Open Journal of Discrete Mathematics, 2017, 7, 185-199
http://www.scirp.org/journal/ojdm
ISSN Online: 2161-7643
ISSN Print: 2161-7635

d-Distance Coloring of Generalized Petersen Graphs *P*(*n*, *k*)

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How to cite this paper: Shaheen, R., Kanaya, Z. and Jakhlab, S. (2017) *d-Distance* Coloring of Generalized Petersen Graphs *P(n, k)*. *Open Journal of Discrete Mathematics*, **7**, 185-199.
https://doi.org/10.4236/ojdm.2017.74017

Received: July 18, 2017
Accepted: September 10, 2017
Published: September 13, 2017

Abstract

A coloring $f: V(G) \to \{1, \cdots, k\}$ of G is *d-distance* if any two vertices at distance at most d from each other get different colors. The minimum number of colors in *d-distance colorings* of G is its *d-distance chromatic number*, denoted by $\chi_d(G)$. In this paper, we give the exact value of $\chi_d(G)$ ($d = 1, 2$), for some types of generalized Petersen graphs *P(n, k)* where $k = 1, 2, 3$ and arbitrary n.

Keywords

Distance Coloring, Generalized Petersen Graphs

1. Introduction

Let $G = (V, E)$ be simple graph. A *vertex k-coloring* of G is a mapping from $V(G)$ to the set $\{1, 2, \cdots, k\}$ such that any two adjacent vertices are mapped to different integers. The smallest integer k for which a *k-coloring* exists is called the *chromatic number* of G, denoted by $\chi(G)$. The *d-distance* between two distinct vertices u and v, $d(u, v)$ is the number of edges of the shortest path joining them. The *d-distance k-coloring*, also called distance (d, k)-*coloring*, is a *k-coloring* of the graph G, that is, any two vertices within distance d in G receive different colors. The *d-distance* chromatic number of G is exactly the chromatic number of G under the *d-distance* condition, denoted by $\chi_d(G)$. For a simple graph G, the *d*th power of G, (G^d of G) is defined such that $V(G^d) = V(G)$ and two vertices u and v are adjacent in G^d if and only if the distance between u and v in G is at most d. Clearly, the following inequality is holds:

$$\chi(G) = \chi_1(G) \le \chi(G^2) = \chi_2(G) \le \chi(G^d) = \chi_d(G) \quad \text{for } d \ge 2.$$

The theory of plane graph coloring has a long history, extending back to the middle of the 19[th]century. In 1969, Florica Kramer and Horst Kramer [1] [2] defined the chromatic number $\chi_d(G)$ relative to distance d of a graph $G(V, E)$ to be the minimum number of colors which are sufficient for coloring the vertices of G in such a way that any two vertices of G of distance not greater than d have distinct colors. In 1977, Wegner [3], studied the problem of distance coloring of planar graphs. Alon and Mohar [4] considered the maximum possible chromatic number of G^2, as G ranges over all graphs with maximum degree d and girth g. Bonamy *et al.* [5], studied the 2-distance coloring of sparse graphs. They proved that every graph with maximum degree Δ at least 4 and maximum average degree less that 7/3 admits a 2-distance (Δ + 1)-coloring. Okamoto and Zhang [6], considered the 2-distance chromatic number of graphs when deleted an edge or a vertex. In [7], Jacko gave the exact value of $\chi_d(G)$ of hexagonal lattice graph when d is odd and some value when d is even. Borodin and Ivanova [8], proved that every planar graph with $g \geq 6$ and $\Delta \geq 18$ is $(\Delta + 2)$-colorable. Dantas *et al.* [9], studied the total coloring of generalized Petersen graphs and shown that "almost all" generalized Petersen graphs have a total chromatic number 4. Miao and Fan, [10], gave an upper bound of the chromatic number $\chi_d(G)$. Many papers have been devoted to it during the last decade, see for example [11] [12] [13] [14] [15].

In this paper, all graphs are finite, simple and undirected. For a graph G, we denote by $V(G)$, $E(G)$, $d(u,v)$, $\Delta(G)$, $\mathrm{diam}(G)$, G^d and $\chi_d(G)$ its vertex set, edge set, the distance between u and v which is the length of shortest path connecting them, the maximum vertex degree, the diameter of G, the power of G and *d-distance* coloring of G.

Theorem 1.1. [1]: For a graph $G=(V,E)$ we have $\chi_d(G)=d+1$ if and only if the graph G is satisfying one of the following conditions:

1) $|V|=d+1$.

2) G is a path of length greater than d.

3) G is a cycle of length a multiple of $(d+1)$. □

Theorem 1.2. [10]: When $\Delta=2$, there exist only two connected graphs of order n:

The path P_n and the cycle C_n:

1). $\chi_d(P_n)=\min\{n,d+1\}$.

2). $\chi_d(C_n)=\begin{cases} d+1: n\equiv 0(\mathrm{mod}(d+1)), \\ \min\left\{i+1\geq d+2: n \bmod i \leq \dfrac{n}{i}\right\}. \end{cases}$

Theorem 1.3. [10]: Let G be a graph. Then

$$\chi_d(G)\leq \Delta\frac{(\Delta-1)^d-1}{\Delta-2}+1.$$

□

Lemma 1.1. [6]: Let G be a nontrivial graph and d a positive integer.

1) If H a subgraph of G, then $\chi_d(H)\leq\chi_d(G)$.

2) $\chi_d(G)$ equals the order of G if and only if G is connected and

$\operatorname{diam}(G) \le d$. □

Definition 1.1. [9]**:** For integers n and k with $2 \le 2k < n$. The Generalized Petersen Graph $P(n,k)$ has vertices and respectively Edges given by:

$$V\left(P(n,k)\right) = \left\{a_i, b_i : 1 \le i \le n, 1 \le k \le \left[\frac{n-1}{2}\right]\right\},$$

$$E\left(P(n,k)\right) = \left\{a_i a_{i+1}, a_i b_i, b_i b_{i+k} : 1 \le i \le n\right\}$$

□

We will call $A(n,k)$ (respectively $B(n,k)$) the outer (respectively inner) subgraph of $P(n,k)$. Note that we take the skip $k \le \left\lfloor \frac{n-1}{2} \right\rfloor$, because of the obvious isomorphism $P(n,k) \cong P(n,n-k)$.

2. Main Results

Our main results here are to establish the exact chromatic number $\chi_d\left(P(n,k)\right)$ (d = 1, 2) for k = 1, 2, 3 and arbitrary n.

Theorem 2.1. $\chi_1\left(P(n,1)\right) = \begin{cases} 2 : n \text{ even}, \\ 3 : n \text{ odd}. \end{cases}$

Proof. Let $G = P(n,1)$, observe from Definition 1.1, that Generalized Petersen Graphs composed of one outer cycle and several inner cycles dependent on k. So, when k = 1 there is one inner cycle, then G composed of two cycles of size n. There are two cases:

Case 1: n is even, immediately from Theorem 1.1 we have $\chi_1(C_n) = 2$ (because d = 1) then

$$\chi_1(G) \ge 2 \tag{1}$$

We define a function f with colors in the set {1, 2} for a_i and b_i as follows:

$$f(a_i) = \begin{cases} 1 : i \text{ odd}, \\ 2 : i \text{ even}. \end{cases}, \quad f(b_i) = \begin{cases} 2 : i \text{ odd}, \\ 1 : i \text{ even}. \end{cases}$$

Then

$$\chi_1(G) \le 2 \tag{2}$$

By (1) and (2) we get $\chi_1(G) = 2$.

Case 2: n is odd, from Theorem 1.2, we have $\chi_1(C_n) = 3$. Then

$$\chi_1(G) \ge 3 \tag{3}$$

We define a function f with colors in the set {1, 2, 3} for a_i and b_i as follows:

$$f(a_i) = \begin{cases} 1 : i \text{ odd and } i < n, \\ 2 : i \text{ even}, \\ 3 : i = n. \end{cases}, \quad f(b_i) = \begin{cases} 2 : i \text{ odd and } i < n, \\ 1 : i \text{ even and } i = n \text{ such that } i \ne n-1, \\ 3 : i = n-1. \end{cases}$$

So,

$$\chi_1(G) \le 3. \tag{4}$$

From (3) and (4), gets $\chi_1(G)=3$. As example see Figure 1.

Theorem 2.2: $\chi_2(P(n,1))=\begin{cases}4: n\equiv 0(\mathrm{mod}\,4),\\ 6: n=3,6,\\ 5:\text{otherwise.}\end{cases}$

Proof. Let $G=P(n,1)$. We have C_4 is an induced subgraph of G and from Theorem 1.2, gets $\chi_2(C_4)=4$. So,

$$\chi_2(G)\geq 4 \tag{5}$$

We define a function f with colors in the set $\{1,2,3,4\}$ for a_i and b_i as follows:

$$f(a_1)=1,\quad f(b_1)=3,\quad f(a_2)=2,\quad f(b_2)=4.$$

By follow-up the coloring to the right, for a_3 there is only a single color as $f(a_3)=3$.

So, for each vertex there is only a single color:

$$f(b_3)=1,\quad f(a_4)=4,\quad f(b_4)=2,\quad f(a_5)=1,$$

$$f(b_5)=3,\quad f(a_6)=2,\quad f(b_6)=4.$$

Observe that we have a repeat of the same order of the colors for each 4-inner (4-outer) vertices. Consider G with $n\geq 4$. Assume that $n=4q+r:\ 0\leq r<4$ for each $j\in\{0,4,\cdots,4(q-1)\}$ we define a subset S_j of $V(G)$ by $S_j=\{a_i,a_{i+1},a_{i+2},a_{i+3},b_i,b_{i+1},b_{i+2},b_{i+3}\}$ then there is a function f with colors in the set $\{1,2,3,4\}$ define as follows:

$$f(a_i)=\begin{cases}1: i\equiv 1(\mathrm{mod}\,4),\\ 2: i\equiv 2(\mathrm{mod}\,4),\\ 3: i\equiv 3(\mathrm{mod}\,4),\\ 4: i\equiv 0(\mathrm{mod}\,4).\end{cases},\quad f(b_i)=\begin{cases}3: i\equiv 1(\mathrm{mod}\,4),\\ 4: i\equiv 2(\mathrm{mod}\,4),\\ 1: i\equiv 3(\mathrm{mod}\,4),\\ 2: i\equiv 0(\mathrm{mod}\,4).\end{cases}$$

We have four cases according to the value of n modulo 4:

Case 1: $r=0$. Then $V(G)=\bigcup_{j=0}^{4(q-1)}S_j$. By function f is

$$\chi_2(G)\leq 4 \tag{6}$$

From (5) and (6), we get $\chi_2(G)=4: n\equiv 0(\mathrm{mod}\,4)$.

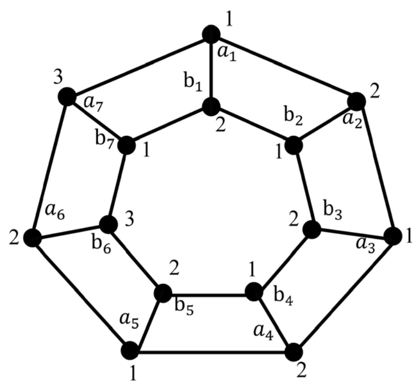

Figure 1. $\chi_1(P(7,1))=3$.

Case 2: $r = 1$. There are two leftover vertices in $V(G)-\bigcup S_j=\{a_n,b_n\}$. By function f we have $f(a_n)=1$, $f(b_n)=3$ which is a contradiction with a_1 and b_1. Moreover $d(a_n,b_n)=1$. So, each of a_n and b_n needs deferent color then $\chi_2(G)>4$. We define $f_1 = f\setminus\{a_n,a_{n-1},a_{n-2},b_n\}\cup f_2$, where f_2 is a function with colors in the set {3, 4, 5} define as follows:

$$f_2(v)=\begin{cases}4: v=a_n,\\ 3: v=a_{n-1},\\ 5: v=a_{n-2},b_n.\end{cases}$$

Then we get $\chi_2(G)=5$ = when $n\equiv 1(\mathrm{mod}\,4)$.

Case 3: For $r = 2$, we have two subcases:

Case 3.1: $r = 2$ and $n > 6$, a similar argument, there is a contradiction for a_n, b_n, a_{n-1}, b_{n-1}. Then, $\chi_2(G)>4$. We define

$$f_1 = f\setminus\{a_n,b_n,a_{n-1},b_{n-1},a_{n-2},b_{n-2},a_{n-3},a_{n-4},b_1,b_2\}\cup f_2.$$

f_2 is a function with colors in the set $\{2,3,4,5\}$ define as follows:

$$f_2(v)=\begin{cases}2: v=a_{n-3},b_{n-1},\\ 3: v=a_{n-2},b_n,\\ 4: v=a_{n-1},b_1,\\ 5: v=a_n,a_{n-4},b_{n-2},b_2.\end{cases}$$

Then we get $\chi_2(G)=5$ when $n\equiv 2(\mathrm{mod}\,4)$, see **Figure 2**.

Case 3.2: $r = 2$ and $n = 6$. There are two cycles of order 6 and know that $\chi_2(C_6)=3$. Without loss of generality, assuming that $f(a_1)=1$. Then the vertices a_2, a_3, a_5, a_6, b_1, b_2, b_6, can't take the color 1. Moreover, at most one of a_4, b_3, b_4, b_5 can be coloring by 1. This implies that each color has only two vertices from $P(6,1)$. So, needs 6 colors for 12 vertices. Furthermore, $\chi_2(P(6,1))=6$.

Case 4: For $r = 3$, there are two subcases:

Case 4.1: $n \ge 7$. For a function coloring f there is a contradiction in a_n and b_n. Then $\chi_2(G)>4$. We define

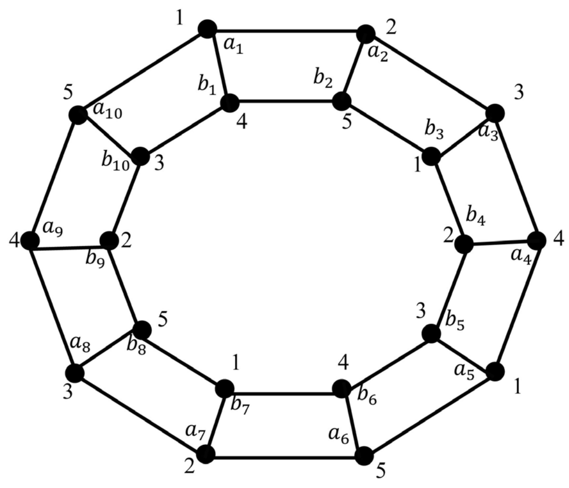

Figure 2. $\chi_2(P(10,1))=5$.

$$f_1 = f \setminus \{a_n, b_n, a_{n-1}, b_{n-2}\} \cup f_2$$

where f_2 is a function with colors in the set $\{2,3,5\}$ define as follows:

$$f_2(v) = \begin{cases} 2 : v = b_n, \\ 3 : v = a_{n-1}, \\ 5 : v = a_n, b_{n-2}. \end{cases}$$

Then, $\chi_2(G) = 5$ when $n \equiv 3(\bmod 4)$.

Case 4.2: n = 3. Then, $\operatorname{diam}(P(6,1)) = 2$. By Lemma 1.1, we get $\chi_2(G) = 6$. □

Theorem 2.3: $\chi_1(P(n,2)) = 3$.

Proof. Let $G = P(n,2)$. We have C_5 is an induced subgraph of *G*. Then by Theorem 1.2, is $\chi_1(C_5) = 3$. So,

$$\chi_1(G) \geq 3 \tag{7}$$

We define a function *f* as follows:

$$f(a_i) = \begin{cases} 1 : i \equiv 1(\bmod 3), \\ 2 : i \equiv 2(\bmod 3), \\ 3 : i \equiv 0(\bmod 3). \end{cases}, \quad f(b_i) = \begin{cases} 2 : i \equiv 1(\bmod 3), \\ 3 : i \equiv 2(\bmod 3), \\ 1 : i \equiv 0(\bmod 3). \end{cases}$$

We have three cases according to the value of n modulo 3:

Case 1: r = 0. By definition *f* we have

$$\chi_1(G) \leq 3 \tag{8}$$

By (7) together with (8), gets $\chi_1(G) = 3$ when $n \equiv 0(\bmod 3)$.

Case 2: r = 1. Then there is a contradiction for a_n. We define

$$f_1 = f \setminus \{a_1\} \cup f_2$$

where $f_2(a_1) = 3$. This implies that $\chi_1(G) = 3$ when $n \equiv 1(\bmod 3)$.

Case 3: r = 2. There is a problem with colors the vertices $\{b_n, b_{n-1}\}$.

We define $f_1 = f \setminus \{a_n, b_n, a_{n-1}, b_{n-1}\} \cup f_2$, where f_2 is a function with colors in the set $\{1,2,3\}$ define as follows:

$$f_2(v) = \begin{cases} 1 : v = b_{n-1}, \\ 3 : v = a_n, \\ 2 : v = a_{n-1}, b_n. \end{cases}$$

By the last result together with (7), we get $\chi_2(G) = 3$ when $n \equiv 2(\bmod 3)$. □

Theorem 2.4: $\chi_2(P(n,2)) = \begin{cases} 5 : n \equiv 0(\bmod 10), \\ 10 : n = 5, \\ 6 : \text{otherwise}. \end{cases}$

Proof. Let $G = P(n,2)$. G including C_5 as an induced subgraph. We have $\operatorname{diam}(C_5) = 2$. Then by Lemma 1.1, we get $\chi_2(C_5) = 5$. Furthermore

$$\chi_2(G) \geq 5. \tag{9}$$

We define a function *f* with colors in the set $\{1,2,3,4,5\}$ for a_i and b_i as follows:

$$f(a_1)=2,\ f(a_2)=2,\ f(a_3)=3,\ f(b_1)=4,\ f(b_3)=5$$

Then for b_2 there are two cases:

Case a: $f(b_2)=4$. (By coloring to the right). So, a_4 has only a single color as $f(a_4)=1$ and $f(b_4)=5$. Follow-up coloring inner (outer) vertices each vertex will have only a single color as follows:

$$f(b_5)=2,\ f(a_5)=4,\ f(b_6)=2,\ f(a_6)=3,\ f(b_7)=1,\ f(a_7)=5,$$

$$f(b_8)=1,\ f(a_8)=4,\ f(b_9)=3,\ f(a_9)=2,\ f(b_{10})=3,\ f(a_{10})=5$$

By continue we will have a repeat of the same order of the colors for each 10-inner (10-outer) vertices.

Case b: $f(b_2)=5$. By coloring to the left. So, we back to consider the Case 1.

We will consider G with $n\geq 10$. Assume that $n=10q+r:\ 0\leq r<10$. Now, for each $j\in\{0,10,\cdots,10(q-1)\}$ we define a subset S_j of $V(G)$ by

$$S_j=\{a_j,a_{j+1},\cdots,a_{j+9},b_j,b_{j+1},\cdots,b_{j+9}\}$$

Then there is a function f define as follows:

$$f(a_i)=\begin{cases}1:i\equiv 1,4(\text{mod}10),\\ 2:i\equiv 2,9(\text{mod}10),\\ 3:i\equiv 3,6(\text{mod}10),\\ 4:i\equiv 5,8(\text{mod}10),\\ 5:i\equiv 7,0(\text{mod}10).\end{cases},\quad f(b_i)=\begin{cases}4:i\equiv 12(\text{mod}10),\\ 5:i\equiv 3,4(\text{mod}10),\\ 2:i\equiv 5,6(\text{mod}10),\\ 1:i\equiv 7,8(\text{mod}10),\\ 3:i\equiv 0,9(\text{mod}10).\end{cases}$$

We have ten cases according to the value of n modulo 10:

Case 1: r = 0. Then $V(G)=\bigcup_{j=0}^{10(q-1)}S_j$. Moreover, by define f we have

$$\chi_2(G)\leq 5 \tag{10}$$

From (9) and (10) we get $\chi_2(G)=5$ when $n\equiv 0(\text{mod}10)$.

Case 2: r = 1. There are two leftover vertices in $V(G)-\bigcup S_j=\{a_n,b_n\}$. By function f, $f(a_n)=1$, $f(b_n)=4$ which is a contradiction with a_1 and b_1, and $d(a_n,b_n)=1$. So each of a_n and b_n needs deferent color then $\chi_2(G)>5$. We define

$$f_1=f\setminus\{a_n,b_n,a_{n-1},a_{n-2},a_{n-3}\}\cup f_2$$

where f_2 is a function with colors in the set $\{2,4,5,6\}$ define as follows:

$$f_2(v)=\begin{cases}2:v=a_{n-1},\\ 4:v=a_{n-2},\\ 5:v=a_n,\\ 6:v=b_n,a_{n-3}.\end{cases}$$

Then we get $\chi_2(G)=6$ when $n\equiv 1(\text{mod}10)$.

Case 3: r = 2. There are four leftover vertices in $V(G)-\bigcup S_j=\{a_n,b_n,a_{n-1},b_{n-1}\}$, which are a contradiction with $\{a_1,a_2,b_1,b_2\}$. This implies that $\chi_2(G)>5$.

Let

$$f_1 = f \setminus \{b_n, b_{n-1}, a_1, a_2, a_3, a_4, a_6\} \cup f_2$$

where f_2 is a function with colors in the set $\{1,3,6\}$ define as follows:

$$f_2(v) = \begin{cases} 1: v = a_2, \\ 3: v = a_1, a_4, \\ 6: v = a_3, a_6, b_n, b_{n-1}. \end{cases}$$

Then $\chi_2(G) = 6$ when $n \equiv 2 \pmod{10}$. See Figure 3.

Case 4: r = 3. By same argument there is a contradiction for b_n, b_{n-1}, b_{n-2}. Which implies that $\chi_2(G) > 5$. So, we define

$$f_1 = f \setminus \{b_n, b_{n-1}, b_{n-2}, a_{n-3}, a_{n-5}\} \cup f_2$$

where that f_2 is a function with colors in the set $\{4,5,6\}$ define as follows:

$$f_2(v) = \begin{cases} 4: v = a_{n-3}, \\ 5: v = b_{n-2}, \\ 6: v = b_n, b_{n-1}, a_{n-5}. \end{cases}$$

Then we get $\chi_2(G) = 6$ when $n \equiv 3 \pmod{10}$.

Case 5: r = 4. There is a contradiction for $b_n, b_{n-1}, b_{n-2}, b_{n-3}, a_n$. We define

$$f_1 = f \setminus \{a_1, a_n, a_{n-3}, b_n, b_{n-1}, b_{n-2}, b_{n-3}\} \cup f_2$$

where f_2 is a function with colors in the set $\{1,4,5,6\}$ define as follows:

$$f_2(v) = \begin{cases} 1: v = b_n, b_{n-1}, \\ 4: v = a_{n-3}, \\ 5: v = a_n, \\ 6: v = a_1, b_{n-2}, b_{n-3}. \end{cases}$$

We get $\chi_2(G) = 6$ when $n \equiv 4 \pmod{10}$.

Case 6: When r = 5 we have two subcases:

Case 6.1: r = 5 and $n > 5$. The contradiction is for $b_n, b_{n-1}, b_{n-3}, a_{n-1}, a_n$. We will need at least three new deferent colors for them, then $\chi_2(G) > 5$. We define

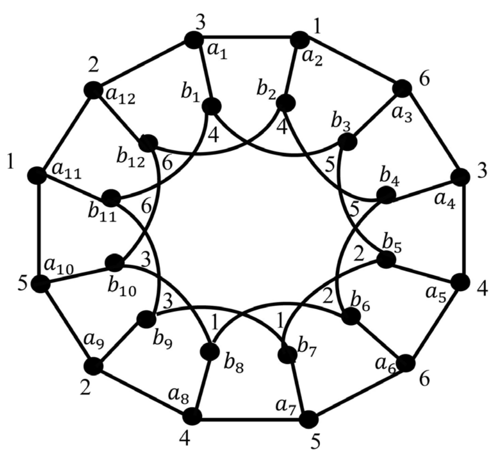

Figure 3. $\chi_2(P(12,2)) = 6$.

$$f_1 = f \setminus \{a_n, b_n, a_{n-1}, b_{n-1}, a_{n-2}, b_{n-3}, a_{n-4}, b_{n-4}, a_{n-5}, b_{n-6}, b_{n-7}, a_2, a_3\} \cup f_2$$

where f_2 is a function with colors in the set $\{1,2,3,4,5,6\}$ define as follows:

$$f_2(v) = \begin{cases} 1 : v = a_{n-2}, a_{n-5}, \\ 2 : v = a_n, \\ 3 : v = a_{n-1}, a_2, b_{n-4}, \\ 4 : v = a_{n-4}, \\ 5 : v = b_{n-3}, \\ 6 : v = a_3, b_n, b_{n-1}, b_{n-6}, b_{n-7}. \end{cases}$$

Then $\chi_2(G) = 6$ when $n \equiv 5(\mathrm{mod}\,10)$. See **Figure 4**.

Case 6. 2: $r = 5$ and $n = 5$. We have diam(G) = 2. So, Lemma 1.1, gets $\chi_2(G) = 10$.

Case 7: $r = 6$. A contradiction for b_n, a_{n-1}. We define $f_1 = f \setminus \{b_1, b_n\} \cup f_2$ and f_2 is a function with color 6. So, $f_2(b_1) = f_2(b_n) = 6$, gets $\chi_2(G) = 6$ when $n \equiv 6(\mathrm{mod}\,10)$.

Notice when $n = 6$ we have the same argument but $q = 0$, so the vertices will take the sequence of colors for (outer, inner)vertices as follows (1, 2, 3, 1, 2, 3, 4, 4, 5, 5, 6, 6).

Case 8: $r = 7$. A contradiction is only for b_n. Let $f_1 = f \setminus \{b_n\} \cup f_2$. Where $f_2(b_n) = 6$. Moreover, $\chi_2(G) = 6$ when $n \equiv 7(\mathrm{mod}\,10)$. Also when $n = 7$ we get $\chi_2(G) = 6$ by the same condition with sequence of colors (outer, inner) vertices as following (1, 2, 3, 1, 4, 3, 5, 6, 4, 5, 5, 2, 2, 6).

Case 9: $r = 8$. A contradiction is for b_{n-1}, b_n, a_n, then $\chi_2(G) > 5$. We define $f_1 = f \setminus \{a_n, b_n, b_{n-1}, a_{n-2}, a_{n-3}\} \cup f_2$ with f_2 is a function with colors in the set $\{3,4,6\}$ define as follows:

$$f_2(v) = \begin{cases} 3 : v = b_n, b_{n-1}, \\ 4 : v = a_{n-2}, \\ 6 : v = a_n, a_{n-3}. \end{cases}$$

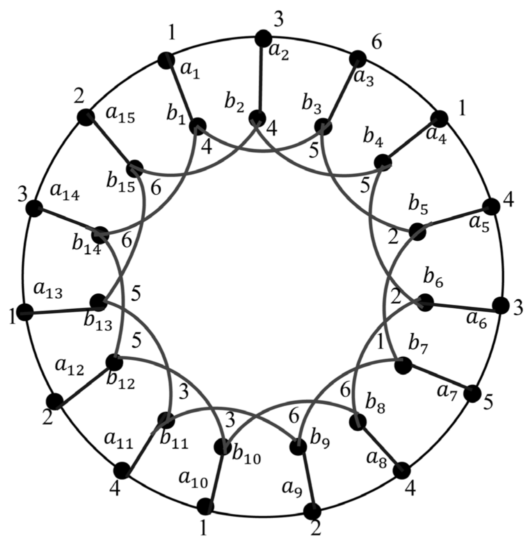

Figure 4. $\chi_2(P(15,2)) = 6$.

And so, gets $\chi_2(G)=6$ when $n\equiv 8(\text{mod}10)$.

Also notice when $n = 8$ we have the same argument but $q=0$, so the vertices will take the sequence of colors for (outer, inner) vertices as follows (1, 2, 3, 1, 6, 4, 5, 6, 4, 4, 5, 5, 2, 2, 3, 3).

Case 10: $r = 9$. There is a contradiction for b_{n-1}, a_{n-1}, a_n then $\chi_2(G)>5$. Let us define $f_1 = f\setminus\{a_n,b_n,a_{n-1},b_{n-1},a_{n-2},a_{n-4}\}\cup f_2$, with f_2 is a function with colors in the set $\{3,4,5,6\}$ define as follows:

$$f_2(v)=\begin{cases}3: v=a_n,\\ 4: v=a_{n-2},\\ 5: v=a_{n-1},\\ 6: v=b_n,b_{n-1},a_{n-4}.\end{cases}$$

Furthermore, $\chi_2(G)=6$ when $n\equiv 9(\text{mod}10)$.

As before when $n = 9$ we get $\chi_2(G)=6$, by the same condition with sequence of colors (outer, inner) vertices as follows (1, 2, 3, 1, 6, 3, 4, 5, 3, 4, 4, 5, 5, 2, 2, 1, 6, 6).

Finally, we conclude that:

$$\chi_2(P(n,2))=\begin{cases}5: n\equiv 0(\text{mod}10),\\ 10: n=5,\\ 6: \text{otherwise.}\end{cases}$$

Theorem 2.5: $\chi_1(P(n,3))=\begin{cases}2: n\equiv 0(\text{mod}\,2),\\ 3: n\equiv 1(\text{mod}\,2).\end{cases}$

Proof. Let $G=P(n,3)$. There are two cases:

Case 1: $n\equiv 0(\text{mod}\,2)$. From Theorem 1.2, we have $\chi_1(C_n)=2$. Then

$$\chi_1(G)\ge 2 \tag{11}$$

We define a function *f* with colors in the set {1, 2} for a_i and b_i ($1\le i\le n$),

$$f(a_i)=\begin{cases}1: i\text{ odd},\\ 2: i\text{ even}.\end{cases},\quad f(b_i)=\begin{cases}2: i\text{ odd},\\ 1: i\text{ even}.\end{cases}$$

Then

$$\chi_1(G)\le 2 \tag{12}$$

From (11) and (12), gets $\chi_1(G)=2$.

Case 2: $n\equiv 1(\text{mod}\,2)$. From Theorem 1.2, we have $\chi_1(C_n)=3$. Moreover,

$$\chi_1(G)\ge 3 \tag{13}$$

Let $f:V(G)\to\{1,2,3\}$ for a_i and b_i, where

$$f(a_i)=\begin{cases}1: i\text{ odd}, i=n-1 \text{ such that } i\ne n,n-2,\\ 2: i\text{ even}, i=n,n-2 \text{ such that } i\ne n-1,n-3,\\ 3: i=n-3.\end{cases}$$

$$f(b_i)=\begin{cases}1: i\text{ even}, i<n-2,\\ 2: i\text{ odd}, i<n-2,\\ 3: i=n,n-1,n-2.\end{cases}$$

Then

$$\chi_1(G) \le 3. \tag{14}$$

From (13) and (14) is $\chi_1(G)=3$. □

Theorem 2.6: $\chi_2(P(n,3))=\begin{cases}4: n\equiv 0(\bmod 4),\\ 6: n=7 \text{ or } 2(\bmod 4),\\ 5: \text{otherwise.}\end{cases}$

Proof. Let $G=P(n,3)$. $K(1,3)$ is an induced subgraph of G and $\chi_2(K(1,3))=4$. This implies that

$$\chi_2(G) \ge 4. \tag{15}$$

Without loss of generality, we define a function f as follows: $f(a_1)=1$, $f(a_2)=2$, $f(a_3)=3$, $f(b_2)=4$. By follow-up the coloring to the right, for b_1 there are two cases $f(b_1)=4$ or $f(b_1)=3$.

Case 1: $f(b_1)=4$. Then, there are two cases for b_3, $f(b_3)=4$ or $f(b_3)=1$. So, if $f(b_3)=4$ then absolutely $f(a_4)=1$ and $f(b_1)=3$. For coloring a_5 we need another color because it has the four colors as neighbors. If $f(b_3)=1$, then $f(a_4)=1$ and $f(b_4)=2$. Furthermore, for coloring a_5 we need another color. So to avoiding the fifth color we have to take the second case.

Case 2: $f(b_1)=3$. There are two cases for b_3, $f(b_3)=4$ or $f(b_3)=1$, we have two subcases:

Case 2.1: $f(b_3)=4$. Absolutely $f(a_4)=1$ and $f(b_4)=4$ or $f(b_4)=2$. If we take $f(b_4)=4$ then $f(a_5)=2$, $f(b_5)=3$, but we need another color for a_6. Also if we take $f(b_4)=2$ then we need new color for a_5.

Case 2.2: $f(b_3)=1$. For each vertex there is only a single color: $f(a_4)=4$, $f(b_4)=2$, $f(a_5)=1$, $f(b_5)=3$, $f(a_6)=2$, $f(b_6)=4$. Observe that, we have a repeat of the same order of the colors for each (4-outer) and (4-inner) vertices as respectively for colors $\{1,2,3,4\}$ and $\{3,4,1,2\}$. Consider G with $n\ge 4$. Assume that $n=4q+r$: $0\le r<4$ for each $j\in\{0,4,\cdots,4(q-1)\}$, we define a subset S_j of $V(G)$ by $S_j=\{a_i,a_{i+1},a_{i+2},a_{i+3},b_i,b_{i+1},b_{i+2},b_{i+3}\}$ then there is a function f define as follows:

$$f(a_i)=\begin{cases}1: i\equiv 1(\bmod 4),\\ 2: i\equiv 2(\bmod 4),\\ 3: i\equiv 3(\bmod 4),\\ 4: i\equiv 0(\bmod 4).\end{cases},\quad f(b_i)=\begin{cases}3: i\equiv 1(\bmod 4),\\ 4: i\equiv 2(\bmod 4),\\ 1: i\equiv 3(\bmod 4),\\ 2: i\equiv 0(\bmod 4).\end{cases}$$

We have four cases according to the value of n modulo 4:

Case 2.2.1: $r=0$. Then $V(G)=\bigcup_{j=0}^{4(q-1)} S_j$. By function f we have

$$\chi_2(G) \le 4 \tag{16}$$

From (15) and (16) we get $\chi_2(G)=4: n\equiv 0(\bmod 4)$.

Case 2.2.2: $r=1$. Then there are two leftover vertices in $V(G)=\bigcup_{j=0}^{4(q-1)} S_j=\{a_n,b_n\}$, by function f we get $f(a_n)=1$, $f(b_n)=3$ which is a contradiction with a_1 and a_3. So each of a_n and b_n needs deferent color then $\chi_2(G) > 4$. We define

$$f_1 = f \setminus \{a_n, a_{n-1}, a_{n-2}, a_{n-3}, b_1, b_n, b_{n-1}, b_{n-2}, b_{n-3}\} \cup f_2$$

where f_2 is a function with colors in the set {2, 3, 4, 5} define as follows:

$$f_2(v) = \begin{cases} 2: v = a_{n-1}, b_{n-3}, \\ 3: v = a_n, b_{n-2}, \\ 4: v = a_{n-2}, b_n, \\ 5: v = a_{n-3}, b_{n-1}, b_1. \end{cases}$$

Then gets $\chi_2(G) = 5$ when $n \equiv 1 \pmod 4$.

Case 2.2.3: $r = 2$. Here, we will consider $\chi_2(P(10,3))$, {we delete the details of the general case because they are too long}.

We have $\chi_2(P(10,3)) \geq 5$. Suppose $\chi_2(P(10,3)) = 5$. It is easy to prove that each color can be given at most to four vertices. This implies that each color has exactly four vertices. {If drawing $P_1(10,3)$ as following form: (outer cycle, inner cycle) respectively, $b_1b_4b_7b_{10}b_3b_6b_9b_2b_5b_8$, $a_1a_4a_7a_{10}a_3a_6a_9a_2a_5a_8$ such that $b_ia_i \in E(P_1(10,3))$ we gets the same graph ($P(10,3)$, *i.e.*, $P_1(10,3) \cong P(10,3)$ }. Furthermore, no more three vertices from (outer cycle, inner cycle) respectively, can be take the same color.

Assume that there are five sets of colors, D_1, D_2, D_3, D_4, D_5, *i.e.*, $f(v) = i$ if and only $v \in D_i$ ($1 \leq i \leq 5$). We will study the cases for one of D_i. If D_i contain r vertices of outer cycle and q vertices of inner cycle, then we called D_i is (r-outer, q-inner). Without loss of generality, we consider D_1. Thus, we distinguish two cases:

Case a: D_1 is (3-outer, 1-inner).

(This Case is similar by symmetry to D_1 is (1-outer, 3-inner).

Let's start with a_1 then we have (up to isomorphism) $D_1 = \{a_1, a_4, a_7, b_9\}$, $a_2 \in D_2$, and $a_3 \in D_3$. Thus, $b_2 \in D_4$ or $b_2 \in D_5$ and $b_3 \in D_4$ or $b_3 \in D_5$. We have two cases:

Case a.1: $b_2 \in D_4$ and $b_3 \in D_5$ or $b_2 \in D_5$ and $b_3 \in D_4$. Two cases are similar by symmetry. Let $b_2 \in D_4$ and $b_3 \in D_5$. Then $b_6 \in D_2$, $a_5 \in D_5$, $b_5 \in D_3$, $b_8 \in D_2$, $b_4 \in D_4$. Then $b_{10} \notin D_i, (1 \leq i \lesssim 5)$, a contradiction with our hypothesis, $\chi_2(P(10,3)) = 5$.

Case a.2: b_2 and b_3 are belonging to the same set, let $b_2, b_3 \in D_4$. There are two subcases $a_5 \in D_2$ or $a_5 \in D_5$:

Case a.2.1: $a_5 \in D_2$. Then $b_6 \in D_5$, $b_{10} \in D_2$, $a_6 \in D_3$, $b_5 \in D_5$, $b_7 \in D_5$, $b_4 \in D_4$, $b_1 \in D_3$. So, $b_8 \notin D_i, (1 \leq i \leq 5)$, a contradiction with $\chi_2(P(10,3)) = 5$.

Case a.2.2: $a_5 \in D_5$. Then $b_5 \in D_3$, $b_6 \in D_2$. This implies that $a_6 \notin D_i, (1 \leq i \leq 5)$, again gets a contradiction with $\chi_2(P(10,3)) = 5$.

Case b: D_1 is (2-outer, 2-inner).

Assume that $a_1 \in D_1$. We have three cases to choose the second vertex from outer cycle.

Case b.1: $a_4 \in D_1$. (We have the same result if we take $a_8 \in D_1$). Just one of b_6, b_9 can belongs to D_1. So, D_1 has three vertices and that means a contradiction with our proof that each set is from size 4.

Case b.2: $a_5 \in D_1$. (We have the same result if we take $a_7 \in D_1$). Then $D_1 = \{a_1, a_5, b_7, b_9\}$, $a_2 \in D_2$, $a_3 \in D_3$, $a_4 \in D_4$, $b_3 \in D_5$. Also, $b_4 \in D_2$ or $b_4 \in D_5$.

Case b.2.1: $b_4 \in D_2$. Then $b_{10} \in D_4$, $b_6 \in D_2$, $a_6 \in D_3$, $b_5 \in D_5$, $a_7 \in D_5$, $b_1 \in D_3$, $b_8 \in D_4$. Thus, $b_2 \notin D_i, (1 \le i \le 5)$, a contradiction with $\chi_2(P(10,3)) = 5$.

Case b.2.2: $b_4 \in D_5$. Then $b_1 \in D_3$, $a_{10} \in D_4$, $b_{10} \in D_2$, $b_5 \in D_5$, $b_2 \in D_4$. Thus, we get $b_6 \notin D_i, (1 \le i \le 5)$, a contradiction with $\chi_2(P(10,3)) = 5$.

Case b.3: $a_6 \in D_1$. Then no vertex in inner cycle can take the color 1. We get a contradiction with our proof that each set is from size 4.

Finally, we conclude that $\chi_2(P(10,3)) > 5$. To prove that $\chi_2(P(10,3)) \le 6$, we take a function $f : V(G) \to \{1,2,3,4,5,6\}$ as follows:

$$f(v) = \begin{cases} 1: v = a_1, a_5, a_8, b_3, \\ 2: v = a_2, a_6, a_9, b_4, \\ 3: v = a_3, a_7, b_1, \\ 4: v = a_4, a_{10}, b_2, \\ 5: v = b_5, b_6, b_7, \\ 6: v = b_8, b_9, b_{10}. \end{cases}$$

Then we get $\chi_2(G) = 6$ when $n \equiv 2 \pmod 4$. See Figure 5.

Case 2.2.4: $r = 3$. we have two subcases:

Case 2.2.4.1: $r = 3$ and $n > 7$. The contradiction in a_n, b_{n-1}, b_{n-2} and b_n. We define

$$f_1 = f \setminus \{a_n, b_n, b_{n-1}, b_{n-2}\} \cup f_2$$

where f_2 is a function with colors in the set {4, 5}, define as follows:

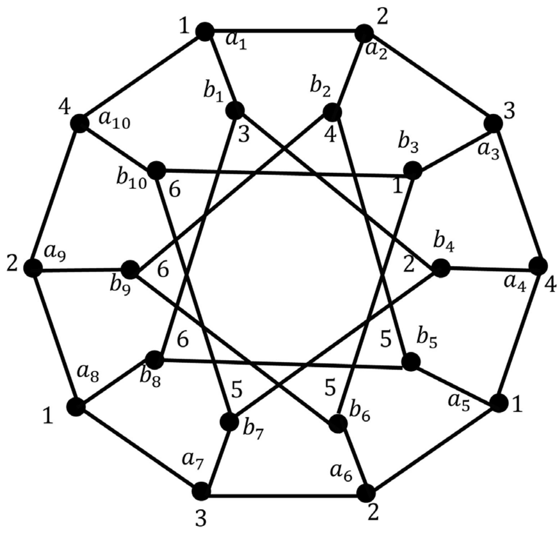

Figure 5. $\chi_2(P(10,3)) = 6$.

$$f_2(v)=\begin{cases}4: v=a_n,\\ 5: v=b_n,b_{n-1},b_{n-2}.\end{cases}$$

Then we get $\chi_2(G)=5$ when $n\equiv 3(\bmod 4)$ for $n>7$.

Case 2.2.4.2: r = 3 and n = 7. In this case we have C_5 induced subgraph from $P(7,3)$. Furthermore, $\chi_2(P(7,3))\geq 5$. Let take the cycle $a_1a_2b_2b_5b_1$ and give it the fife color as follows: $f(a_1)=1$, $f(a_2)=2$, $f(b_2)=3$, $f(b_5)=4$, $f(b_1)=5$, so for a_3 there are two cases $f(a_3)=4$ or 5.

Case 2.2.4.2.a: $f(a_3)=4$. Then for b_3 we have two choices 1 or 5. For the first choice $f(b_3)=1$ we get $f(a_4)=3$, $f(b_4)=2$, $f(a_5)=1$. But for a_6 there are two colors 2 or 5. If $f(a_6)=5$, then we will need a new color for b_6. Also, if $f(a_6)=2$ then $f(b_6)=5$. Obviously, we need a new color for b_7. For second choice $f(b_3)=5$ then $f(a_4)=1$ or $f(a_4)=3$. If $f(a_4)=1$ we have for b_4 two colors 2 or 3 if we take the color 2 then needs a new color for the vertices a_5. Also, if we take the color 3 we will need a new color for a_6 because a_5 can only take the color 2. If $f(a_4)=3$ then $f(b_4)=2$, $f(a_5)=1$, $f(a_6)=2$. Moreover, we will need a new color for b_6.

Case 2.2.4.2.b: $f(a_3)=5$ then for b_3 we have two choices 1 or 4. For $f(b_3)=1$ we get $f(a_4)=3$, $f(b_4)=2$, $f(a_5)=1$, $f(a_6)=5$. Then we need a new color for b_6. For second choice $f(b_3)=4$ then $f(a_4)=1$ or $f(a_4)=3$. If $f(a_4)=1$, then we have for b_4 two colors 2 or 3. If we take the color 2 we will need a new color for the vertices a_5. Also, if we take the color 3 we will need a new color for a_7. If $f(a_4)=3$, then $f(b_4)=2$,so we will need a new color for b_7. We conclude that for all the cases, needs six colors. Furthermore, $\chi_2(P(7,3))>5$. To prove that $\chi_2(P(7,3))\leq 6$, we take a function $f: V(G)\to\{1,2,3,4,5,6\}$ as follows:

$$f(a_1)=f(a_5)=f(b_3)=1,\ f(a_2)=f(a_6)=f(b_4)=2,\ f(b_1)=f(a_3)=3,$$

$$f(b_2)=f(a_4)=f(a_7)=4,\ f(b_5)=f(b_7)=5,\ f(b_6)=6.$$

Finally, we get $\chi_2(P(7,3))=6$.

Acknowledgements

The authors would like to thank the referees for their careful reading of the paper and their helpful comments.

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