right)+2\mathrm{Re}\left\{zq(0)\right\}\right)\\&=i\omega_0\tau_k z+\bar{q}^*(0)\cdot f_0\left(z,\bar{z}\right).\end{aligned} \tag{45}$$

We rewrite this equation as

$$\dot{z}=i\omega_0\tau_k z+g\left(z,\bar{z}\right), \tag{46}$$

where

$$g(z,\bar{z}) = \bar{q}^*(0)\cdot f_0(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \tag{47}$$

It follows from (42) and (44) that

$$\chi_t(\Theta) = W(z,\bar{z},\Theta) + 2\mathrm{Re}\{zq(\Theta)\}, \tag{48}$$

$$= W_{20}(\Theta)\frac{z^2}{2} + W_{11}(\Theta)z\bar{z} + W_{02}(\Theta)\frac{\bar{z}^2}{2} + z(1,a_1,a_2,a_3)^{\mathrm{T}}\,\mathrm{e}^{i\omega_0\tau_k\Theta} + \bar{z}(1,\bar{a}_1,\bar{a}_2,\bar{a}_3)^{\mathrm{T}}\,\mathrm{e}^{-i\omega_0\tau_k\Theta} + \cdots. \tag{49}$$

Also we have

$$g(z,\bar{z}) = \bar{q}^*(0)\cdot f_0(0,\chi_t) = \tau_k\bar{D}\left(1,\bar{a}_1^*,\bar{a}_2^*,\bar{a}_3^*\right)^{\mathrm{T}}\begin{pmatrix}V_1\\V_2\\V_3\\0\end{pmatrix}$$

where

$$V_1 = -\left(\frac{r}{K} - \frac{\alpha_1 A_r I^*}{\left(1+B^*\right)^2}\right)\chi_{1t}^2(0) + \left\{\frac{\alpha_1 A_r B^*}{\left(1+B^*\right)^2} - \frac{\alpha_1 A_r}{1+B^*}\right\}\chi_{1t}(0)\chi_{2t}(0) + \frac{\alpha_1 A_r}{\left(1+B^*\right)^2}\chi_{1t}^2(0)\,\chi_{2t}(0)$$
$$-\gamma_1\chi_{1t}(0)\chi_{3t}(0) + \phi_1\chi_{1t}(0)\chi_{4t}(0),$$

$$V_2 = -\frac{r_1}{L}\chi_{2t}^2(0) - \frac{\alpha_2 A_r I^*}{\left(1+B^*\right)^2}\chi_{1t}^2(0) + \left\{\frac{\alpha_2 A_r}{1+B^*} - \frac{\alpha_2 A_r B^*}{\left(1+B^*\right)^2}\right\}\chi_{1t}(0)\chi_{2t}(0) - \frac{\alpha_2 A_r}{\left(1+B^*\right)^2}\chi_{1t}^2(0)\chi_{2t}(0)$$

$$V_3 = -\gamma_2\chi_{1t}(0)\chi_{3t}(0).$$

So that

$$\chi_{1t}(\Theta) = W_{20}^{(1)}(\Theta)\frac{z^2}{2} + W_{11}^{(1)}(\Theta)z\bar{z} + W_{02}^{(1)}(\Theta)\frac{\bar{z}^2}{2} + z\mathrm{e}^{i\omega_0\tau_k\Theta} + \bar{z}\mathrm{e}^{-i\omega_0\tau_k\Theta} + \cdots$$

$$\chi_{2t}(\Theta) = W_{20}^{(2)}(\Theta)\frac{z^2}{2} + W_{11}^{(2)}(\Theta)z\bar{z} + W_{02}^{(2)}(\Theta)\frac{\bar{z}^2}{2} + a_1 z\mathrm{e}^{i\omega_0\tau_k\Theta} + \bar{a}_1\bar{z}\mathrm{e}^{-i\omega_0\tau_k\Theta} + \cdots$$

$$\chi_{3t}(\Theta) = W_{20}^{(3)}(\Theta)\frac{z^2}{2} + W_{11}^{(3)}(\Theta)z\bar{z} + W_{02}^{(3)}(\Theta)\frac{\bar{z}^2}{2} + a_2 z\mathrm{e}^{i\omega_0\tau_k\Theta} + \bar{a}_2\bar{z}\mathrm{e}^{-i\omega_0\tau_k\Theta} + \cdots$$

$$\chi_{4t}(\Theta) = W_{20}^{(4)}(\Theta)\frac{z^2}{2} + W_{11}^{(4)}(\Theta)z\bar{z} + W_{02}^{(4)}(\Theta)\frac{\bar{z}^2}{2} + a_3 z\mathrm{e}^{i\omega_0\tau_k\Theta} + \bar{a}_3\bar{z}\mathrm{e}^{-i\omega_0\tau_k\Theta} + \cdots.$$

Thus

$$\chi_{1t}(0) = z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \cdots$$

$$\chi_{2t}(0) = a_1 z + \bar{a}_1\bar{z} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + \cdots$$

$$\chi_{3t}(0) = a_2 z + \bar{a}_2\bar{z} + W_{20}^{(3)}(0)\frac{z^2}{2} + W_{11}^{(3)}(0)z\bar{z} + W_{02}^{(3)}(0)\frac{\bar{z}^2}{2} + \cdots$$

$$\chi_{4t}(0) = a_3 z + \bar{a}_3\bar{z} + W_{20}^{(4)}(0)\frac{z^2}{2} + W_{11}^{(4)}(0)z\bar{z} + W_{02}^{(4)}(0)\frac{\bar{z}^2}{2} + \cdots.$$

Now

$$g(z,\bar{z}) = \tau_k \bar{D}\left(1, \bar{a}_1^*, \bar{a}_2^*, \bar{a}_3^*\right)^{\mathrm{T}} \cdot \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ 0 \end{pmatrix},$$

$$= \tau_k \bar{D}\left[-\left(\frac{r}{K} - \frac{\alpha_1 A_r I^*}{\left(1+B^*\right)^2}\right) - \frac{\bar{a}_1^* \alpha_2 A_r I^*}{\left(1+B^*\right)^2}\right]\chi_{1t}^2(0)$$

$$+\tau_k \bar{D}\left[\left\{\frac{\alpha_1 A_r B^*}{\left(1+B^*\right)^2} - \frac{\alpha_1 A_r}{1+B^*}\right\} + \bar{a}_1^* \left\{\frac{\alpha_2 A_r}{1+B^*} - \frac{\alpha_2 A_r B^*}{\left(1+B^*\right)^2}\right\}\right] \cdot \chi_{1t}(0)\chi_{2t}(0)$$

$$+\tau_k \bar{D}\left[\frac{\alpha_1 A_r}{\left(1+B^*\right)^2} - \frac{\bar{a}_1^* \alpha_2 A_r}{\left(1+B^*\right)^2}\right]\chi_{1t}^2(0)\chi_{2t}(0)$$

$$-\tau_k \bar{D}\left\{\gamma_1 + \bar{a}_2^* \gamma_2\right\}\chi_{1t}(0)\chi_{3t}(0)$$

$$+\tau_k \bar{D}\phi_1 \chi_{1t}(0)\chi_{4t}(0)$$

$$-\tau_k \bar{D}\bar{a}_1^* \frac{r_1}{L}\chi_{2t}^2(0).$$

Comparing the coefficients in (37) with those in (50), we get

$$g_{20} = 2\tau_k \bar{D}\left[-\left(\frac{r}{K} - \frac{\alpha_1 A_r I^*}{\left(1+B^*\right)^2} + \frac{\bar{a}_1^* \alpha_2 A_r I^*}{\left(1+B^*\right)^2}\right) + a_1\left\{\frac{\alpha_1 A_r B^*}{\left(1+B^*\right)^2} - \frac{\alpha_1 A_r}{1+B^*}\right\}\right.$$
$$\left. + a_1 \bar{a}_1^* \left\{\frac{\alpha_2 A_r}{1+B^*} - \frac{\alpha_2 A_r B^*}{\left(1+B^*\right)^2}\right\} + a_2\left\{-\gamma_1 - \bar{a}_2^* \gamma_2\right\} + a_3 \phi_1 - \bar{a}_1^* \frac{r_1}{L} a_1^2\right],$$

$$g_{11} = 2\tau_k \bar{D}\left[-\left\{\frac{r}{K} - \frac{\alpha_1 A_r I^*}{\left(1+B^*\right)^2} + \frac{\bar{a}_1^* \alpha_2 A_r I^*}{\left(1+B^*\right)^2}\right\} + \mathrm{Re}(a_1)\left\{\frac{\alpha_1 A_r B^*}{\left(1+B^*\right)^2} - \frac{\alpha_1 A_r}{1+B^*}\right\}\right.$$
$$+ \bar{a}_1^* \left\{\frac{\alpha_2 A_r}{1+B^*} - \frac{\alpha_2 A_r B^*}{\left(1+B^*\right)^2}\right\}\mathrm{Re}(a_1)$$
$$\left. - \mathrm{Re}(a_2)\left\{\gamma_1 + \bar{a}_2^* \gamma_2\right\} + \phi_1 \mathrm{Re}(a_3) - \frac{\bar{a}_1^* r_1}{L}\left|a_1\right|^2\right],$$

$$g_{02} = 2\tau_k \bar{D}\left[-\left(\frac{r}{K} - \frac{\alpha_1 A_r I^*}{\left(1+B^*\right)^2} + \frac{\bar{a}_1^* \alpha_2 A_r I^*}{\left(1+B^*\right)^2}\right) + \bar{a}_1\left\{\frac{\alpha_1 A_r B^*}{\left(1+B^*\right)^2} - \frac{\alpha_1 A_r}{1+B^*}\right\}\right.$$
$$\left. + \bar{a}_1 \bar{a}_1^* \left\{\frac{\alpha_2 A_r}{1+B^*} - \frac{\alpha_2 A_r B^*}{\left(1+B^*\right)^2}\right\} - \bar{a}_2\left\{-\gamma_1 - \bar{a}_2^* \gamma_2\right\} + \bar{a}_3 \phi_1 - \bar{a}_1^* \frac{r_1}{L} \bar{a}_1^2\right],$$

$$g_{21}=2\tau_k\bar{D}\Bigg[-\left(\frac{r}{K}-\frac{\alpha_1 A_r I^*}{\left(1+B^*\right)^2}+\frac{\bar{a}_1^*\alpha_2 A_r I^*}{\left(1+B^*\right)^2}\right)\left(W_{20}^{(1)}(0)+2W_{11}^{(1)}(0)\right)$$
$$+\left\{\frac{\alpha_1 A_r B^*}{\left(1+B^*\right)^2}-\frac{\alpha_1 A_r}{1+B^*}+\frac{\alpha_2 A_r\bar{a}_1^*}{1+B^*}-\frac{\alpha_2 A_r B^*\bar{a}_1^*}{\left(1+B^*\right)^2}\right\}\left(a_1W_{11}^{(1)}(0)+\bar{a}_1\frac{W_{20}^{(1)}(0)}{2}+\frac{W_{20}^{(2)}(0)}{2}+W_{11}^{(2)}(0)\right)$$
$$+\left\{\frac{\alpha_1 A_r}{\left(1+B^*\right)^2}-\frac{\bar{a}_1^*\alpha_2 A_r}{\left(1+B^*\right)^2}\right\}\left(2a_1+\bar{a}_1\right)-\left(\gamma_1+\bar{a}_2^*\gamma_2\right)\left(a_2W_{11}^{(1)}(0)+\frac{\bar{a}_2}{2}W_{20}^{(1)}(0)+\frac{W_{20}^{(3)}(0)}{2}+W_{11}^{(3)}(0)\right)$$
$$+\phi_1\left\{a_3W_{11}^{(1)}(0)+\frac{\bar{a}_3}{2}W_{20}^{(1)}(0)+\frac{W_{20}^{(4)}(0)}{2}+W_{11}^{(4)}(0)\right\}-\frac{\bar{a}_1^* r_1}{L}\left\{\bar{a}_1W_{20}^{(2)}(0)+2a_1W_{11}^{(2)}(0)\right\}\Bigg].$$

In order to compute g_{21}, we need to $W_{20}(\Theta)$ and $W_{11}(\Theta)$. From Equations (42) and (45) we have

$$\dot{W}=\dot{\chi}_t-\dot{z}q-\dot{\bar{z}}\bar{q}$$
$$=\begin{cases}AW-2\mathrm{Re}\left\{\bar{q}^*(0)\cdot f_0q(\Theta)\right\}, & \Theta\in[-1,0),\\ AW-2\mathrm{Re}\left\{\bar{q}^*(0)\cdot f_0q(0)\right\}+f_0, & \Theta=0,\end{cases}\tag{51}$$
$$=AW+H(z,\bar{z},\Theta)\tag{52}$$

with

$$H(z,\bar{z},\Theta)=H_{20}(\Theta)\frac{z^2}{2}+H_{11}(\Theta)z\bar{z}+H_{02}(\Theta)\frac{\bar{z}^2}{2}+\cdots.\tag{53}$$

Also, on C_0, using chain rule, we get

$$\dot{W}=W_z\dot{z}+W_{\bar{z}}\dot{\bar{z}}.\tag{54}$$

It follows from (46), (52) and (54)

$$\left(A-2i\omega_0\tau_k\right)W_{20}=-H_{20},\tag{55}$$
$$AW_{11}=-H_{11}\tag{56}$$

etc. Now for $\Theta\in[-1,0)$

$$\begin{aligned}H(z,\bar{z},\Theta)&=-\bar{q}^*(0)\cdot f_0q(\Theta)-\bar{q}^*(0)\cdot\bar{f}_0\bar{q}(\Theta)\\&=-g(z,\bar{z})q(\Theta)-\bar{g}(z,\bar{z})\bar{q}(\Theta)\\&=-\left(g_{20}q(\Theta)+\bar{g}_{02}\bar{q}(\Theta)\right)\frac{z^2}{2}-\left(g_{11}q(\Theta)+\bar{g}_{11}\bar{q}(\Theta)\right)z\bar{z}+\cdots,\end{aligned}\tag{57}$$

which on comparing the coefficients with (53) gives

$$H_{20}(\Theta)=-g_{20}q(\Theta)-\bar{g}_{02}\bar{q}(\Theta),\tag{58}$$
$$H_{11}(\Theta)=-g_{11}q(\Theta)-\bar{g}_{11}\bar{q}(\Theta).\tag{59}$$

From (56), (58) and the definition of A, we have

$$W_{20}'(\Theta)=2i\omega_0\tau_kW_{20}(\Theta)+g_{20}q(\Theta)+\bar{g}_{02}\bar{q}(\Theta).\tag{60}$$

Note that $q(\Theta)=q(0)\mathrm{e}^{i\omega_0\tau_k\Theta}$, hence

$$W_{20}(\Theta)=\frac{ig_{20}}{\omega_0\tau_k}q(\Theta)+\frac{i\bar{g}_{20}}{3\omega_0\tau_k}\bar{q}(\Theta)+F_1\mathrm{e}^{2i\omega_0\tau_k\Theta}.\tag{61}$$

Similarly from (56), (59) and the definition of A, we have

$$W_{11}'(\Theta) = g_{11}q(\Theta) + \bar{g}_{11}\bar{q}(\Theta) \tag{62}$$

$$W_{11}(\Theta) = -\frac{ig_{11}}{\omega_0\tau_k}q(\Theta) + \frac{i\bar{g}_{11}}{\omega_0\tau_k}\bar{q}(\Theta) + F_2 \tag{63}$$

where $F_1 = \left(F_1^{(1)}, F_1^{(2)}, F_1^{(3)}, F_1^{(4)}\right)$ and $F_2 = \left(F_2^{(1)}, F_2^{(2)}, F_2^{(3)}, F_2^{(4)}\right) \in R^4$ are constant vectors, to be determined.

It follows from the definition of A and (45) that

$$\int_{-1}^{0} \mathrm{d}\eta(\Theta) W_{20}(\Theta) = 2i\omega_0\tau_k W_{20}(0) - H_{20}(0) \tag{64}$$

$$\int_{-1}^{0} \mathrm{d}\eta(\Theta) W_{11}(\Theta) = -H_{11}(0). \tag{65}$$

From Equations (61) and (63) we get

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\tau_k \begin{pmatrix} -\left(\frac{r}{K} - \frac{\alpha_1 A_r I^*}{(1+B^*)^2} - \frac{\alpha_1 a_1 A_r B^*}{(1+B^*)^2} + \frac{\alpha_1 a_1 A_r}{1+B^*} + a_2\gamma_1 - a_3\phi_1\right) \\ -\frac{\alpha_2 A_r I^*}{(1+B^*)^2} + a_1\left(\frac{\alpha_2 A_r}{1+B^*} - \frac{\alpha_2 A_r B^*}{(1+B^*)^2}\right) - \frac{r_1 a_1^2}{L} \\ -a_2\gamma_2 \\ 0 \end{pmatrix}, \tag{66}$$

and

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + 2\tau_k \begin{pmatrix} -\left(\frac{r}{K} - \frac{\alpha_1 A_r I^*}{(1+B^*)^2} - \left(\frac{\alpha_1 A_r B^*}{(1+B^*)^2} - \frac{\alpha_1 A_r}{1+B^*}\right)\mathrm{Re}(a_1) + \mathrm{Re}(a_2)\gamma_1 - \mathrm{Re}(a_3)\phi_1\right) \\ -\frac{\alpha_2 A_r I^*}{(1+B^*)^2} + \mathrm{Re}(a_1)\left(\frac{\alpha_2 A_r}{1+B^*} - \frac{\alpha_2 A_r B^*}{(1+B^*)^2}\right) - \frac{r_1 |a_1|^2}{L} \\ -\mathrm{Re}(a_2)\gamma_2 \\ 0 \end{pmatrix}. \tag{67}$$

Using (61) and (66) in (64) and noting that $q(\Theta)$ is eigenvector of A, we have

$$\left(2i\omega_0\tau_k I - \int_{1}^{0} \mathrm{e}^{2i\omega_0\tau_k\Theta}\mathrm{d}\eta\ (\Theta)\right)F_1 = 2\tau_k \begin{pmatrix} -\left(\frac{r}{K} - \frac{\alpha_1 A_r I^*}{(1+B^*)^2} - \frac{\alpha_1 a_1 A_r B^*}{(1+B^*)^2} + \frac{\alpha_1 a_1 A_r}{1+B^*} + a_2\gamma_1 - a\phi_1\right) \\ -\frac{\alpha_2 A_r I^*}{(1+B^*)^2} + a_1\left(\frac{\alpha_2 A_r}{1+B^*} - \frac{\alpha_2 A_r B^*}{(1+B^*)^2}\right) - \frac{r_1 a_1^2}{L} \\ -a_2\gamma_2 \\ 0 \end{pmatrix} \tag{68}$$

i.e.

$$\begin{pmatrix} 2i\omega_0+B^*\left(\frac{r}{K}-\frac{A_r\alpha_1 I^*}{\left(1+B^*\right)^2}\right) & \frac{\alpha_1 B^* A_r}{1+B^*} & \gamma_1 B^* & -\phi_1 B^* \\ -\frac{\alpha_2 A_r I^*}{1+B^*}+\frac{\alpha_2 A_r B^* I^*}{\left(1+B^*\right)^2} & 2i\omega_0+\frac{r_1 I^*}{L} & 0 & 0 \\ \gamma_2 T^* & Q_1 & 2i\omega_0+\delta_0+\gamma_2 B^* & 0 \\ \phi e^{-2i\omega_0\tau_k} & 0 & 0 & 2i\omega_0+\phi_0 \end{pmatrix}\begin{pmatrix} F_1^{(1)} \\ F_1^{(2)} \\ F_1^{(3)} \\ F_1^{(4)} \end{pmatrix}$$
$$=2\begin{pmatrix} -\left(\frac{r}{K}-\frac{\alpha_1 A_r I^*}{\left(1+B^*\right)^2}-\frac{\alpha_1 a_1 A_r B^*}{\left(1+B^*\right)^2}+\frac{\alpha_1 a_1 A_r}{1+B^*}+a_2\gamma_1-a_3\phi_1\right) \\ -\frac{\alpha_2 A_r I^*}{\left(1+B^*\right)^2}+a_1\left(\frac{\alpha_2 A_r}{1+B^*}-\frac{\alpha_2 A_r B^*}{\left(1+B^*\right)^2}\right)-\frac{r_1 a_1^2}{L} \\ -a_2\gamma_2 \\ 0 \end{pmatrix}. \tag{69}$$

Similarly using (63) and (67) in (65), we get

$$\begin{pmatrix} B^*\left(\frac{r}{K}-\frac{A_r\alpha_1 I^*}{\left(1+B^*\right)^2}\right) & \frac{\alpha_1 B^* A_r}{1+B^*} & \gamma_1 B^* & -\phi_1 B^* \\ -\frac{\alpha_2 A_r I^*}{1+B^*}+\frac{\alpha_2 A_r B^* I^*}{\left(1+B^*\right)^2} & \frac{r_1 I*}{L} & 0 & 0 \\ \gamma_2 T^* & Q_1 & \delta_0+\gamma_2 B^* & 0 \\ \phi & 0 & 0 & \phi_0 \end{pmatrix}\begin{pmatrix} F_2^{(1)} \\ F_2^{(2)} \\ F_2^{(3)} \\ F_2^{(4)} \end{pmatrix}$$
$$=2\begin{pmatrix} -\left(\frac{r}{K}-\frac{\alpha_1 A_r I^*}{\left(1+B^*\right)^2}-\left(\frac{\alpha_1 A_r B^*}{\left(1+B^*\right)^2}-\frac{\alpha_1 A_r}{1+B^*}\right)\mathrm{Re}\left(a_1\right)+\mathrm{Re}\left(a_2\right)\gamma_1-\mathrm{Re}\left(a_3\right)\phi_1\right) \\ -\frac{\alpha_2 A_r I^*}{\left(1+B^*\right)^2}+\mathrm{Re}\left(a_1\right)\left(\frac{\alpha_2 A_r}{1+B^*}-\frac{\alpha_2 A_r B^*}{\left(1+B^*\right)^2}\right)-\frac{r_1\left|a_1\right|^2}{L} \\ -\mathrm{Re}\left(a_2\right)\gamma_2 \\ 0 \end{pmatrix}. \tag{70}$$

We solve system (69) for F_1 and (70) for F_2 and using these values are determine W_{20} and W_{11} and hence g_{21}. Now to determine the direction, stability and period of bifurcating periodic solutions from critical point at the critical value $\tau=\tau_k$ we can compute the following necessary quantities as given by [10].

$$C_1(0)=\frac{i}{2\omega_0\tau_k}\left(g_{11}g_{20}-2\left|g_{11}\right|^2-\frac{\left|g_{02}\right|^2}{3}\right)+\frac{g_{21}}{2} \tag{71}$$

$$\mu_2=-\frac{\mathrm{Re}\left\{c_1(0)\right\}}{\mathrm{Re}\left\{\psi'\left(\tau_k\right)\right\}} \tag{72}$$

$$b_2=2\mathrm{Re}\left\{c_1(0)\right\} \tag{73}$$

$$T_2 = -\frac{I_m\{c_1(0)\} + \mu\, I_m\{\psi'(\tau_k)\}}{\omega_0 \tau_k}. \tag{74}$$

Hence, using the results of [10]. We have the following theorem.

Theorem (3.2.1): If $\mu_2 > 0$ $(\mu_2 < 0)$, then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau > \tau_k$ $(\tau < \tau_k)$. The bifurcating periodic solution is stable (unstable) if $b_2 < 0$ $(b_2 > 0)$ and the period increase (decrease) if $T_2 > 0$ $(T_2 < 0)$.

3. Numerical Support

In this section, we present numerical simulation to illustrate the results obtained in the previous sections. The system (4) is solved using the MATLAB software package under the following set of parameters.

(a) $r = 2$, $K = 100$, $\alpha_1 = 0.2$, $A_r = 0.6$, $\gamma_1 = 0.01$, $L = 100$, $\alpha_2 = 0.1$, $Q_0 = 1$, $Q_1 = 0.2$, $\delta_0 = 0.5$, $\gamma_2 = 0.01$, $r_1 = 1$, $\phi = 0.01$, $\phi_0 = 0.02$, $\phi_1 = 0.8$.

The interior equilibrium point of system (4) with data (a) is

$$B^* = 99.1014, \quad I^* = 145.94, \quad T^* = 20.2466, \quad T_c^* = 0.449306.$$

Then, we can easily obtain that (H_1) to be satisfied. By computation, we have $\omega = 0.395806$, $\tau_0 = 5.82023$. The transversality condition (27) is satisfied as $\left[\frac{\mathrm{d}(\mathrm{Re}\,\psi)}{\mathrm{d}\tau}\right]_{\tau=\tau_0} = 1.20963 \neq 0$.

The stability behavior of the system (4) for $\tau = 0$ can be depicted by **Figure 1**. To check the dynamic behavior of the system (4) for $\tau \neq 0$ can be seen by **Figure 2** and **Figure 3**. A Hopf bifurcation occurs at $E_3(B^*, I^*, T^*, T_c^*)$ when $\tau = \tau_0 = 5.82023$ and small amplitude periodic solution around $E_3(B^*, I^*, T^*, T_c^*)$ and this can be visualized from **Figure 2** and **Figure 3**. From **Figure 2** and **Figure 3** we can see that when $\tau = 1 < \tau_0 = 5.82023$ the system is stable and for $\tau = 8 > \tau_0 = 5.82023$ the system becomes unstable. An alternative resource has a strong impact on the depletion of forestry biomass which can be seen from **Figure 4**. From this we can see increasing the value of alternative resource the concentration of forestry biomass increases and also controlling the instability of the system (4) when $\tau = 8 > \tau_0 = 5.82023$ see **Figure 4(b)**. Now see impact of other factors on forestry biomass. Increasing the value of α_1 the concentration of forest biomass increases

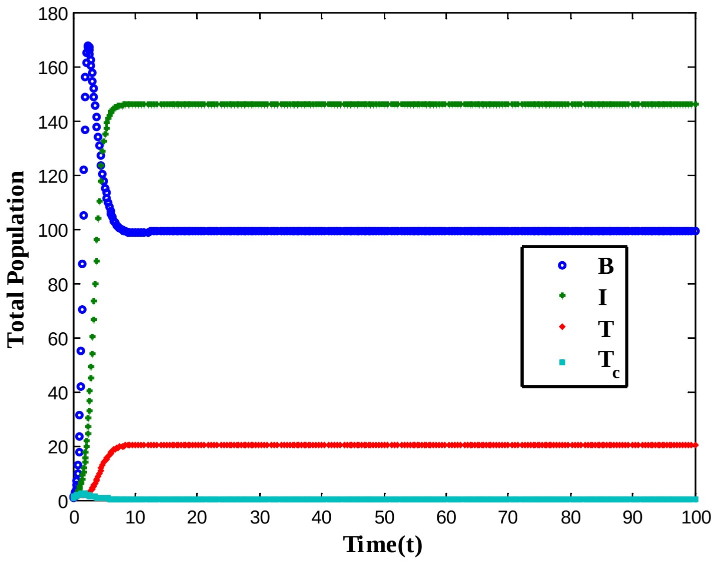
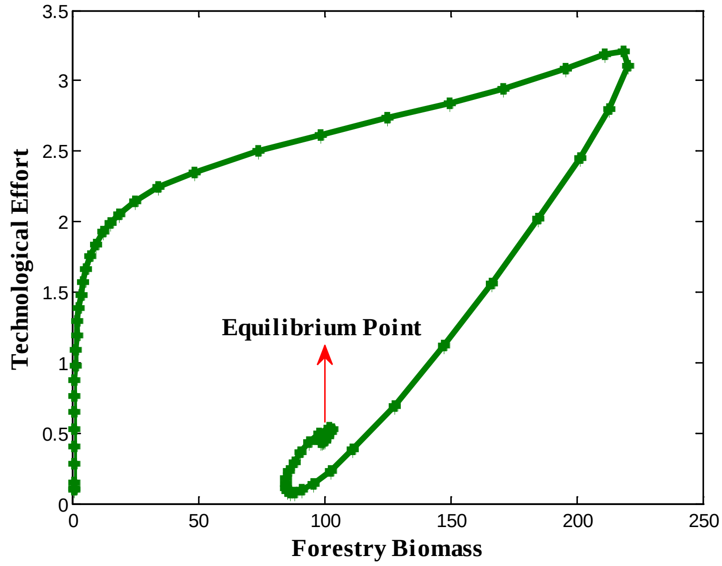
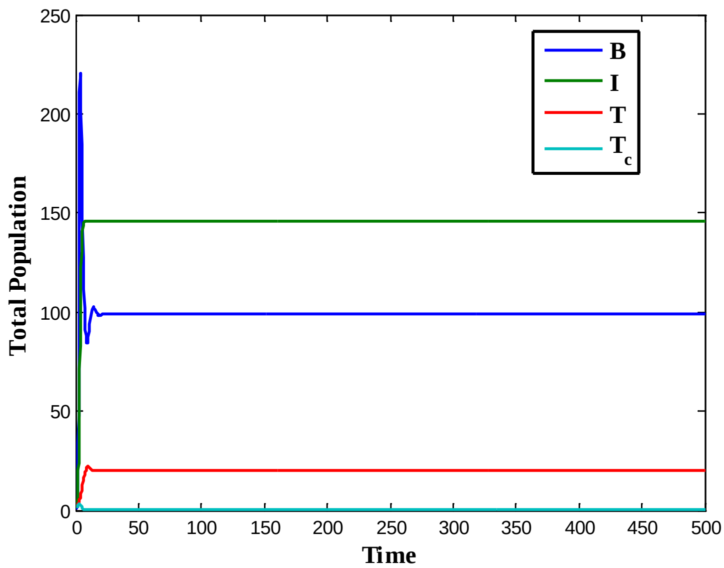


Figure 1. Stable behavior of B, I, T and T_c with time, when $\tau = 0$ and other parameter values are same as (a).



(a)

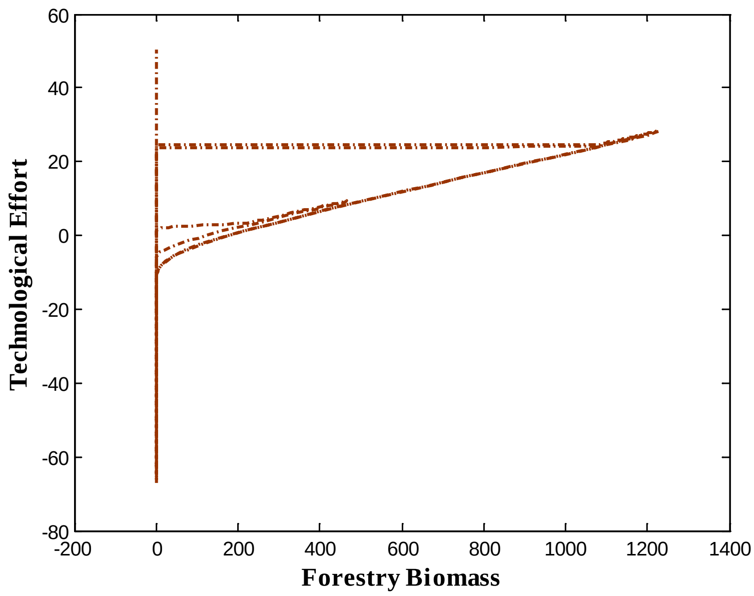


(b)

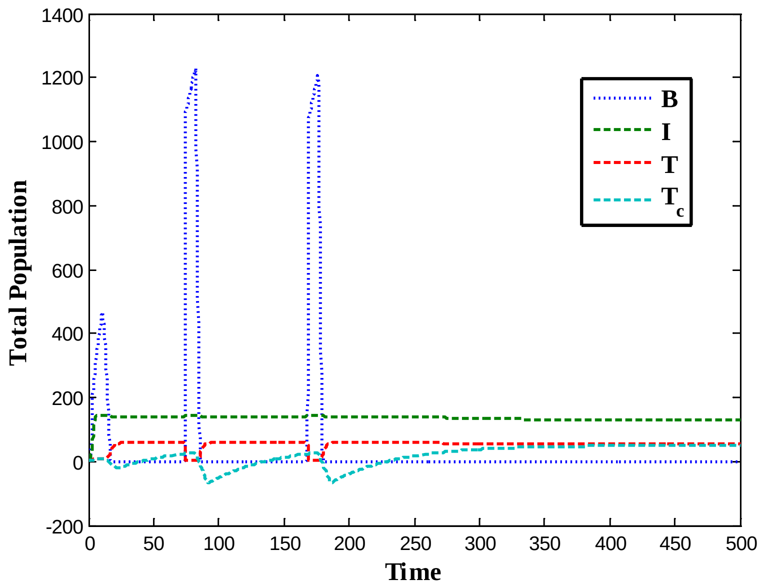
Figure 2. Trajectory portrait and phase portrait of system (4) for $\tau = 1 < \tau_0 = 5.82023$ and other parameters are same as (a).

refer **Figure 5**. From **Figure 6** we see that the concentration of forestry biomass decreases when the value of γ_1 increases. Increasing the values of ϕ_1 and ϕ the concentration of forestry biomass increases refer **Figure 7** and **Figure 8**.

Now to verify the result of Theorem (3.2.1), we have shown the variation of variables B, I, T and T_c for $\tau = 1$ and $\tau = 8$ in **Figure 2** and **Figure 3** respectively. Also for the above set of parameter values, we get $c_1(0) = -0.19028 - 15.1071i$, $\mu_2 = 0.157305$, $b_2 = -0.38056$ and $T_2 = 6.00118$. Since $\mu_2 > 0$, the Hopf bifurcation is supercritical and the direction of the bifurcation $\tau > \tau_0$. Also $b_2 < 0$ and $T_2 > 0$, this implies that the bifurcating periodic solutions arising from $E_3\left(B^*, I^*, T^*, T_c^*\right)$ at τ_0 are stable and the periods of limit cycle increases.



(a)

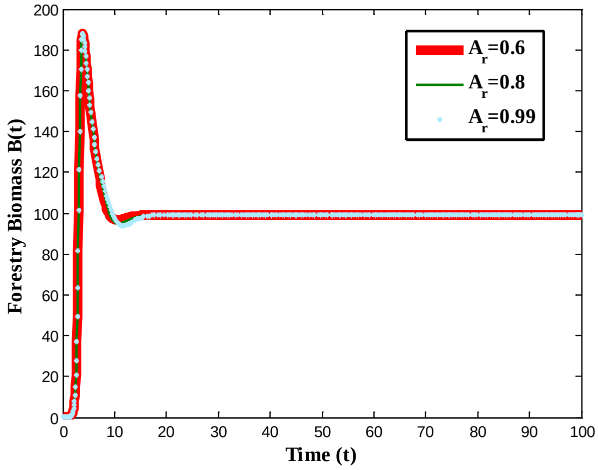


(b)

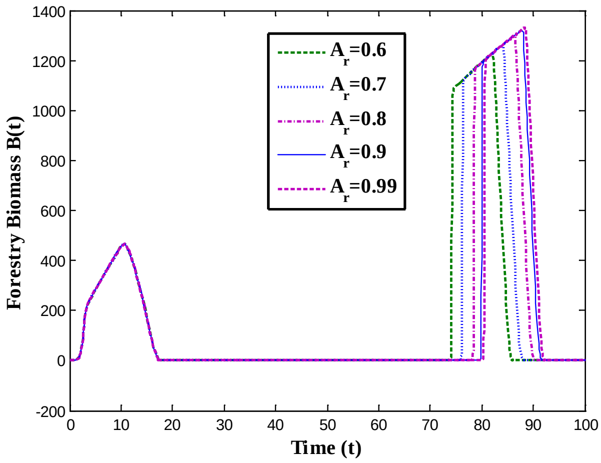
Figure 3. Trajectory portrait and phase portrait of system (4) for $\tau = 8 > \tau_0 = 5.82023$ and other parameters are same as (a).

4. Conclusion

In this paper, a nonlinear mathematical model is proposed and analyzed to see the effect of alternative resource and time delay on conservation of forestry biomass. We have obtained the explicit formulae that determine the stability and direction of the bifurcating periodic solutions by using the normal form theory and the center manifold theorem. For the given set of parameter values in (a), we found that, the Hopf bifurcation was supercritical with stable periodic solutions and the direction of bifurcation was $\tau > \tau_0$. Forests serve as a source of life for the forest based small and large scale industries. However, due to shrinking forests area, the industries are facing wood crisis. To overcome wood crisis, alternative resources like synthetic, liquid wood, plastic, wood composite lumber etc are good alternative for wood based products. For preserving our forestry biomass we can control the



(a)



(b)

Figure 4. (a) and (b) show variation of the forestry biomass with time for different values of A_r when $\tau = 1 < \tau_0 = 5.82023$ and $\tau = 8 > \tau_0 = 5.82023$ respectively and other parameter values are same as (a).

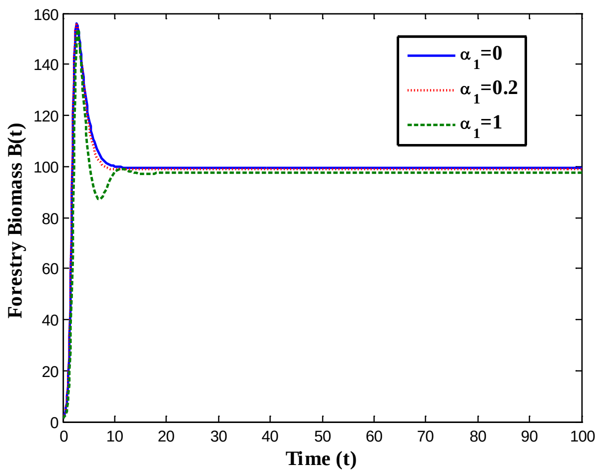


Figure 5. Variation of the forestry biomass with time for different values of α_1 and other parameter values are same as (a).

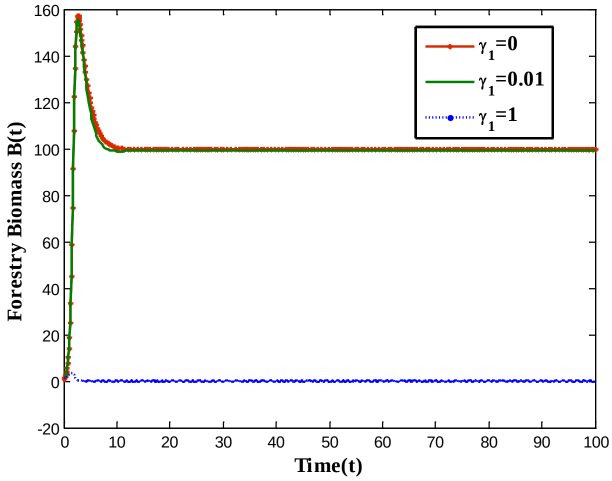


Figure 6. Variation of the forestry biomass with time for different values of γ_1 and other parameter values are same as (a).

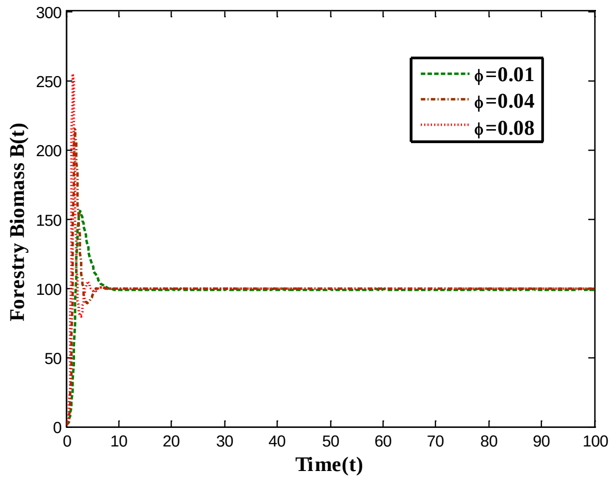


Figure 7. Variation of the forestry biomass with time for different values of ϕ and other parameter values are same as (a).

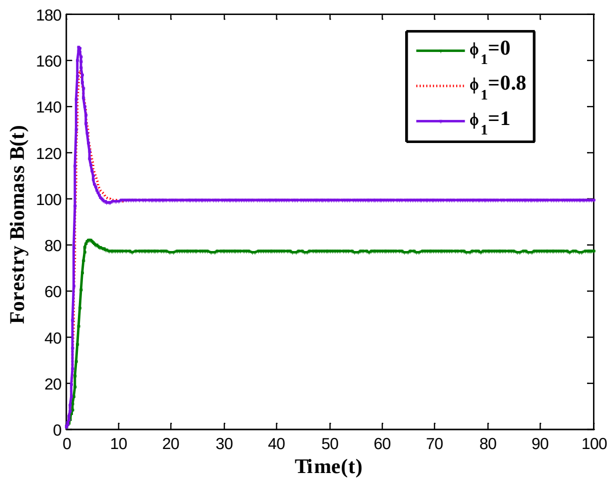


Figure 8. Variation of the forestry biomass with time for different values of ϕ_1 and other parameter values are same as (a).

wood based industries by human awareness or some government action. Hence, we conclude from our analysis that the forestry biomass may be conserved by applying technological effort and alternative resources.

Acknowledgements

Second author thankfully acknowledges the NBHM (2/40(29)/2014/R&D-11/14138) for the financial assistance in the form of PDF.

References

[1] Shukla, J.B., Freedman, H.I., Pal, V.N., Misra, O.P., Agarwal, M. and Shukla, A. (1989) Degradation and Subsequent Regeneration of a Forestry Resource: A Mathematical Model. *Ecological Modelling*, **44**, 219-229. http://dx.doi.org/10.1016/0304-3800(89)90031-8

[2] Li, W.Z., Ju, M.T., Liu, L., Wang, Y.N. and Li, T.L. (2011) The Effects of Biomass Solid Waste Resources Technology in Economic Development. *Energy Procedia*, **5**, 2455-2460. http://dx.doi.org/10.1016/j.egypro.2011.03.422

[3] Shukla, J.B., Sharma, S., Dubey, B. and Sinha, P. (2009) Modelling the Survival of a Resource Dependent Population: Effect of Toxicants (Pollutants) Emitted from the External Sources as Well as Formed by Its Precursors. *Nonlinear Analysis*: *Real World Applications*, **10**, 54-70. http://dx.doi.org/10.1016/j.nonrwa.2007.08.014

[4] Dubey, B. and Narayanan, A.S. (2010) Modelling Effects of Industrialization, Population and Pollution on a Renewable Resource. *Nonlinear Analysis*: *Real World Applications*, **11**, 2833-2848. http://dx.doi.org/10.1016/j.nonrwa.2009.10.007

[5] Dhar, J., Chaudhary, M. and Sahu, G.P. (2013) Mathematical Model of Depletion of Forestry Resource, Effect of Synthetic Based Industries. *International Journal of Biological*, *Life Science and Engineering*, **7**, 1-5.

[6] Misra, A.K. and Lata, K. (2013) Modeling the Effect of Time Delay on the Conservation of Forestry Biomass. *Chaos*, *Solitons & Fractals*, **47**, 1-11. http://dx.doi.org/10.1016/j.chaos.2012.10.002

[7] Misra, A.K., Lata, K. and Shukla, J.B. (2014) A Mathematical Model for Depletion of Forestry Resources Due to Population and Population Pressure Augmented Industrialization. *International Journal of Modeling*, *Simulation and Scientific Computing*, **5**, 1-16.

[8] Misra, A.K., Lata, K. and Shukla, J.B. (2014) Effects of Population and Population Pressure on Forest Resources and Their Conservation: A Modeling Study. *Environment*, *Development and Sustainability*, **16**, 361-374. http://dx.doi.org/10.1007/s10668-013-9481-x

[9] Agarwal, M. and Devi, S. (2011) A Resource-Dependent Competition Model: Effects of Toxicants Emitted from External Sources as Well as Formed by Precursors of Competing Species. *Nonlinear Analysis*: *Real World Applications*, **12**, 751-766. http://dx.doi.org/10.1016/j.nonrwa.2010.08.003

[10] Hassard, B.D., Kzrinoff, N.D. and Wan, W.H. (1981) Theory and Application of Hopf Bifurcation: London Mathematics Society Lecture Note Series. Vol. 41, Cambridge University Press, Cambridge.