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Non-Full Rank Factorization of Finite Abelian Groups

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Abstract

Tilings of p-groups are closely associated with error-correcting codes. In [1], M. Dinitz, attempting to generalize full-rank tilings of $\mathbb{Z}_2^n$ to arbitrary finite abelian groups, was able to show that if $p \geq 5$, then $\mathbb{Z}_p^n$ admits full-rank tiling and left the case $p = 3$, as an open question. The result proved in this paper the settles of the question for the case $p = 3$.

Keywords

Factorization of Abelian Groups, Error-Correcting Codes

1. Introduction

A factorization of a finite abelian group G is a collection of subsets $A_1, \cdots, A_i, \cdots, A_k$ of G such that each element $g \in G$ can be represented in the form $g = a_1 \cdots a_i \cdots a_k$. In this case, we write $G = A_1, \cdots, A_i, \cdots, A_k$ and if each A_i contains the identity element e of G, we say we have a normalized factorization of G.

The notion of factorization of abelian groups arose when G. Hajós [3] found the answer to "Minkowski's conjecture" about lattice tiling of $\mathbb{R}^n$ by unit cubes or clusters of unit cubes. The geometric version of "Minkowski's conjecture" can be explained as follows:

A lattice tiling of $\mathbb{R}^n$ is a collection $\{T_i : i \in I\}$ of subsets of $\mathbb{R}^n$ such that $\bigcup_{i \in I} T_i = \mathbb{R}^n$ and $\text{int}(T_i) \bigcap \text{int}(T_j) = \varnothing$, if $i \neq j$, $i, j \in I$. Two unit cubes are called twins if they share a complete $(n-1)$-dimensional face. Minkowski was wondering if there exists a tiling of $\mathbb{R}^n$ by unit cubes such that there are no twins! Minkowski's conjecture is usually expressed as follows:

Each lattice tiling of $\mathbb{R}^n$ by unit cubes contains twins.

As mentioned above, it was G. Hajós [3] who solved Minkowski' conjecture.

His answer was in the affirmative, after translating the conjecture into an equivalent conjecture about finite abelian groups. Its group—theoretic equivalence reads as follows:

"If G is a finite abelian group and $G = A_1, \cdots, A_i, \cdots, A_k$ is a normalized factorization of G, where each of the subsets A_i is of the form $\{e, a, a^2, \cdots, a^k\}$, where $k < |a|$; here $|a|$ denotes order of a, then at least one of the subsets A_i is a subgroup of G".

Rėdei [4] generalized Hajos's theorem to read as follows:

"If G is a finite abelian group and $G = A_1 \cdots A_i \cdots A_k$ is a normalized factorization of G, where each of the subsets A_i contains a prime number of elements, then at least one of the subsets A_i is a subgroup of G".

2. Preliminaries

A tiling is a special case of normalized factorization in which there are only two subsets, say A and B of a finite abelian groups G, such that $G = AB$ is a factorization of G.

A tiling of a finite abelian group G is called a full-rank tiling if $G = AB$ implies that $\langle A \rangle = \langle B \rangle = G$, where $\langle A \rangle$ denotes the subgroup generated by A. In this case, A and B are called full-rank factors of G. Otherwise, it is called a *non-full-rank* tiling of G. As suggested by M. Dinitz [1] and also in that of *O*. Fraser and *B*. Gordon [2], finding answers to certain questions is sometimes easier in one context than in others. In this connection consider the group, $\mathbb{Z}_p^n$ viewed as a vector space of n-tuples $(x_1, x_2, \cdots, x_n)$ over $\mathbb{Z}_p$. Then subspaces correspond to subgroups. Moreover, $\mathbb{Z}_p^n$ is equipped with a metric, called Hamming distance d_H, which is defined as follows:

For $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$,

$$d_H(x, y) = |\{i : 1 \le i \le n, x_i \ne y_i\}|.$$

With respect to this metric, the sphere $S(x, e)$ with center at x and radius e is the set $S(x, e) = \{y : d_H(x, y) \le e\}$.

A *perfect error-correcting* code is a subset C of $\mathbb{Z}_p^n$ such that $\bigcup_{x \in C} S(x, e) = \mathbb{Z}_p^n$ and $S(x, e) \cap S(y, e) = \varnothing$, if $x \ne y$.

Observe that in the language of tiling, this says that $\mathbb{Z}_p^n = CS(0, e)$ is a *factorization* of $\mathbb{Z}_p^n$ [6].

Factorization and Partition

Let $G = AB$ be a factorization of a finite Abelian group G. Then the sets $\{aB : a \in A\}$ form a partition of G. Also, $|G| = |A||B|$, where $|A|$ as before denotes the number of elements of A.

Definition

Let A and A' be subsets of G. We say that A is replaceable by A', if whenever $G = AB$ is a factorization of G, then so is $G = A'B$.

Redei [4] showed that if $G = AB$ is a factorization of G, where $A = \{e, a_1, a_2, \cdots, a_{p-1}\}$, and p is a prime, then A is replaceable by $\langle a_i \rangle$, for each $i, 1 \le i \le p-1$.

Definition

A subset A of G is *periodic*, if there exists $g \in G$, $g \neq e$ such that $gA = A$. It is easy to see that if A is periodic, then $A = HC$, where H is a proper subgroup of G [5].

Before we show the aim of this paper, we mention the following observation. If $G = AB$ is a factorization of G, then for any $a \in A$, and $b \in B$, then so is $G = a^{-1}Ab^{-1}B$, so we may assume all factorizations $G = AB$ are normalized.

Theorem

Let $G = \mathbb{Z}_3^n$ and assume $G = AB$ is a factorization of G, where $|A| = 3$, then either A or B is a non-full-rank factor of G.

Proof:

Note that $|G| = 3^n$. We induct on n.

If $n = 1$, then $|B| = 1$. Thus, B is a non-full-rank factor of G.

Let $n > 1$ and assume the result is true for all such groups of order less than 3^n.

Let $A = \{e, a, b\}$. Then in $G = AB$, by Rédei [4], A can be replace by $A' = \{e, a, a^2\}$.

If $a^3 = e$, then A is a subgroup of G. Thus, $\langle A \rangle \neq G$, so A is a *non-full-rank factor* of G.

If $a^3 \neq e$, then from $G = \{e, a, a^2\}B$, we get the following partition of G:

$$G = eB \bigcup aB \bigcup a^2 B \cdots (*)$$

from which we get

$$G = aB \bigcup a^2 B \bigcup a^3 B \cdots (**).$$

Comparing $(*)$ with $(**)$, we obtain $B = a^3 B$. Thus, B is periodic, from which it follows that $B = HC$, where H is a a proper subgroup of G. Now, from $G = AB$, we obtain the factorization $G/H = AB/H = (A/H)(B/H)$ of the quotient group G/H, which is of order less than 3^n. So, by inductive assumption, either $\langle AH/H \rangle \neq G/H$ or $\langle BH/H \rangle \neq G/H$ from which it follows that either $\langle A \rangle \neq G$ or $\langle B \rangle \neq G$. That is either A or B is a non-full-rank factor of G QED.

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