

L-Convex Polyominoes: Discrete Tomographical Aspects

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Abstract

This paper uses the geometrical properties of *L*-convex polyominoes in order to reconstruct these polyominoes. The main idea is to modify some clauses to the original construction of Chrobak and Dürr in order to control the *L*-convexity using 2SAT satisfaction problem.

Keywords

Convex Polyominoes, Monotone Paths, Discrete Geometry

1. Introduction

Discrete tomography focuses on the problem of reconstruction of discrete objects from small number of their projections. In order to reduce the number of solutions we could add some convexity conditions to these discrete objects. There are many notions of discrete convexity of polyominoes (namely *HV*-convex [1], *Q*-convex [2], *L*-convex polyominoes [3]) and each one leads to interesting studies. One natural notion of convexity on the discrete plane is the class of *HV*-convex polyominoes that is polyominoes with consecutive cells in rows and columns. Following the work of Del Lungo, Nivat, Barcucci, and Pinzani [1] we are able using discrete tomography to reconstruct polyominoes that are *HV*-convex according to their horizontal and vertical projections.

In addition to that, for an *HV*-convex polyomino *P* every pairs of cells of *P* can be reached using a path included in *P* with only two kinds of unit steps (such a path is called monotone). A polyomino is called *kL*-convex if for every two cells we find a monotone path with at most *k* changes of direction. Obviously a *kL*-convex polyomino is an *HV*-convex polyomino. Thus, the set of *kL*-convex polyominoes for $k \in \mathbb{N}$ forms a hierarchy of *HV*-convex polyominoes according

to the number of changes of direction of monotone paths. This notion of L -convex polyominoes has been considered by several points of view. In [4] combinatorial aspects of L -convex polyominoes are analyzed, giving the enumeration according to the semi-perimeter and the area. In [5] it is given an algorithm that reconstructs an L -convex polyomino from the set of its maximal L -polyominoes. Similarly in [3] it is given another way to reconstruct an L -convex polyomino from the size of some special paths, called bordered L -paths.

The main contribution of this paper is the development of an algorithm that reconstructs all subclasses of L -convex polyominoes by using their geometrical properties and the algorithm of Chrobak and Dürr [6]. In particular, I add and modify some clauses to the original construction of Chrobak and Dürr in order to control the L -convexity using 2SAT satisfaction problem.

This paper is divided into 6 sections. After basics on polyominoes, I present briefly in Section 3 the four geometrical properties between the feet of all subclasses of non-directed L -convex polyominoes. In Section 4, I also introduce the subclasses of directed L -convex polyominoes with the conditions of the L -convexity. In the last Section I give the reconstruction algorithms of all L -convex polyominoes using simple modifications of Chrobak and Dürr's algorithm. The last section is a final comment on my contribution.

2. Definitions and Notation

A planar discrete set is a finite subset of the integer lattice \mathbb{Z}^2 defined up to a translation. A discrete set can be represented either by a set of cells, i.e. unitary squares of the cartesian plane, or by a binary matrix, where the 1's determine the cells of the set (see [Figure 1](#)).

A polyomino P is a finite connected set of adjacent cells, defined up to translations, in the cartesian plane. A row convex polyomino (resp. column-convex) is a self avoiding convex polyomino such that the intersection of any horizontal line (resp. vertical line) with the polyomino has at most two connected components. Finally, a polyomino is said to be convex if it is both row and column-convex (see [Figure 2](#)).

A convex polyomino containing at least one corner of its minimal bounding box is said to be a directed convex polyomino. (see [Figure 3](#)).

To each discrete set S , represented as a $m \times n$ binary matrix, we associate two integer vectors $H = (h_1, \dots, h_m)$ and $V = (v_1, \dots, v_n)$ such that, for each $1 \leq i \leq m, 1 \leq j \leq n$, h_i and v_j are the number of cells of S (elements 1 of the matrix) which lie on row i and column j , respectively. The vectors H and V are called the *horizontal* and *vertical* projections of S , respectively (see [Figure 4](#)). By convention, the origin of the matrix (that is the cell with coordinates $(1,1)$) is in the upper left position.

For any two cells A and B in a polyomino, a *path* Π_{AB} , from A to B , is a sequence $(i_1, j_1), (i_2, j_2), \dots, (i_r, j_r)$ of adjacent disjoint cells $\in P$, with

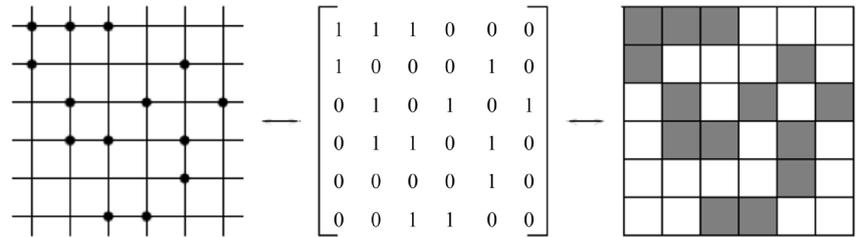


Figure 1. A finite set of $\mathbb{Z} \times \mathbb{Z}$, and its representation in terms of a binary matrix and a set of cells (The origin of this figure is in [7]).

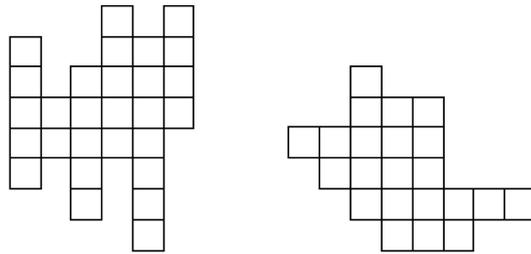


Figure 2. A column convex (left) and a convex (right) polyomino (The origin of this figure is in [3]).

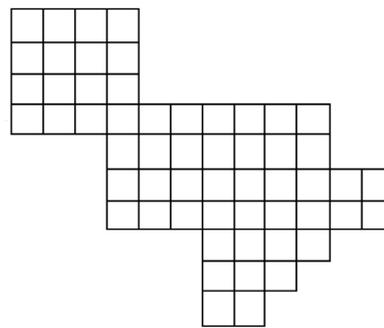


Figure 3. A directed convex polyomino (The origin of this figure is in [8]).

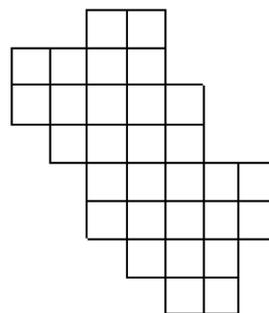


Figure 4. A polyomino P with $H = (2, 4, 5, 4, 5, 5, 3, 2)$ and $V = (2, 3, 6, 7, 6, 4, 2)$ (The origin of this figure is in [8]).

$A = (i_1, j_1)$, and $B = (i_r, j_r)$. For each $1 \leq k \leq r$, we say that the two consecutive cells $(i_k, j_k), (i_{k+1}, j_{k+1})$ form:

- an *east* step if $i_{k+1} = i_k$ and $j_{k+1} = j_k + 1$;

- a *north* step if $i_{k+1} = i_k - 1$ and $j_{k+1} = j_k$;
- a *west* step if $i_{k+1} = i_k$ and $j_{k+1} = j_k - 1$;
- a *south* step if $i_{k+1} = i_k + 1$ and $j_{k+1} = j_k$.

Let us consider a polyomino P . A path in P has a change of direction in the cell (i_k, j_k) , for $2 \leq k \leq r-1$, if

$$i_k \neq i_{k-1} \Leftrightarrow j_{k+1} \neq j_k.$$

Finally, we define a path to be *monotone* if its entirely made of only two of the four types of steps defined above.

Proposition 1 (Gastiglione, Restivo) [5] *A polyomino P is convex if and only if every pair of cells is connected by a monotone path.*

3. Geometrical Properties of L -Convex Polyominoes

In this section, we present the geometrical properties of L -convex polyominoes in terms of monotone paths.

Let (H, V) be two vectors of projections and let P be a convex polyomino, that satisfies (H, V) . By a classical argument P is contained in a rectangle R of size $m \times n$ (called minimal bounding box). Let $[\min(S), \max(S)]$ ($[\min(E), \max(E)]$, $[\min(N), \max(N)]$, $[\min(W), \max(W)]$) be the intersection of P 's boundary on the lower (right, upper, left) side of R (see [1]). By abuse of notation, for each $1 \leq i \leq m$ and $1 \leq j \leq n$, we call $\min(S)$ [resp. $\min(E)$, $\min(N)$, $\min(W)$] the cell at the position $(m, \min(S))$ [resp. $(\min(E), n)$, $(1, \min(N))$, $(\min(W), 1)$] and $\max(S)$ [resp. $\max(E)$, $\max(N)$, $\max(W)$] the cell at the position $(m, \max(S))$ [resp. $(\max(E), n)$, $(1, \max(N))$, $(\max(W), 1)$] (see Figure 5).

Definition 1. *The segment $[\min(S), \max(S)]$ is called the S -foot. Similarly, the segments $[\min(E), \max(E)]$, $[\min(N), \max(N)]$ and $[\min(W), \max(W)]$ are called E -foot, N -foot and W -foot.*

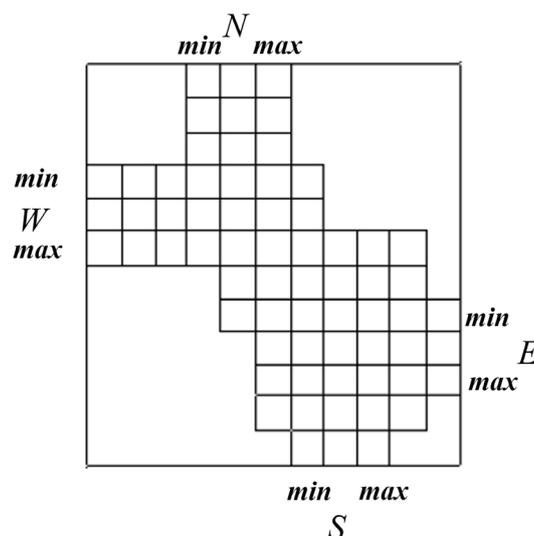


Figure 5. Min and max of the four feet in the rectangle R .

Proposition 2. Let (H, V) be two vectors of projections and let P be a convex polyomino, that satisfies (H, V) . If $H = (n, h_2, \dots, h_m)$ or $H = (h_1, h_2, \dots, n)$ or $V = (m, v_2, \dots, v_n)$ or $V = (v_1, v_2, \dots, m)$ then P is an L -convex polyomino.

Proof. Let P be a convex polyomino such that $H = (n, h_2, \dots, h_m)$ (see **Figure 6**), then the bar allows us to go from the first cell situated at the position $(1, 1)$ to all other cells with at most one change of direction. Thus every two cells is connected by a monotone path with at most one change of direction and hence P is an L -convex polyomino. (Similar reasoning holds for the other three cases).

Let \mathcal{C} (resp. \mathcal{C}_L) be the class of convex polyominoes (resp. L -convex polyominoes) and let P be in \mathcal{C} (resp. \mathcal{C}_L) such that P does not satisfy Proposition 2. Also suppose that P is not a directed polyomino, then one can define the following subclasses of convex polyominoes:

$$\alpha = \{P \in \mathcal{C} \mid \min(N) = \min(S) \text{ and } \min(W) = \min(E)\}.$$

$$\beta = \{P \in \mathcal{C} \mid \min(N) = \min(S) \text{ and } (\min(W) < \min(E) \text{ or } \min(W) > \min(E))\}.$$

$$\gamma = \{P \in \mathcal{C} \mid (\min(N) < \min(S) \text{ or } \min(N) > \min(S)) \text{ and } \min(W) = \min(E)\}.$$

$$\mu = \{P \in \mathcal{C} \mid (\min(N) < \min(S) \text{ or } \min(N) > \min(S)) \text{ and } (\min(W) < \min(E) \text{ or } \min(W) > \min(E))\}.$$

$$\alpha_L = \{P \in \mathcal{C}_L \mid \min(N) = \min(S) \text{ and } \min(W) = \min(E)\}.$$

$$\beta_L = \{P \in \mathcal{C}_L \mid \min(N) = \min(S) \text{ and } (\min(W) < \min(E) \text{ or } \min(W) > \min(E))\}.$$

$$\gamma_L = \{P \in \mathcal{C}_L \mid (\min(N) < \min(S) \text{ or } \min(N) > \min(S)) \text{ and } \min(W) = \min(E)\}.$$

$$\mu_L = \{P \in \mathcal{C}_L \mid (\min(N) < \min(S) \text{ or } \min(N) > \min(S)) \text{ and } (\min(W) < \min(E) \text{ or } \min(W) > \min(E))\}.$$

(See **Figure 7**).

Let us define the following sets:

- $WN = \{(i, j) \in P \mid i < \min(W) \text{ and } j < \min(N)\},$
- $SE = \{(i, j) \in P \mid i > \max(E) \text{ and } j > \max(S)\}.$
- $NE = \{(i, j) \in P \mid i < \min(E) \text{ and } j > \max(N)\},$
- $WS = \{(i, j) \in P \mid i > \max(W) \text{ and } j < \min(S)\}.$

The following characterizations hold for convex polyominoes in the class μ_L, α_L, β_L and γ_L .

Proposition 3. Let P be an L convex polyomino in the class μ_L (resp. α_L, β_L and γ_L), then there exist an L -path from $\min(N)$ to $\max(E)$ with a south step followed by an east step, and an L -path from $\min(W)$ to $\max(S)$ with an east step followed by a south step.

Proposition 4. Let P be an L -convex polyomino in the class μ_L , then at least one of the four following affirmations is true.

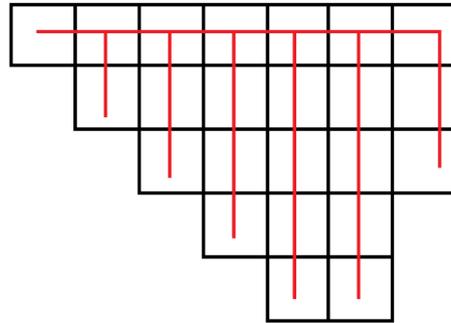


Figure 6. An L -convex polyomino with $H = (7, 6, 5, 3, 2)$.

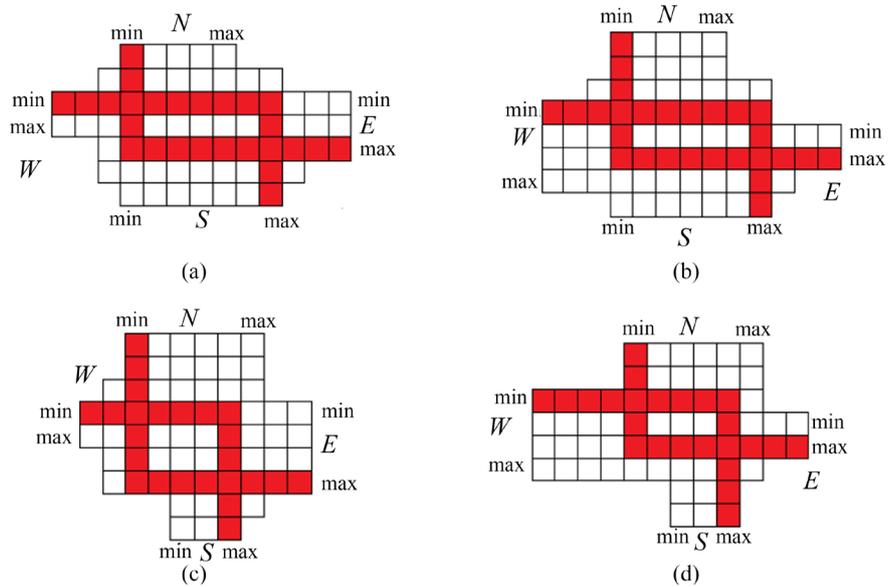


Figure 7. (a) an element of the class α_L ; (b) an element of the class β_L ; (c) an element of the class γ_L , and (d) an element of the class μ_L .

1) The feet of P are connected by an L -path from $\min(N)$ to $\max(S)$ with an east step followed by a south step and an L -path from $\min(W)$ to $\max(E)$ with a south step followed by an east step.

2) The feet of P are connected by an L -path from $\min(N)$ to $\max(S)$ with an east step followed by a south step and an L -path from $\max(W)$ to $\min(E)$ with an east step followed by a north step.

3) The feet of P are connected by an L -path from $\min(W)$ to $\max(E)$ with a south step followed by an east step and an L -path from $\min(S)$ to $\max(N)$ with an east step followed by a north step.

4) The feet of P are connected by an L -path from $\max(W)$ to $\min(E)$ with an east step followed by a north step and an L -path from $\min(S)$ to $\max(N)$ with an east step followed by a north step (see **Figure 8**).

Now if P is an L -convex polyomino (P is not directed), then the feet of P are characterized by the geometries shown in the **Figure 9**.

Case (1) is the first geometry ($GEO1$ in the algorithm).

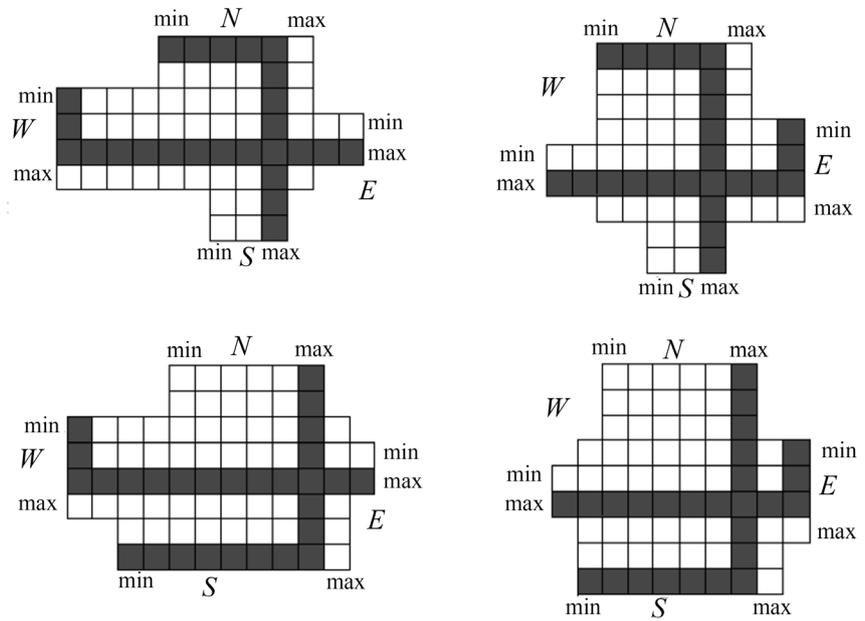


Figure 8. The four different L -paths between the feet in the class μ_L .

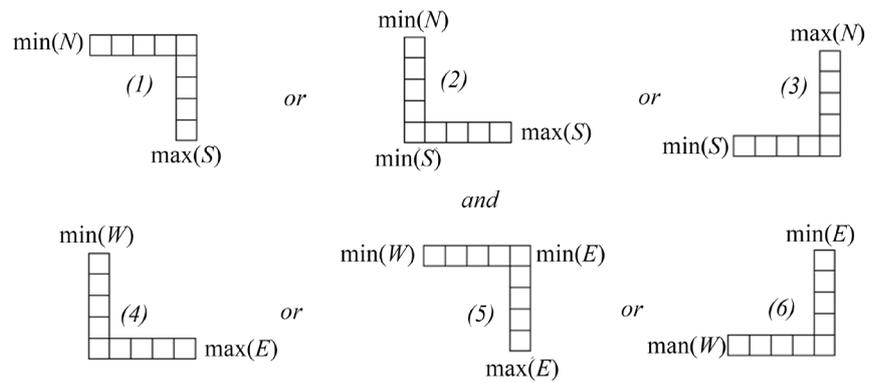


Figure 9. The three types of L -paths between each two opposite feet.

Case (2) is the second geometry ($GEO2$ in the algorithm).

Case (3) is the third geometry ($GEO3$ in the algorithm).

Case (4) is the fourth geometry ($GEO4$ in the algorithm).

Proposition 5. Let P be an L -convex polyomino (P is not directed), then the feet of P are connected at least by one of the nine following geometries of the L -paths in **Figure 9**.

- $(2) \cap (5) \in \alpha_L$
- $(2) \cap (4) \in \beta_L$
- $(2) \cap (6) \in \beta_L$
- $(1) \cap (5) \in \gamma_L$
- $(3) \cap (5) \in \gamma_L$
- $(1) \cap (4) \in \mu_L$

- $(1) \cap (6) \in \mu_L$
- $(3) \cap (4) \in \mu_L$
- $(3) \cap (6) \in \mu_L$.

Remark 1. *The geometries $(1) \cap (4)$, $(2) \cap (5)$, $(2) \cap (6)$, and $(2) \cap (5)$ mentioned in Proposition 5 give directly the two L -paths mentioned in Proposition 3.*

The geometries $(2) \cap (4)$, $(3) \cap (4)$, and $(3) \cap (6)$ in Proposition 5 give directly the L -path from $\min(N)$ to $\max(E)$ with a south step followed by an east step.

The geometries $(1) \cap (5)$ and $(1) \cap (6)$ in Proposition 5 give directly the L -path from $\min(W)$ to $\max(S)$ with an east step followed by a south step.

Now, we define the cells on the SE and WS borders to define the sets X, Z, X' and Z' from these cells.

Let P be a convex polyomino in the class μ (resp. α, β and γ) (P is not directed) and let $I = \{(i_1, j_1), (i_2, j_2), \dots, (i_r, j_r)\}$ be the set of cells belonging to P such that $(i_1, j_1) = (m, \max(S))$, $(i_r, j_r) = (\max(E), n)$, and for $2 \leq k \leq r-1$, let (i_k, j_k) be the cells situated on the border of the set SE .

Similarly, let $J = \{(i'_1, j'_1), (i'_2, j'_2), \dots, (i'_s, j'_s)\}$ be the set of cells belonging to P such that $(i'_1, j'_1) = (m, \min(S))$, $(i'_s, j'_s) = (\max(W), 1)$, and for $2 \leq l \leq s-1$, let (i'_l, j'_l) be the cells situated on the border of the set WS .

Now let $X = \{x_1, \dots, x_k, \dots, x_r\}$ be the set of cells such that

$$x_1 = (m - v_{\max(S)} + 1, \max(S)), \dots, x_k = (i_k - v_{j_k} + 1, j_k), \dots, x_r = (\min(E), n)$$

and $Z = \{z_1, \dots, z_k, \dots, z_r\}$ be the set of cells such that

$$z_1 = (m, \min(S)), \dots, z_k = (i_k, j_k - h_{i_k} + 1), \dots, z_r = (\max(E), n - h_{\max(E)} + 1).$$

Similarly, let $X' = \{x'_1, \dots, x'_l, \dots, x'_s\}$ be the set of cells such that

$$x'_1 = (m - v_{\min(S)} + 1, \min(S)), \dots, x'_l = (i_l - v_{j_l} + 1, j_l), \dots, x'_s = (\min(W), 1)$$

and $Z' = \{z'_1, \dots, z'_l, \dots, z'_s\}$ be the set of cells such that

$$z'_1 = (m, \max(S)), \dots, z'_l = (i'_l, j'_l + h_{i'_l} - 1), \dots, z'_s = (\max(W), 1 + h_{\max(W)} - 1)$$

(see **Figure 10**).

Theorem 1. *Let P be a convex polyomino such that P satisfies at least one of the following geometries*

- $(2) \cap (5) \in \alpha$
- $(2) \cap (4) \in \beta$
- $(2) \cap (6) \in \beta$
- $(1) \cap (1) \in \gamma$
- $(3) \cap (5) \in \gamma$
- $(1) \cap (4) \in \mu$

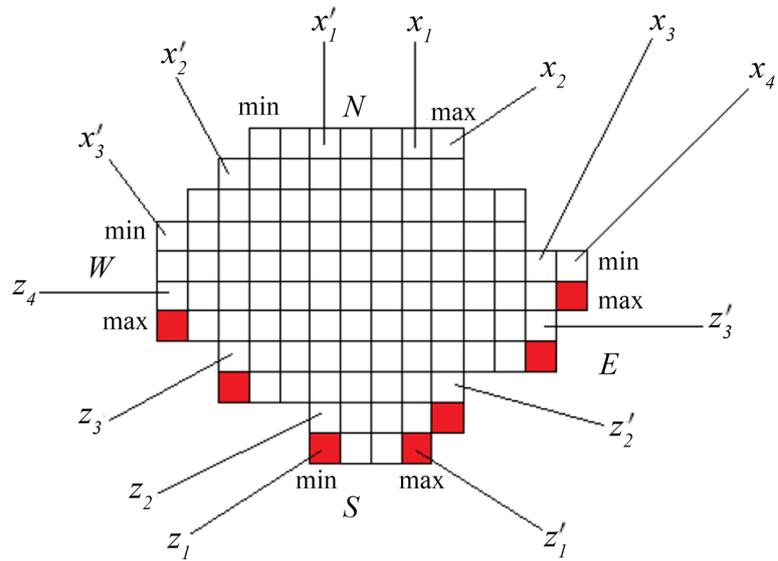


Figure 10. Red cells are the cells situated on the border of SE and WS with $(m, \max(S))$, $(\max(E), n)$, $(m, \min(S))$ and $(\max(W), 1)$.

- $(1) \cap (6) \in \mu$
- $(3) \cap (4) \in \mu$
- $(3) \cap (6) \in \mu$.

Then P is an L -convex polyomino if and only if for $2 \leq k \leq r-1$, $2 \leq l \leq s-1$ the cells situated at the positions

$$(m - v_{\max(S)}, \min(S) - 1), \dots, (i_k - v_{j_k}, j_k - h_{i_k}), \dots, (\min(E) - 1, n - h_{\max(E)})$$

and

$$(m - v_{\min(S)}, \max(S) + 1), \dots, (i_l - v_{j_l}, j_l + h_{i_l}), \dots, (\min(W) - 1, 1 + h_{\max(W)})$$

do not belong to P .

Proof. Suppose that P is a convex polyomino. The intersections control the geometries and the L -path between feet.

\Rightarrow If P is an L -convex then obviously the cells situated at the positions

$$(m - v_{\max(S)}, \min(S) - 1), \dots, (i_k - v_{j_k}, j_k - h_{i_k}), \dots, (\min(E) - 1, n - h_{\max(E)})$$

and

$$(m - v_{\min(S)}, \max(S) + 1), \dots, (i_l - v_{j_l}, j_l + h_{i_l}), \dots, (\min(W) - 1, 1 + h_{\max(W)})$$

do not belong to P . Indeed, these cells could be attained only by using a $2L$ -path from the SE or WS borders.

\Leftarrow The cells situated at the positions

$$(m - v_{\max(S)}, \min(S) - 1), \dots, (i_k - v_{j_k}, j_k - h_{i_k}), \dots, (\min(E) - 1, n - h_{\max(E)})$$

and

$$(m - v_{\min(S)}, \max(S) + 1), \dots, (i_l - v_{j_l}, j_l + h_{i_l}), \dots, (\min(W) - 1, 1 + h_{\max(W)})$$

control maximal rectangles from SE and WS . Thus they control the L -convexity of the polyomino (see **Figure 11**).

Simplification of the Nine Geometries of L -Convex Polyominoes

In this subsection, we show that the four geometries mentioned in Proposition 4 are sufficient to reconstruct non-directed L -convex polyominoes in the subclasses α_L, β_L and γ_L and so the nine geometries can be simplified to obtain only four geometries.

If $\min(W) = \min(E)$ then the geometry could be defined by a point on the larger foot between W -foot and S -foot. If the length of E -foot is larger than the length of W -foot, then we use an L -path between $\max(W)$ and $\min(E)$ thus we use the second geometry $(1 \cap 6)$. If the length of E -foot is smaller than the length of W -foot then we use a L -path between $\min(W)$ and $\max(E)$ thus we use the first geometry $(1 \cap 4)$.

If $\min(N) = \min(S)$ the same arguments give that we use the third geometry $(3 \cap 4)$ or the fourth geometry $(3 \cap 6)$ depending on the relative length of N -foot and S -foot.

So to reconstruct a non-directed L -convex polyomino we use the combinations of the four L -paths (**Figure 12**).

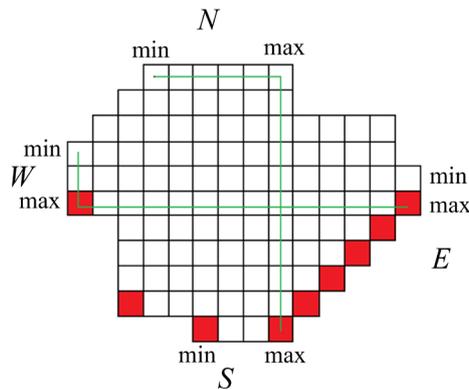


Figure 11. An L -convex polyomino satisfying Theorem 1.

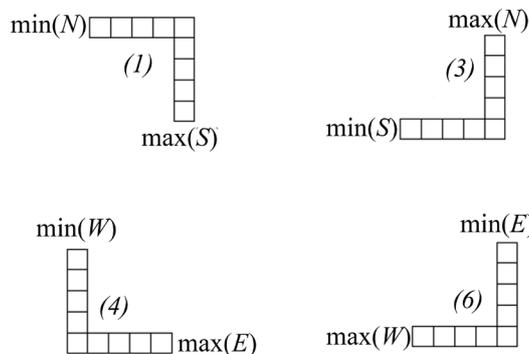


Figure 12. The four L -paths between the feet.

4. Directed L-Convex Polyominoes

Let P be a convex polyomino such that P does not satisfy Proposition 2. From the definition of directed convex polyominoes, let us define the following classes.

- $\delta = \{P \in \mathcal{C} \mid (1, \min(N)) = (\min(W), 1)\}$.
- $\psi = \{P \in \mathcal{C} \mid (\max(W), 1) = (m, \min(S))\}$.
- $\delta' = \{P \in \mathcal{C} \mid (m, \max(S)) = (\max(E), n)\}$.
- $\psi' = \{P \in \mathcal{C} \mid (1, \max(N)) = (\min(E), n)\}$.
- $\delta_L = \{P \in \mathcal{C}_L \mid (1, \min(N)) = (\min(W), 1)\}$ (see Figure 13).
- $\psi_L = \{P \in \mathcal{C}_L \mid (\max(W), 1) = (m, \min(S))\}$.
- $\delta'_L = \{P \in \mathcal{C}_L \mid (m, \max(S)) = (\max(E), n)\}$ (see Figure 13).
- $\psi'_L = \{P \in \mathcal{C}_L \mid (1, \max(N)) = (\min(E), n)\}$.

Let us define the horizontal transformation (symmetry)

$$S_H : (i, j) \rightarrow (m - i + 1, j)$$

which transforms the polyomino P from δ to ψ , δ' to ψ' , δ_L to ψ_L , and δ'_L to ψ'_L . Indeed the transformation acts on the feet of the polyomino as it is shown in the following table (see Figure 14). Thus we only investigate the properties of the classes δ_L and δ'_L .

Proposition 6. *Let P be an L-convex polyomino in the class δ_L , then there exist two L-paths from $\min(N) = \min(W)$ to $\max(E)$ with a south step followed by an east step, and from $\min(N) = \min(W)$ to $\max(S)$ with an east step followed by a south step.*

Theorem 2. *Let P be a convex polyomino in the class δ such that here exist two L-paths from $\min(N) = \min(W)$ to $\max(E)$ with a south followed by an east step, and from $\min(N) = \min(W)$ to $\max(S)$ with an east followed by a south step. Then P is an L-convex polyomino if and only if the cell at the position $(\max(W) + 1, \max(N) + 1)$ does not belong to P (see Figure 15).*

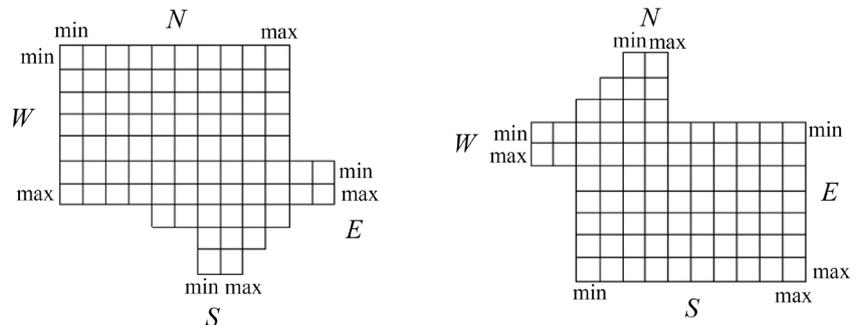


Figure 13. An element of the class δ_L on the left and one of the class δ'_L on the right.

N, S	W, E
$S \rightarrow N$	$W \rightarrow W$
$N \rightarrow S$	$E \rightarrow E$
$\min \rightarrow \min$	$\min \rightarrow \max$
$\max \rightarrow \max$	$\max \rightarrow \min$

Figure 14. The horizontal transformation S_H on the feet of P .

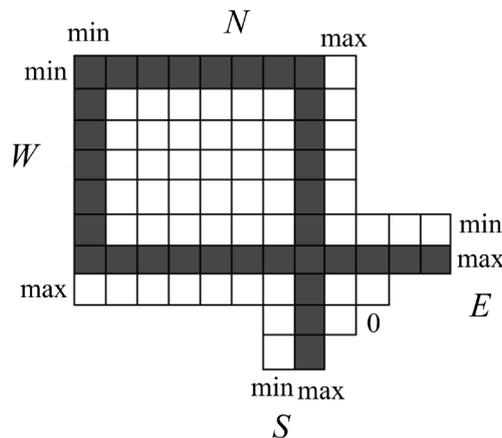


Figure 15. An L -convex polyomino in the class δ_L .

Proposition 7. Let P be an L -convex polyomino in the class δ'_L , then there exist two L -paths from $\max(E) = \max(S)$ to $\min(N)$ with a west step followed by a north step, and from $\max(E) = \max(S)$ to $\min(W)$ with a north step followed by a west step.

Theorem 3. Let P be a convex polyomino in the class δ' such that there exist two L -paths from $\max(E) = \max(S)$ to $\min(N)$ with a west step followed by a north step, and from $\max(E) = \max(S)$ to $\min(W)$ with a north step followed by a west step. Then P is an L -convex polyomino if and only if the cell at the position $(\min(E) - 1, \min(S) - 1)$ does not belong to P (see Figure 16).

5. Reconstruction Algorithms

One main problem in discrete tomography consists on the reconstruction of discrete objects according to their vectors of projections. In order to restrain the number of solutions, we could add convexity constraints to these discrete objects. The present section uses the theoretical material presented in the above sections in order to reconstruct all subclasses of L -convex polyominoes. Some modifications are made in the reconstruction algorithm of Chrobak and Dürr for HV -convex

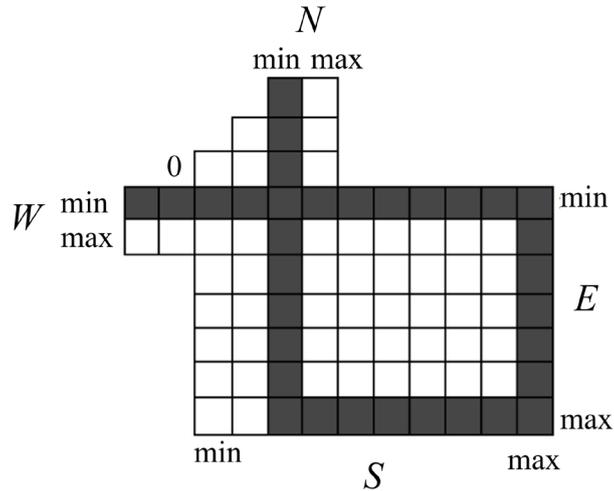


Figure 16. An L -convex polyomino in the class δ'_L .

polyominoes in order to impose our geometries. All the clauses that have been added and the modifications of the original algorithm are well explained in the proofs of each subclass.

5.1. Chrobak and Dürr's Algorithm

Assume that H, V denote strictly positive row and column sum vectors. We also assume that $\sum_i h_i = \sum_j v_j$, since otherwise (H, V) do not have a realization.

The idea of Chrobak and Dürr [6] for the control of the HV -convexity is in fact to impose convexity on the four corner regions outside of the polyomino.

An object A is called an *upper-left corner region* if $(i+1, j) \in A$ or $(i, j+1) \in A$ implies $(i, j) \in A$. In an analogous fashion they can define other corner regions. Let \bar{P} be the complement of P . The definition of HV -convex polyominoes directly implies the following lemma.

Lemma 1. P is an HV -convex polyomino if and only if $\bar{P} = A \cup B \cup C \cup D$, where A, B, C, D are disjoint corner regions (upper-left, upper-right, lower-left and lower-right, respectively) such that 1) $(i-1, j-1) \in A$ implies (i, j) not in D , and 2) $(i-1, j+1) \in B$ implies $(i, j) \notin C$.

Given an HV -convex polyomino P and two row indices $1 \leq k, l \leq m$. P is anchored at (k, l) if $(k, 1), (l, n) \in P$. The idea of Chrobak and Dürr is, given (H, V) , to reconstruct a 2SAT expression (a boolean expression in conjunctive normal form with at most two literals in each clause) $F_{k,l}(H, V)$ with the property that $F_{k,l}(H, V)$ is satisfiable iff there is an HV -convex polyomino realization P of (H, V) that is anchored at (k, l) . $F_{k,l}(H, V)$ consists of several sets of clauses, each set expressing a certain property: "Corners" (Cor), "Disjointness" (Dis), "Connectivity" (Con), "Anchors" (Anc), "Lower bound on column sums" (LBC) and "Upper bound on row sums" (UBR).

$$Cor = \bigwedge_{i,j} \left\{ \begin{array}{llll} A_{i,j} \Rightarrow A_{i-1,j} & B_{i,j} \Rightarrow B_{i-1,j} & C_{i,j} \Rightarrow C_{i+1,j} & D_{i,j} \Rightarrow D_{i+1,j} \\ A_{i,j} \Rightarrow A_{i,j-1} & B_{i,j} \Rightarrow B_{i,j+1} & C_{i,j} \Rightarrow C_{i,j-1} & D_{i,j} \Rightarrow D_{i,j+1} \end{array} \right\}$$

The set of clauses Cor means that the corners are convex, that is for the corner A if the cell (i, j) belongs to A then cells $(i-1, j)$ and $(i, j-1)$ belong also to A . Similarly for corners B, C , and D .

$$Dis = \bigwedge_{i,j} \{X_{i,j} \Rightarrow \bar{Y}_{i,j} : \text{for symbols } X, Y \in \{A, B, C, D\}, X \neq Y\}$$

The set of clauses Dis means that all four corners are pairwise disjoint, that is $X \cap Y = \emptyset$ for $X, Y \in \{A, B, C, D\}$.

$$Con = \bigwedge_{i,j} \{A_{i,j} \Rightarrow \bar{D}_{i+1,j+1} \quad B_{i,j} \Rightarrow \bar{C}_{i+1,j-1}\}$$

The set of clauses Con means that if the cell (i, j) belongs to A then the cell $(i+1, j+1)$ does not belong to D , and similarly if the cell (i, j) belongs to B then the cell $(i+1, j-1)$ does not belong to C .

$$Anc = \{\bar{A}_{k,1} \wedge \bar{B}_{k,1} \wedge \bar{C}_{k,1} \wedge \bar{D}_{k,1} \wedge \bar{A}_{l,n} \wedge \bar{B}_{l,n} \wedge \bar{C}_{l,n} \wedge \bar{D}_{l,n}\}$$

The set of clauses Anc means that we fix two cells on the west and east feet of the polyomino P , for $k, l = 1, \dots, m$ the first one at the position $(k, 1)$ and the second one at the position (l, n) .

$$LBC = \bigwedge_{i,j} \left\{ \begin{array}{l} A_{i,j} \Rightarrow \bar{C}_{i+v_j,j} \quad A_{i,j} \Rightarrow \bar{D}_{i+v_j,j} \\ B_{i,j} \Rightarrow \bar{C}_{i+v_j,j} \quad B_{i,j} \Rightarrow \bar{D}_{i+v_j,j} \end{array} \right\} \wedge \bigwedge_j \{ \bar{C}_{v_j,j} \quad \bar{D}_{v_j,j} \}$$

The set of clauses LBC implies that for each column j , we have that $\sum_i P_{i,j} \geq v_j$.

$$UBR = \bigwedge_j \left\{ \begin{array}{l} \bigwedge_{i \leq \min\{k,l\}} \bar{A}_{i,j} \Rightarrow B_{i,j+h_i} \quad \bigwedge_{k \leq i \leq l} \bar{C}_{i,j} \Rightarrow B_{i,j+h_i} \\ \bigwedge_{l \leq i \leq k} \bar{A}_{i,j} \Rightarrow D_{i,j+h_i} \quad \bigwedge_{\max\{k,l\} \leq i} \bar{C}_{i,j} \Rightarrow D_{i,j+h_i} \end{array} \right\}$$

The set of clauses UBR implies that for each row i , we have that $\sum_j P_{i,j} \leq h_i$.

Define $F_{k,l}(H, V) = Cor \wedge Dis \wedge Con \wedge Anc \wedge LBC \wedge UBR$. All literals with indices outside the set $\{1, \dots, m\} \times \{1, \dots, n\}$ are assumed to have value 1.

Algorithm 1.

Input: $H \in \mathbb{N}^m, V \in \mathbb{N}^n$

W.l.o.g assume: $\forall i: h_i \in [1, n], \forall j: v_j \in [1, m], \sum_i h_i = \sum_j v_j$ and $m \leq n$.

For $k, l = 1, \dots, m$ **do begin**

If $F_{k,l}(H, V)$ is satisfiable,

then output $P = \overline{A \cup B \cup C \cup D}$ and **halt**.

end

output “failure”.

The following theorem allows to link the existence of HV -convex solution and the evaluation of $F_{k,l}(H, V)$. The crucial part of this algorithm comes from the constraints on the two sets of clauses LBC and UBR .

Theorem 4 (Chrobak, Durr) $F_{k,l}(H, V)$ is satisfiable if and only if (H, V) have a realization P that is an HV -convex polyomino anchored at (k, l) .

Theorem 5 (Chrobak, Durr) Algorithm 1 solves the reconstruction problem for HV -convex polyominoes in time $O(mn \min(m^2, n^2))$.

5.2. Reconstruction of L -Convex Polyominoes

In this subsection, we add the clauses $Anc1$, $COND1$, $COND2$, $GEO1$, $GEO2$, $GEO3$, $GEO4$, $For1$ and we modify the clause Anc of the original Chrobak and Dürr’s algorithm in order to reconstruct if it is possible all polyominoes in the subclass $\alpha_L, \beta_L, \gamma_L$ and μ_L .

$$Cor = \bigwedge_{i,j} \left\{ \begin{array}{llll} A_{i,j} \Rightarrow A_{i-1,j} & B_{i,j} \Rightarrow B_{i-1,j} & C_{i,j} \Rightarrow C_{i+1,j} & D_{i,j} \Rightarrow D_{i+1,j} \\ A_{i,j} \Rightarrow A_{i,j-1} & B_{i,j} \Rightarrow B_{i,j+1} & C_{i,j} \Rightarrow C_{i,j-1} & D_{i,j} \Rightarrow D_{i,j+1} \end{array} \right\}$$

$$Dis = \bigwedge_{i,j} \{ X_{i,j} \Rightarrow \bar{Y}_{i,j} : \text{for symbols } X, Y \in \{A, B, C, D\}, X \neq Y \}$$

$$Con = \bigwedge_{i,j} \{ A_{i,j} \Rightarrow \bar{D}_{i+1,j+1} \quad B_{i,j} \Rightarrow \bar{C}_{i+1,j-1} \}$$

$$Anc = \left\{ \begin{array}{l} \bar{A}_{\min(W),1} \wedge \bar{A}_{\min(E),n} \wedge \bar{B}_{\min(W),1} \wedge \bar{B}_{\min(E),n} \wedge \\ \bar{C}_{\min(W),1} \wedge \bar{C}_{\min(E),n} \wedge \bar{D}_{\min(W),1} \wedge \bar{D}_{\min(E),n} \wedge \\ \bar{A}_{1,\min(N)} \wedge \bar{A}_{m,\min(S)} \wedge \bar{B}_{1,\min(N)} \wedge \bar{B}_{m,\min(S)} \wedge \\ \bar{C}_{1,\min(N)} \wedge \bar{C}_{m,\min(S)} \wedge \bar{D}_{1,\min(N)} \wedge \bar{D}_{m,\min(S)} \wedge \\ \bar{A}_{\max(W),1} \wedge \bar{A}_{\max(E),n} \wedge \bar{B}_{\max(W),1} \wedge \bar{B}_{\max(E),n} \wedge \\ \bar{C}_{\max(W),1} \wedge \bar{C}_{\max(E),n} \wedge \bar{D}_{\max(W),1} \wedge \bar{D}_{\max(E),n} \wedge \\ \bar{A}_{1,\max(N)} \wedge \bar{A}_{m,\max(S)} \wedge \bar{B}_{1,\max(N)} \wedge \bar{B}_{m,\max(S)} \wedge \\ \bar{C}_{1,\max(N)} \wedge \bar{C}_{m,\max(S)} \wedge \bar{D}_{1,\max(N)} \wedge \bar{D}_{m,\max(S)} \end{array} \right\}$$

$$Anc1 = \{ A_{1,1} \wedge B_{1,n} \wedge C_{m,1} \wedge D_{m,n} \}$$

$Anc1$ is added in order to consider non-directed convex polyominoes by positioning exterior cells of the polyomino in the four corners of the minimal bounding box.

$$LBC = \bigwedge_i \left\{ \begin{array}{ll} \bigwedge_{j < \min(N)} A_{i,j} \Rightarrow \bar{C}_{i+v_j,j} & \bigwedge_{j > \max(S)} B_{i,j} \Rightarrow \bar{D}_{i+v_j,j} \\ \bigwedge_{\max(N) < j < \min(S)} B_{i,j} \Rightarrow \bar{C}_{i+v_j,j} & \bigwedge_{\min(S) \leq j \leq \max(S)} B_{i,j} \Rightarrow \bar{C}_{i+v_j,j} \\ \bigwedge_{\min(N) \leq j \leq \max(N)} C_{i+v_j,j} \Rightarrow \bar{A}_{i,j} & \end{array} \right\}$$

$$\wedge \bigwedge_j \{ \bar{C}_{v_j,j} \quad \bar{D}_{v_j,j} \}$$

$$UBR = \bigwedge_j \left\{ \begin{array}{ll} \bigwedge_{i < \min(W)} \bar{A}_{i,j} \Rightarrow B_{i,j+h_i} & \bigwedge_{i > \max(E)} \bar{C}_{i,j} \Rightarrow D_{i,j+h_i} \\ \bigwedge_{\max(W) < i < \min(E)} \bar{C}_{i,j} \Rightarrow B_{i,j+h_i} & \bigwedge_{\min(E) \leq i \leq \max(E)} \bar{B}_{i,j+h_i} \Rightarrow C_{i,j} \\ \bigwedge_{\min(W) \leq i \leq \max(W)} \bar{A}_{i,j} \Rightarrow B_{i,j+h_i} & \end{array} \right\}$$

$$COND1 = \{ \bar{A}_{\max(E),\min(N)} \wedge \bar{B}_{\max(E),\min(N)} \wedge \bar{C}_{\max(E),\min(N)} \wedge \bar{D}_{\max(E),\min(N)} \}$$

$COND1$ controls the L -path between E -foot and N -foot (see proposition 3).

$$COND2 = \{ \bar{A}_{\min(W),\max(S)} \wedge \bar{B}_{\min(W),\max(S)} \wedge \bar{C}_{\min(W),\max(S)} \wedge \bar{D}_{\min(W),\max(S)} \}$$

$COND2$ controls the L -path between W -foot and S -foot (see proposition 3).

$$GEO1 = \left\{ \begin{array}{l} \bar{A}_{1,\max(S)} \wedge \bar{B}_{1,\max(S)} \wedge \bar{C}_{1,\max(S)} \wedge \bar{D}_{1,\max(S)} \wedge \\ \bar{A}_{\max(E),1} \wedge \bar{B}_{\max(E),1} \wedge \bar{C}_{\max(E),1} \wedge \bar{D}_{\max(E),1} \end{array} \right\}$$

GEO1 controls the first geometry ($1 \cap 4$).

$$GEO2 = \left\{ \begin{array}{l} \bar{A}_{1,\max(S)} \wedge \bar{B}_{1,\max(S)} \wedge \bar{C}_{1,\max(S)} \wedge \bar{D}_{1,\max(S)} \wedge \\ \bar{A}_{\max(W),n} \wedge \bar{B}_{\max(W),n} \wedge \bar{C}_{\max(W),n} \wedge \bar{D}_{\max(W),n} \end{array} \right\}$$

GEO2 controls the second geometry ($1 \cap 6$).

$$GEO3 = \left\{ \begin{array}{l} \bar{A}_{m,\max(N)} \wedge \bar{B}_{m,\max(N)} \wedge \bar{C}_{m,\max(N)} \wedge \bar{D}_{m,\max(N)} \wedge \\ \bar{A}_{1,\max(E)} \wedge \bar{B}_{1,\max(E)} \wedge \bar{C}_{1,\max(E)} \wedge \bar{D}_{1,\max(E)} \end{array} \right\}$$

GEO3 controls the third geometry ($3 \cap 4$).

$$GEO4 = \left\{ \begin{array}{l} \bar{A}_{m,\max(N)} \wedge \bar{B}_{m,\max(N)} \wedge \bar{C}_{m,\max(N)} \wedge \bar{D}_{m,\max(N)} \wedge \\ \bar{A}_{\max(W),n} \wedge \bar{B}_{\max(W),n} \wedge \bar{C}_{\max(W),n} \wedge \bar{D}_{\max(W),n} \end{array} \right\}$$

GEO4 controls the fourth geometry ($3 \cap 6$).

$$For1 = \bigwedge_{i,j} \left\{ \begin{array}{ll} B_{SEi,j} \Rightarrow \bar{D}_{i,j} & B_{SEi,j} \Rightarrow D_{i,j+1} \\ B_{SEi,j} \Rightarrow D_{i+1,j} & B_{SEi,j} \Rightarrow A_{i-v_j,j-h_i} \\ B_{WSi,j} \Rightarrow \bar{C}_{i,j} & B_{WSi,j} \Rightarrow C_{i,j-1} \\ B_{WSi,j} \Rightarrow C_{i+1,j} & B_{WSi,j} \Rightarrow B_{i-v_j,j+h_i} \end{array} \right\}$$

For1 controls the cells in the *SE* and *WS* borders of *P* and imposes that the cells of Theorem 1 are outside the polyomino *P*. In order to reconstruct and to obtain all *L*-convex polyominoes, we use the set of clauses:

$$F_{L,GEO1}(H, V) = Cor \wedge Dis \wedge Con \wedge Anc \wedge Anc1 \wedge LBC \wedge UBR \\ \wedge COND1 \wedge COND2 \wedge GEO1 \wedge For1$$

$$F_{L,GEO2}(H, V) = Cor \wedge Dis \wedge Con \wedge Anc \wedge Anc1 \wedge LBC \wedge UBR \\ \wedge COND1 \wedge COND2 \wedge GEO2 \wedge For1$$

$$F_{L,GEO3}(H, V) = Cor \wedge Dis \wedge Con \wedge Anc \wedge Anc1 \wedge LBC \wedge UBR \\ \wedge COND1 \wedge COND2 \wedge GEO3 \wedge For1$$

$$F_{L,GEO4}(H, V) = Cor \wedge Dis \wedge Con \wedge Anc \wedge Anc1 \wedge LBC \wedge UBR \\ \wedge COND1 \wedge COND2 \wedge GEO4 \wedge For1$$

Algorithm 2.

Input: $H \in \mathbb{N}^m, V \in \mathbb{N}^n$

W.l.o.g assume: $\forall i : h_i \in [1, n], \forall j : v_j \in [1, m], \sum_i h_i = \sum_j v_j$.

For $\min(W), \min(E) = 1, \dots, m$ and

$\min(N), \min(S) = 1, \dots, n$ **do begin**

If $F_{L,GEO1}(H, V)$ or $F_{L,GEO2}(H, V)$ or $F_{L,GEO3}(H, V)$ or $F_{L,GEO4}(H, V)$ is satisfiable,

then output $P = \overline{A \cup B \cup C \cup D}$ and **halt**.

end

output “failure”.

Proof. The feet of all *L*-convex polyominoes that are not directed are characterized by at least one of the four geometries described in Theorem 1 and

by the property that the cells situated at the positions

$$\left(m - v_{\max(S)}, \max(S) - h_m\right), \dots, \left(i_k - v_{j_k}, j_k - h_{i_k}\right), \dots, \left(\max(E) - v_n, n - h_{\max(E)}\right)$$

do not belong to these polyominoes. Thus we combine all geometries and conditions using suitable set of clauses in order to reconstruct L -convex polyominoes. We make the following modifications of the original algorithm of Chrobak and Dürr [6] in order to add the geometrical constraints.

The set *COND1* (resp. *COND2*) implies that we put a cell in the interior of the polyomino at the position $(\max(E), \min(N))$ (resp. $(\min(W), \max(S))$) and then by convexity an L -path between $\max(E)$ and $\min(N)$ (resp. $\min(W)$ and $\max(S)$).

The set *GEO1* implies that we put a cell in the interior of the polyomino at the position $(1, \max(S))$ (resp. $(\max(E), 1)$) and then by convexity an L -path between $\min(N)$ and $\max(S)$ with an east step followed by a south step (resp. $\min(W)$ and $\max(E)$ with a south step followed by an east step).

The set *GEO2* implies that we put a cell in the interior of the polyomino at the position $(m, \min(N))$ (resp. $(\min(W), n)$) and then by convexity an L -path between $\min(N)$ and $\max(S)$ with a south step followed by then an east step (resp. $\min(W)$ and $\max(E)$ with an east step followed by a south step).

The set *GEO3* implies that we put a cell in the interior of the polyomino at the position $(m, \min(N))$ (resp. $(\max(E), 1)$) and then by convexity an L -path between $\min(N)$ and $\max(S)$ with an east followed by a south step (resp. $\min(W)$ and $\max(E)$ with a south step followed by an east step).

The set *GEO4* implies that we put a cell in the interior of the polyomino at the position $(1, \max(S))$ (resp. $(\min(W), n)$) and then by convexity an L -path between $\min(N)$ and $\max(S)$ with an east step followed by a south step (resp. $\min(W)$ and $\max(E)$ with an east step followed by a south step).

The set *For1* implies that the cell (i, j) is situated on the border of SE with the two cells $(m, \max(S))$ and $(\max(E), n)$. In fact, $B_{SEi,j} \Rightarrow \bar{D}_{i,j}$, $B_{SEi,j} \Rightarrow D_{i,j+1}$ and $B_{SEi,j} \Rightarrow D_{i+1,j}$ imply that (i, j) is on the border and $B_{SEi,j} \Rightarrow A_{i-v_j, j+h_i}$ implies that for each (i, j) situated on the border the cell at the position $(i - v_j, j + h_i)$ does not belong to P .

Using the conjunction of the whole set of clauses, if one of the

$F_{L,GEO1}(H, V)$ or $F_{L,GEO2}(H, V)$ or $F_{L,GEO3}(H, V)$ or $F_{L,GEO4}(H, V)$ is satisfiable, then we are able to reconstruct a L -convex polyomino which is not directed.

5.3. Clauses for the Subclass δ_L

In this subsection, we add the clauses *Pos*, *GEO*, *For2* and we modify the clause *Anc* of the original Chrobak and Dürr's algorithm in order to reconstruct if it is possible all polyominoes in the class δ_L .

$$Pos = \left\{ B_{(1, \max(N)+1)} \wedge C_{(m, \min(S)-1)} \wedge \bar{A}_{1,1} \right\}$$

$$\begin{aligned}
 Cor &= \bigwedge_{i,j} \left\{ \begin{array}{lll} B_{i,j} \Rightarrow B_{i-1,j} & C_{i,j} \Rightarrow C_{i+1,j} & D_{i,j} \Rightarrow D_{i+1,j} \\ B_{i,j} \Rightarrow B_{i,j+1} & C_{i,j} \Rightarrow C_{i,j-1} & D_{i,j} \Rightarrow D_{i,j+1} \end{array} \right\} \\
 Dis &= \bigwedge_{i,j} \{ X_{i,j} \Rightarrow \bar{Y}_{i,j} : \text{for symbols } X, Y \in \{B, C, D\}, X \neq Y \} \\
 Con &= \bigwedge_{i,j} \{ B_{i,j} \Rightarrow \bar{C}_{i+1,j-1} \} \\
 Anc &= \left\{ \begin{array}{l} \bar{B}_{1,1} \wedge \bar{B}_{\min(E),n} \wedge \bar{C}_{1,1} \wedge \bar{C}_{\min(E),n} \wedge \\ \bar{D}_1 \wedge \bar{D}_{\min(E),n} \wedge \bar{B}_{1,1} \wedge \bar{B}_{m,\min(S)} \wedge \\ \bar{C}_{1,1} \wedge \bar{C}_{m,\min(S)} \wedge \bar{D}_{1,1} \wedge \bar{D}_{m,\min(S)} \wedge \\ \bar{B}_{\max(W),1} \wedge \bar{B}_{\max(E),n} \wedge \bar{C}_{\max(W),1} \wedge \bar{C}_{\max(E),n} \wedge \\ \bar{D}_{\max(W),1} \wedge \bar{D}_{\max(E),n} \wedge \bar{B}_{1,\max(N)} \wedge \bar{B}_{m,\max(S)} \wedge \\ \bar{C}_{1,\max(N)} \wedge \bar{C}_{m,\max(S)} \wedge \bar{D}_{1,\max(N)} \wedge \bar{D}_{m,\max(S)} \end{array} \right\} \\
 LBC &= \bigwedge_i \left\{ \begin{array}{ll} \bigwedge_{j > \max(S)} B_{i,j} \Rightarrow \bar{D}_{i+v_j,j} & \bigwedge_{\min(S) \leq j \leq \max(S)} B_{i,j} \Rightarrow \bar{D}_{i+v_j,j} \\ \bigwedge_{\min(N) \leq j \leq \max(N)} C_{i+v_j,j} \Rightarrow \bar{B}_{i,j} & \bigwedge_{\max(N) < j < \min(S)} B_{i,j} \Rightarrow \bar{C}_{i+v_j,j} \end{array} \right\} \\
 &\quad \wedge \bigwedge_j \{ \bar{C}_{v_j,j} \quad \bar{D}_{v_j,j} \} \\
 UBR &= \bigwedge_j \left\{ \begin{array}{ll} \bigwedge_{i > \max(E)} \bar{C}_{i,j} \Rightarrow D_{i,j+h_i} & \bigwedge_{\min(E) \leq i \leq \max(E)} \bar{D}_{i,j+h_i} \Rightarrow C_{i,j} \\ \bigwedge_{\min(W) \leq i \leq \max(W)} \bar{C}_{i,j} \Rightarrow B_{i,j+h_i} & \bigwedge_{\max(W) < i < \min(E)} \bar{C}_{i,j} \Rightarrow B_{i,j+h_i} \end{array} \right\} \\
 GEO &= \left\{ \begin{array}{l} \bar{A}_{1,\max(S)} \wedge \bar{B}_{1,\max(S)} \wedge \bar{C}_{1,\max(S)} \wedge \bar{D}_{1,\max(S)} \wedge \\ \bar{A}_{\max(E),1} \wedge \bar{B}_{\max(E),1} \wedge \bar{C}_{\max(E),1} \wedge \bar{D}_{\max(E),1} \end{array} \right\} \\
 For2 &= \{ D_{\max(W)+1, \max(N)+1} \}
 \end{aligned}$$

In order to reconstruct all L -convex polyominoes in the class δ_L , we use the set of clauses:

$$\delta_L(H, V) = Pos \wedge Cor \wedge Dis \wedge Con \wedge Anc \wedge LBC \wedge UBR \wedge GEO \wedge For2.$$

Algorithm 3.

Input: $H \in \mathbb{N}^m, V \in \mathbb{N}^n$

W.l.o.g assume: $\forall i : h_i \in [1, n], \forall j : v_j \in [1, m], \sum_i h_i = \sum_j v_j$.

For $\min(W) = 1, \min(E) = 1, \dots, m$ and

$\min(N) = 1, \min(S) = 1, \dots, n$ **do begin**

If $\delta_L(H, V)$ is satisfiable,

then output $P = \overline{B \cup C \cup D}$ and **halt**.

end

output “failure”.

Proof. We make the following modifications of the original algorithm of Chrobak and Dürr in order to add the constraints and the properties of the class δ_L . The set Pos imposes the constraints of the relative positions of the feet in δ_L . The set GEO implies that the cells at the position $(\max(E), \min(N))$ and $(\min(W), \max(S))$ belong to P and thus, by convexity there exist L -paths from $\max(E)$ to $\min(N)$ and from $\min(W)$, to $\max(S)$. The set $For2$ implies

that the cell at the position $(\max(W)+1, \max(N)+1)$ does not belong to P (see **Figure 17**).

5.4. Clauses for the Class δ'_L

$$\begin{aligned}
 Pos &= \{C_{(\max(W)+1,1)} \wedge C_{(1,\max(N)+1)} \wedge \bar{D}_{m,n}\} \\
 Cor &= \bigwedge_{i,j} \left\{ \begin{array}{lll} A_{i,j} \Rightarrow A_{i-1,j} & B_{i,j} \Rightarrow B_{i-1,j} & C_{i,j} \Rightarrow C_{i+1,j} \\ A_{i,j} \Rightarrow A_{i,j-1} & B_{i,j} \Rightarrow B_{i,j+1} & C_{i,j} \Rightarrow C_{i,j-1} \end{array} \right\} \\
 Dis &= \bigwedge_{i,j} \{X_{i,j} \Rightarrow \bar{Y}_{i,j} : \text{for symbols } X, Y \in \{A, B, C\}, X \neq Y\} \\
 Con &= \bigwedge_{i,j} \{B_{i,j} \Rightarrow \bar{C}_{i+1,j-1}\} \\
 Anc &= \left\{ \begin{array}{l} \bar{A}_{\min(W),1} \wedge \bar{A}_{\min(E),n} \wedge \bar{B}_{\min(W),1} \wedge \bar{B}_{\min(E),n} \wedge \\ \bar{C}_{\min(W),1} \wedge \bar{C}_{\min(E),n} \wedge \bar{A}_{1,\min(N)} \wedge \bar{A}_{m,\min(S)} \wedge \\ \bar{B}_{1,\min(N)} \wedge \bar{B}_{m,\min(S)} \wedge \bar{C}_{1,\min(N)} \wedge \bar{C}_{m,\min(S)} \wedge \\ \bar{A}_{\max(W),1} \wedge \bar{A}_{m,n} \wedge \bar{B}_{\max(W),1} \wedge \bar{B}_{m,n} \wedge \\ \bar{C}_{\max(W),1} \wedge \bar{C}_{m,n} \wedge \bar{A}_{1,\max(N)} \wedge \bar{A}_{m,n} \wedge \\ \bar{B}_{1,\max(N)} \wedge \bar{B}_{m,n} \wedge \bar{C}_{1,\max(N)} \wedge \bar{C}_{m,n} \end{array} \right\} \\
 LBC &= \bigwedge_i \left\{ \begin{array}{ll} \bigwedge_{j < \min(N)} A_{i,j} \Rightarrow \bar{C}_{i+v_j,j} & \bigwedge_{\min(S) \leq j \leq \max(S)} B_{i,j} \Rightarrow \bar{C}_{i+v_j,j} \\ \bigwedge_{\min(N) \leq j \leq \max(N)} C_{i+v_j,j} \Rightarrow \bar{A}_{i,j} & \bigwedge_{\max(N) < j < \min(S)} B_{i,j} \Rightarrow \bar{C}_{i+v_j,j} \end{array} \right\} \\
 &\quad \wedge \bigwedge_j \{\bar{C}_{v_j,j}\} \\
 UBR &= \bigwedge_j \left\{ \begin{array}{ll} \bigwedge_{i < \min(W)} \bar{A}_{i,j} \Rightarrow B_{i,j+h_i} & \bigwedge_{\min(E) \leq i \leq \max(E)} \bar{B}_{i,j+h_i} \Rightarrow C_{i,j} \\ \bigwedge_{\min(W) \leq i \leq \max(W)} \bar{A}_{i,j} \Rightarrow B_{i,j+h_i} & \bigwedge_{\max(W) < i \leq \min(E)} \bar{C}_{i,j} \Rightarrow B_{i,j+h_i} \end{array} \right\}
 \end{aligned}$$

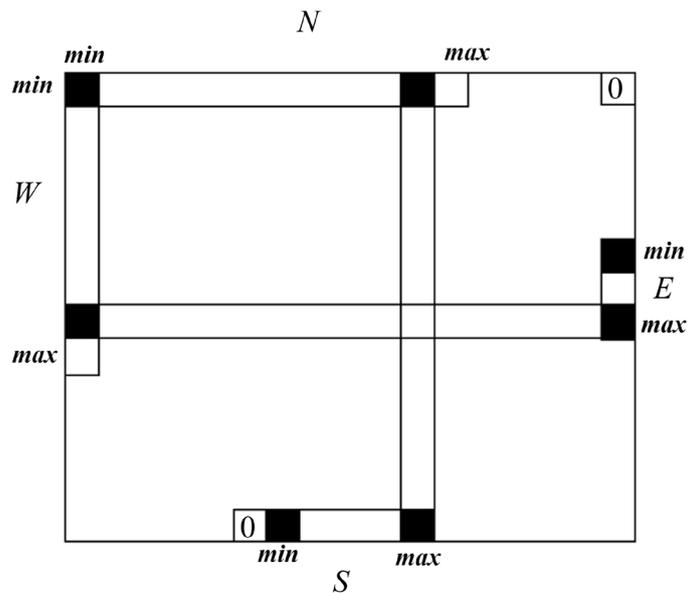


Figure 17. Position, anchored and GEO of the feet in the class δ_L .

$$GEO' = \left\{ \begin{array}{l} \bar{A}_{m, \min(N)} \wedge \bar{B}_{m, \min(N)} \wedge \bar{C}_{m, \min(N)} \wedge \bar{D}_{m, \min(N)} \wedge \\ \bar{A}_{\min(W), n} \wedge \bar{B}_{\min(W), n} \wedge \bar{C}_{\min(W), n} \wedge \bar{D}_{\min(W), n} \end{array} \right\}$$

$$For3 = \left\{ A_{\min(E)-1, \min(S)-1} \right\}$$

In order to reconstruct all L -convex polyominoes in the class δ'_L , we use the set of clauses

$$\delta'_L(H, V) = Pos \wedge Cor \wedge Dis \wedge Con \wedge Anc \wedge LBC \wedge UBR \wedge GEO' \wedge For3.$$

Algorithm 4

Input: $H \in \mathbb{N}^m, V \in \mathbb{N}^n$

W.l.o.g assume: $\forall i: h_i \in [1, n], \forall j: v_j \in [1, m], \sum_i h_i = \sum_j v_j$.

For $\max(E) = m, \min(W), \min(E) = 1, \dots, m$ and

$\max(S) = n, \min(N), \min(S) = 1, \dots, n$ **do begin**

If $\delta'_L(H, V)$ is satisfiable,

then output $P = A \cup B \cup C$ and **halt**.

end output "failure".

6. Final Comment

The contribution of this paper will be used to investigate the geometrical and tomographical aspects of kL -convex polyominoes for $k \in \mathbb{N}$.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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