Scientific Research Publishing

Open Journal of Discrete Mathematics, 2016, 6, 340-350
http://www.scirp.org/journal/ojdm
ISSN Online: 2161-7643
ISSN Print: 2161-7635

A Dynamic Programming Approach for the Max-Min Cycle Packing Problem in Even Graphs

Peter Recht

Operations Research und Wirtschaftsinformatik, Dortmund, Germany
Email: peter.recht@tu-dortmund.de

How to cite this paper: Recht, P. (2016) A Dynamic Programming Approach for the Max-Min Cycle Packing Problem in Even Graphs. *Open Journal of Discrete Mathematics*, **6**, 340-350.
http://dx.doi.org/10.4236/ojdm.2016.64027

Received: August 26, 2016
Accepted: October 25, 2016
Published: October 28, 2016

Abstract

Let $G=(V(G),E(G))$ be an undirected graph. The maximum cycle packing problem in G then is to find a collection $\{C_1,C_2,\cdots,C_s\}$ of edge-disjoint cycles C_i in G such that s is maximum. In general, the maximum cycle packing problem is NP-hard. In this paper, it is shown for even graphs that if such a collection satisfies the condition that it minimizes the quantity $\sum_{i=1}^{s}|E(C_i)|^2+(|E(G)|-|E(\mathcal{C})|)^2$ on the set of all edge-disjoint cycle collections, then it is a maximum cycle packing. The paper shows that the determination of such a packing can be solved by a dynamic programming approach. For its solution, an A^*-shortest path procedure on an appropriate acyclic network $\vec{N}$ is presented. It uses a particular monotonous node potential.

Keywords

Maximum Edge-Disjoint Cycle Packing, Extremal Problems in Graph Theory, Dynamic Programming, A^*-Shortest Path Procedure

1. Introduction

We consider a finite and undirected graph G with vertex set $V=V(G)$ and edge-set $E=E(G)$ that contain no loops.

For a finite sequence $(v_{i_1},e_1,v_{i_2},e_2,\cdots,e_{r-1},v_{i_r})$ of vertices v_{i_j} and pairwise distinct edges $e_j=(v_{i_j},v_{i_{j+1}})$ the subgraph W of G with vertices $V(W)=\{v_{i_1},v_{i_2},\cdots,v_{i_r}\}$ and edges $E(W)=\{e_1,e_2,\cdots,e_{r-1}\}$ is called a *walk* with *start vertex* v_{i_1} and *end vertex* v_{i_r}.

If W is closed, *i.e.* $v_{i_1}=v_{i_r}$, we call it a *circuit* in *G*. A *path* is a walk in which all vertices v have degree $d_W(v)\le 2$. A *cycle* is a closed path. The length $|E(C)|$ of a cycle $C\subset G$ is denoted by $l(C)$. An *even graph* is a graph G in which all vertices v

have even degree $d_G(v) \geq 2$. An *Eulerian* graph is a connected even graph.

For $1 \leq i \leq k$, let $G_i \subset G$ be subgraphs of *G*. We say G is *induced by* $\{G_1, G_2, \cdots, G_k\}$ if $V(G) = V(G_1) \cup V(G_2) \cup \cdots \cup V(G_k)$ and $E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_k)$. Two subgraphs $G' = (V', E')$, $G'' = (V'', E'') \subset G$ are called *edge-disjoint* if $E' \cap E'' = \varnothing$. For $E' \subseteq E$, we define $G \setminus E' = (V, E \setminus E')$. For $V' \subset V$, we define $G \setminus V' = G|_{V \setminus V'}$, where $V(G|_{V \setminus V'}) = V \setminus V'$ and $E(G|_{V \setminus V'}) = \{e \in E(G) \mid \text{both end vertices of } e \text{ belong to } V \setminus V'\}$.

A *packing* $\mathcal{Z}(G) = \{G_1, \cdots, G_q\}$ of *G* is a collection of subgraphs G_i of G ($i = 1, \cdots, q$) such that all G_i are mutually edge-disjoint and G is induced by $\{G_1, \cdots, G_q\}$. If exactly *s* of the G_i is cycles, $\mathcal{Z}(G)$ is called a cycle packing of cardinality S. The family of cycle-packings of cardinality *s* is denoted by $\mathcal{C}_s(G)$. If the cardinality of a cycle packing $\mathcal{Z}(G)$ is maximum, it is called a *maximum cycle packing*. Its cardinality is denoted by $\nu(G)$. If no confusion is possible, we will write $\mathcal{Z}$ instead of $\mathcal{Z}(G)$ and $\mathcal{C}_s$ instead of $\mathcal{C}_s(G)$, respectively.

Packing edge-disjoint cycles in graphs is a classical graph-theoretical problem. There is a large amount of literature concerning conditions that are sufficient for the existence of certain numbers of disjoint cycles which may satisfy some further restrictions. An overview of related references is given in [1]. Practical applications of cycle packings are mentioned in the papers [2] [3] [4] [5]. The algorithmic problems concerning the construction of maximum edge-disjoint cycle packings are typically hard (e.g. see [6] [7] [8]). A simple greedy-type heuristic for the problem is presented in [7], which iteratively looks for cycles of small length and removes the corresponding edges from the current graph until there is no cycle left. A different approach to tackle the problem is to relate maximum cycle packings of *G* to maximum cycle packings of subgraphs of *G*. In [1] it is described how $\nu(G)$ can be obtained if *G* has a vertex cut-set *S* of cardinality *k*. In this case, $\nu(G)$ can be determined by the values $\nu(H)$ for at most $2^{\binom{k}{2}+1}$ graphs of order smaller than *G*. Let $\mu(G)$ denote the cyclomatic number of *G*, *i.e.* $\mu(G) = |E(G)| - |V(G)| + c$, where *c* denotes the number of connected components of *G*. If $\mu(G) - \nu(G)$ is known, then [9] shows how to construct *G* from one of a finite number of graphs by a series of simple graph operations. The paper [10] investigates a relation between a maximum cycle packing and maximum local traces for the case that *G* is Eulerian. For $v \in V$, an Eulerian subgraph $T(v)$ of *G* is called *a local trace* (*at v*) if every walk $W \subset T(v)$ with start vertex *v* can be extended to an Eulerian tour in $T(v)$. Traces were first considered in [11] and [12].

In [13] bounds on $\nu(G)$ are presented if *G* is a polyhedral graph. These bounds depend on the size, the order or the number of faces of *G*, respectively. Polyhedral graphs are constructed that attain these bounds.

In the present paper, we will consider even graphs and tackle the cycle packing problem by a dynamic programming approach. The main idea is, instead of regarding the length $l(C)$ of a cycle $C \subset G$, to consider its square $l^2(C)$. Doing so, a cycle packing $\mathcal{Z}$ of cardinality *s* with cycles $\{C_1, C_2, C, \cdots, C_s\}$ can be scored by

$$L(\mathcal{Z}) = \sum_{i=1}^{s} l^2(C_i) + \left(\left|E(G)\right| - \sum_{i=1}^{s} l(C_i)\right)^2.$$

In Section 2, we prove a max-min theorem that relates a minimizer $\mathcal{Z}^*$ of L to a maximum cycle packing of G. This theorem gives reason to consider maximum cycle packing problems of G within the framework of dynamic programming. In section 3, therefore, the problem is transformed into a shortest path problem on some appropriate acyclic networks $\vec{N}$. In order to avoid unnecessary excessive calculations in $\vec{N}$, suitable bounds on the length of an optimal paths are used. These bounds can be incorporated into an A^*-algorithm. The algorithmic scheme of the procedure is presented in Section 3.2.

2. A Max-Min Theorem

Let $\nu(G) \geq 1$. A cycle packing $\mathcal{Z} \in \mathcal{C}_s(G)$ then can be represented by

$$\mathcal{Z} = \left\{C_1, C_2, \cdots, C_s, \bar{G}_s(\mathcal{Z})\right\}$$

(if $s > 0$) where the C_i are the s cycles and $\bar{G}_s(\mathcal{Z})$ is the uniquely determined remainder graph induced by $E(G) \setminus \bigcup_{i=1}^{s} E(C_i)$. If no confusion is possible, we will write $\bar{G}_s$ instead of $\bar{G}_s(\mathcal{Z})$. For $s = 0$, $\mathcal{Z}_0 = \left\{\bar{G}_0\right\} = \{G\}$. For $s < \nu(G)$, $\bar{G}_s$ might still contain cycles of G. For an even graph G, it may occur that $E(G_s) = \varnothing$ also in cases that $s < \nu(G)$. In these cases, we will write $\mathcal{Z} = \{C_1, C_2, \cdots, C_s\}$. If G is non-even, $E(\bar{G}_s) \neq \varnothing$ for all $0 \leq s \leq \nu(G)$.

For $\mathcal{Z}_s \in \mathcal{C}_s(G)$, define

$$L(\mathcal{Z}_s) = \sum_{i=1}^{s} l^2(C_i) + \left|E(\bar{G}_s)\right|^2.$$

For $s = 0$, set $L(\mathcal{Z}_0) = \left|E(G)\right|^2$.

For the purpose of proving the crucial Lemma 1, consider particular subsets $\bar{\mathcal{C}}_s$ of $\mathcal{C}_s$, defined by

1) $\bar{\mathcal{C}}_{\nu(G)} = \mathcal{C}_{\nu(G)}$

2) For $0 \leq s \leq \nu(G) - 1$, a packing $\mathcal{Z}_s \in \bar{\mathcal{C}}_s$ if and only if its reminder graph $\bar{G}_s(\mathcal{Z})$ contains a cycle.

We then get

Lemma 1. *Let G be even,* $\nu(G) \geq 2$, *and let* $G' = G \cup K_2$ *be the graph induced by G and a single edge as an additional component. For* $s \in \{0, 1, 2, \cdots, \nu(G)\}$, *define*

$$m_s(G) = \min\left\{L(\mathcal{Z}_s) \mid \mathcal{Z}_s \in \bar{\mathcal{C}}_s(G')\right\}.$$

Then

$$m_0(G) > m_1(G) > m_2(G) > \cdots > m_{\nu(G)-1} > m_{\nu(G)}.$$

Proof. It can easily be seen that $\bar{\mathcal{C}}_1 = \mathcal{C}_1$ and $\left|E(G')\right|^2 = m_0(G) > m_1(G)$.

We will use induction on $r = \nu(G)$.

$\nu(G) = 2$. Let $\mathcal{Z}_1 = \left\{C_1, \bar{G}_1'\right\} \in \bar{\mathcal{C}}_1(G')$ such that $m_1(G) = L(\mathcal{Z}_1)$. Since $\bar{G}_1'$ is the graph induced by $G' \setminus C_1$ and $\nu(G) = \nu(G') = 2$, $\bar{G}_1'$ must contain a cycle C_2 of length $l(C_2)=|E(G)|-l(C_1)\geq 2$. Obviously, $\mathcal{Z}_2=\{C_1,C_2,K_2\}\in\bar{\mathcal{C}}_2(G')$. Then $m_1(G)=L(\mathcal{Z}_1)=l^2(C_1)+l^2(C_2)+2l(C_2)+1=L(\mathcal{Z}_2)+2l(C_2)\geq m_2(G)+2l(C_2)>m_2(G)$.

Now, let $r\geq 2$ and let us assume that for all even graphs G such that $2\leq\nu(G)\leq r$ the strict inequalities $m_0(G)>m_1(G)>m_2(G)>\cdots>m_{r-1}(G)>m_r(G)$ hold.

Let G be an even graph such that $\nu(G)=r+1$. We will show $m_s(G)>m_{s+1}(G)$ for all $1\leq s\leq r$.

Let $\mathcal{Z}_s(G')=\{C_1,C_2,\cdots,C_s,\bar{G}'_s\}\in\bar{\mathcal{C}}_s(G')$ such that $L(\mathcal{Z}_s(G'))=m_s(G)$. Consider the graphs $\tilde{G}=G\setminus C_1$ and $\tilde{G}'=\tilde{G}\cup K_2$. $\tilde{G}$ is even and $s\leq\nu(\tilde{G})\leq r$. Obviously, $\mathcal{Z}=\mathcal{Z}_s(G')\setminus C_1\in\bar{\mathcal{C}}_{s-1}(\tilde{G}')$ and $L(\mathcal{Z})=m_s(G)-l^2(C_1)$. But also $L(\mathcal{Z})=\min\{L(\mathcal{Z}_{s-1})\,|\,\mathcal{Z}_{s-1}\in\bar{\mathcal{C}}_{s-1}(\tilde{G}')\}$ must hold, otherwise $\mathcal{Z}_s(G')$ would not be a minimizer of L on $\bar{\mathcal{C}}_s(G')$.

Hence, $m_{s-1}(\tilde{G})=m_s(G)-l^2(C_1)$. Applying the induction assumption to $\tilde{G}$, we then get:

$$m_s(G)=m_{s-1}(\tilde{G})+l^2(C_1)>m_s(\tilde{G})+l^2(C_1)=L(\mathcal{Z}_s^*(\tilde{G}'))+l^2(C_1)=L(\mathcal{Z}_s^*(\tilde{G}')\cup C_1)\geq m_{s+1}(G).$$

Here $\mathcal{Z}_s^*(\tilde{G}')$ is a minimizer of L on $\bar{\mathcal{C}}_{s-1}(\tilde{G}')$. From this we finally conclude $m_0(G)>m_1(G)>\cdots>m_{r-1}(G)>m_r(G)>m_{r+1}(G)$.

□

By $\mathcal{C}(G)=\bigcup_{s=0}^{\nu(G)}\mathcal{C}_s(G)$, we denote the family of all cycle packings of *G*. We get

Theorem 1. *Let G be even,* $\nu(G)\geq 1$. *Every cycle packing* $\mathcal{Z}^*$ *that minimizes L on* $\mathcal{C}(G)$ *is maximum, i.e.* $\mathcal{Z}^*\in\mathcal{C}_{\nu(G)}$.

Proof. Let $\mathcal{Z}^*=\{C_1^*,\cdots,C_s^*,\bar{G}_s^*\}\in\mathcal{C}_s(G)$ be a minimizer of L on $\mathcal{C}(G)$. We can assume that $E(\bar{G}_s^*)=\varnothing$. We will show that $s=\nu(G)$.

For this, consider the non-even graph $G'=G\cup K_3\cup K_2$ obtained by adding a component K_3 and an additional edge to *G*. Obviously, $G\cup K_3$ is even and $\nu(G\cup K_3)=\nu(G)+1\geq 2$. For the packing $\mathcal{Z}'=\{C_1^*,\cdots,C_s^*,\bar{G}_s'^*\}$ with $\bar{G}_s'^*=K_2\cup K_3$ it holds $\mathcal{Z}'\in\bar{\mathcal{C}}_s(G')$ and $m_s(G\cup K_3)\leq L(\mathcal{Z}')=L(\mathcal{Z}^*)+16$. We will show $L(\mathcal{Z}')=m_s(G\cup K_3)$.

For an arbitrary packing $\mathcal{Z}=\{C_1,\cdots,C_s,\bar{G}'_s\}\in\bar{\mathcal{C}}_s(G')$, the remainder $\bar{G}'_s$ is the disjoint union of K_2 and some even graph *H* that contains at least one cycle $C\subset G'$ of length $l(C)\geq 3$. Let $k=|E(H)|-l(C)\geq 0$.

If the component K_3 is not contained in *H*, then K_3 is one of the cycles $C_1,C_2,\cdots,C_s$, *i.e.* $\mathcal{Z}=\{K_3,C_2,\cdots,C_s,\bar{G}'_s\}$. In this case *C* is a cycle in *G*.

Consider the packing $\tilde{\mathcal{Z}}=\{C,C_2,\cdots,C_s,\tilde{\bar{G}}'_s\}\in\bar{\mathcal{C}}_s(G'),\tilde{\bar{G}}'_s=\bar{G}'_s\setminus C\cup K_3$. Then $|E(\tilde{\bar{G}}'_s)|=k+4$. We get

$$\begin{aligned}L(\mathcal{Z})-L(\tilde{\mathcal{Z}})&=3^2+(k+l(C)+1)^2-\left(l^2(C)+(k+4)^2\right)\\&=2k\cdot(l(C)+1)+2l(C)+1-8k-7\\&\geq 2\cdot k\cdot 4+2\cdot 3-8k-6\\&\geq 0\end{aligned}$$

i.e. $L(\tilde{\mathcal{Z}})\leq L(\mathcal{Z})$

We conclude, that there must be a minimizer $\tilde{\mathcal{Z}}^*=\{\tilde{C}_1^*,\tilde{C}_2^*,\cdots,\tilde{C}_s^*,\tilde{\bar{G}}_s'^*\}$ of L on

$\bar{\mathcal{C}}_s(G')$, such that $K_3 \cup K_2 \subset \bar{\tilde{G}}_s'^*$. Obviously, $\bar{\tilde{G}}_s'^* \setminus (K_3 \cup K_2) = G \setminus \bigcup_{i=1}^{s} \tilde{C}_i^*$, *i.e.* $\mathcal{Z}' = \left\{\tilde{C}_1^*, \tilde{C}_2^*, \cdots, \tilde{C}_s^*, \bar{\tilde{G}}_s\right\} \in \mathcal{C}_s(G)$ with $\bar{\tilde{G}}_s = G \setminus \bigcup_{i=1}^{s} \tilde{C}_i^*$.

We get

$$m_s(G \cup K_3) = L\left(\tilde{\mathcal{Z}}^*\right) = L(\mathcal{Z}') + 8\left|E\left(\bar{\tilde{G}}_s\right)\right| + 16 \geq L\left(\mathcal{Z}^*\right) + 16.$$

Applying Lemma 1 to the even graph $G \cup K_3$, we get for $s < \nu(G')$:

$$m_{\nu(G')}(G \cup K_3) < m_{\nu(G)}(G \cup K_3) \leq m_s(G \cup K_3),$$

where the last inequality is strict if $s < \nu(G)$.

Now, let $\hat{\mathcal{Z}}^* = \left\{K_3, \hat{C}_1^*, \hat{C}_2^*, \cdots, \hat{C}_{\nu(G)}^*, K_2\right\}$ be a maximum cycle packing that minimizes L on $\bar{\mathcal{C}}_{\nu(G')}(G')$, *i.e.* $m_{\nu(G')} = L\left(\hat{\mathcal{Z}}^*\right)$.

Then $\hat{\mathcal{Z}}^* = \left\{\hat{C}_1^*, \hat{C}_2^*, \cdots, \hat{C}_{\nu(G)}^*, \bar{G}_{\nu(G)}'\right\}$ with $\bar{G}_{\nu(G)}' = K_3 \cup K_2$ minimizes L on $\bar{\mathcal{C}}_{\nu(G)}(G')$. Obviously, $\hat{\mathcal{Z}} = \left\{\hat{C}_1^*, \hat{C}_2^*, \cdots, \hat{C}_{\nu(G)}^*\right\} \in \mathcal{C}(G)$. We get

$$L\left(\hat{\mathcal{Z}}\right) + 16 = \bar{m}_{\nu(G)}(G \cup K_3) \leq \bar{m}_s(G \cup K_3) = L\left(\mathcal{Z}^*\right) + 16,$$

where the inequality is strict, if $s < \nu(G)$.

It follows $L(\mathcal{Z}) \leq L\left(\hat{\mathcal{Z}}^*\right)$. But $\mathcal{Z}^*$ was a minimizer on $\mathcal{C}(G)$, *i.e.* $L\left(\hat{\mathcal{Z}}\right) \geq L\left(\mathcal{Z}^*\right)$. Therefore, $L\left(\hat{\mathcal{Z}}\right) = L\left(\mathcal{Z}^*\right)$ and $s = \nu(G)$. □

A maximum cycle packing $\mathcal{Z}^* = \left\{C_1^*, C_2^*, \cdots, C_{\nu(G)}^*\right\}$ of G is said to be *max-min* if it minimizes L on $\mathcal{C}(G)$. The quantity $L^*(G) := L\left(\mathcal{Z}^*\right)$ is the *max-min cycle value of G*. The determination of a max-min cycle packing $\mathcal{Z}^*$ will be called the *max-min cycles packing problem* (mmcp-problem) of G. Clearly, max-min cycle packings, in general, are not unique.

The following theorem relates the determination of $L^*(G)$ to the determination of the max-min cycle values for even subgraphs $H \subset G$.

Theorem 2. *Let G be even. Then*

$$L^*(G) = \min_{H \text{ even}, \mathcal{Z}(G \setminus H) \in \mathcal{C}(G \setminus H)} \left\{L^*(H) + L\left(\mathcal{Z}(G \setminus H)\right)\right\}.$$

Proof. Let $H \subset G$ be an even subgraph of G and $\mathcal{Z}(H) = \{C_1, C_2, \cdots, C_r\} \in \mathcal{C}(H)$ be max-min.

A packing $\mathcal{Z}(G \setminus H) = \left\{C_{r+1}, C_{r+2}, \cdots, C_s, \overline{(G \setminus H)}_{(s-r)}\right\} \in \mathcal{C}(G \setminus H)$ then induces a packing $\mathcal{Z} = \left\{C_1, C_2, \cdots, C_r, C_{r+1}, \cdots, C_s, \overline{G \setminus H}_{(s-r)}\right\} \in \mathcal{C}(G)$. We then get

$$L^*(G) \leq L(\mathcal{Z}) = L\left(\mathcal{Z}(H) \cup \mathcal{Z}(G \setminus H)\right) = L^*(H) + L\left(\mathcal{Z}(G \setminus H)\right).$$

and conclude

$$L^*(G) \leq \min_{H \text{ even}, \mathcal{Z}(G \setminus H) \in \mathcal{C}(G \setminus H)} \left\{L^*(H) + L\left(\mathcal{Z}(G \setminus H)\right)\right\}.$$

Now, let $\mathcal{Z}^*(G) = \left\{C_1^*, C_2^*, \cdots, C_{\nu(G)}^*\right\}$ be max-min and let H^* and $G \setminus H^*$ be induced by $\mathcal{Z}\left(H^*\right) = \left\{C_1^*, C_2^*, \cdots, C_r^*\right\}$ and $\mathcal{Z}\left(G \setminus H^*\right) = \left\{C_{r+1}^*, \cdots, C_{\nu(G)}^*\right\}$, respectively. The packings $\mathcal{Z}\left(H^*\right)$ and $\mathcal{Z}\left(G \setminus H^*\right)$ must also be max-min. We get

$$\begin{aligned} L^*(G) &= L(\mathcal{Z}^*) = L^*(H^*) + L^*(G \setminus H^*) \\ &= L^*(H^*) + L(\mathcal{Z}(G \setminus H^*)) \\ &\geq \min_{H \text{ even}, \, \mathcal{Z}(G\setminus H) \in \mathcal{C}(G\setminus H)} \left\{ L^*(H) + L(\mathcal{Z}(G \setminus H)) \right\}. \end{aligned}$$

□

The proof of Theorem 2 immediately induces

Corollary 1.

1) $L^*(G) \leq L^*(H) + L^*(G \setminus H)$, for every even subgraph H of G.

2) $L^*(G) = \min_{C \subset G, C \text{ cycle}} \left\{ l^2(C) + L^*(G \setminus C) \right\}$.

3. A Shortest Path Approach for the MMCP-Problem

Theorem 2 gives reason to treat the mmcp-problem as a shortest path problem within a suitable weighted acyclic network $\vec{N} = (X, U, \gamma)$.

3.1. The MMCP-Network $\vec{N}$

Let the edges in E be labelled, *i.e.* $E(G) = \{e_1, e_2, \cdots, e_m\}$. In a canonical way, a subgraph $H = (V(H), E(H)) \subset G$ is determined by its $\{0,1\}$ incidence vector

$$s(H) = (s_1, s_2, \cdots, s_m),$$

i.e.,

$$s_i = \begin{cases} 1, & \text{if } e_i \in E(H) \\ 0, & \text{if } e_i \notin E(H) \end{cases}$$

Let $i_1^*(H) = \min\{k \mid s_k(H) = 1\}$ and $i_0^*(H) = \min\{k \mid s_k(H) = 0\}$.

We will identify the set X of nodes[1] in $\vec{N}$ with the set of even subgraphs of G. Each node $x \in X$ corresponds to some specific even subgraph H of G (we will write $H \in X$). Nodes in $\vec{N}$ are also assigned to *stages* $0, 1, 2, \cdots$, *i.e.* $X = X_0 \cup X_1 \cup X_2 \cup \cdots$.

For the construction of $\vec{N}$, the nodes and edges are defined inductively:

- The unique node in X_0 corresponds to the subgraph G_0 of G with empty edge set. Assume that the set of nodes X_{j-1} is given. Then a node belongs to X_j if there is $G_{j-1} \in X_{j-1}$ such that

$$C_j = G_j \setminus G_{j-1} \text{ is a cycle with } i_1^*(C_j) = i_0^*(G_{j-1}).$$

We call $G_j = G_{j-1} \cup C_j$ to be a *successor* of G_{j-1}. The set of all successors of G_{j-1} is called an *expansion of* G_{j-1}, and a specific successor G_j is *generated by expanding* G_{j-1}.

- An edge in U corresponds to some specific cycle in G. Edges exist only between nodes at consecutive stages. An edge $(x_{j-1}, x_j) \in U$ if and only if for the corresponding subgraphs G_j is a successor of G_{j-1}.
- As edge weights we set $\gamma(x_{j-1}, x_j) := l^2(C_j)$.

[1]For $\vec{N}$ we will use "nodes", in G we use "vertices".

Clearly, $\vec{N}=(X,U,\gamma)$ is acyclic and the number of stages in $\vec{N}$ cannot exceed $\nu(G)+1$. An even subgraph $H\subset G$ *is reachable* in $\vec{N}$ if there is a path $P(H)\subset\vec{N}$ with starting node G_0 and end node *H*. In a canonical way, any path $P(H)$ induces a cycle packing $\mathcal{Z}=\{C_1,C_2,\cdots,C_s,G\setminus H\}\in\mathcal{C}_s(G)$. The set $\{C_1,C_2,\cdots,C_s\}$ describes the cycles used in the successive expansions of the corresponding nodes.

Obviously, G is reachable in $\vec{N}$, but not all even subgraphs of G have this property. Hence, not every cycle packing $\mathcal{Z}\in\mathcal{C}(G)$ is induced by some paths $P(H)\in\vec{N}$. We get

Lemma 2. *Let* $H\subset G$ *be reachable in* $\vec{N}$ *and* $\mathcal{Z}(H)=\{C_1,C_2,\cdots,C_s\}$ *be a cycle-packing of H of cardinality s. Then there is a path* $P(H)$ *in* $\vec{N}$ *that induces* $\mathcal{Z}(H)$.

Proof. Let $\mathcal{Z}(H)=\{C_1,C_2,\cdots,C_s\}$ be a cycle packing of H of cardinality $s\le\nu(H)$. Without loss of generality, we can assume that the cycles in $\mathcal{Z}(H)$ are ordered such that $i_1^*(C_1)<i_1^*(C_2)<\cdots<i_1^*(C_s)$. Let $1\le j\le s$ and G_{j-1} be the even subgraph of H induced by $\{C_1,C_2,\cdots,C_{j-1}\}$. Since the cycles are mutually edge-disjoint, the number $i_1^*(C_j)$ coincides with $i_0^*(G_{j-1})$. Therefore, G_j is generated by an expansion of G_{j-1}, *i.e.* there is a path $P(H)\in\vec{N}$ that induces $\mathcal{Z}(H)$. Moreover, if $\mathcal{Z}(H)=\{C_1,C_2,\cdots,C_s\}$ and $\bar{\mathcal{Z}}(H)=\{\bar{C}_1,\bar{C}_2,\cdots,\bar{C}_r\}$ are two different cycle-packings of *H*, then the corresponding paths in $\vec{N}$ must be different. □

Together with Theorem 1, we conclude for the special case $H=G$:

Proposition 3. $\mathcal{Z}^*$ *is a max-min cycle packing of G if and only if* $\mathcal{Z}^*$ *corresponds to a shortest path* $P^*(G)$ *in* $\vec{N}$.

In order to reduce the number of graphs that have to be expanded within the algorithmic procedure, those subgraphs H in $\vec{N}$ have to be identified that cannot be contained in $P^*(G)$. Such an identification, preferably, should be done as early as possible in the calculations. The following proposition gives conditions for such a situation. They can be checked during the shortest path procedure. If such a condition is satisfied, H must never be expanded.

Proposition 4. *Let* $P^*(G)$ *be a shortest path in* $\vec{N}$ *and let* $H\in X_{j_1},\bar{H}\in X_{j_2}$ *be reachable. If*

1) $E(\bar{H})\subset E(H)$ and $j_1<j_2$ or
2) $E(\bar{H})\subsetneq E(H)$ and $j_1=j_2$

holds, then $H\notin P^*(G)$.

Proof. Since $E(\bar{H})\subset E(H)$, $\nu(\bar{H})\le\nu(H)$ holds. Assume that $H\in P^*(G)$. Then there is a cycle packing $\mathcal{Z}(H)\subset\mathcal{Z}^*(G)$ induced by $P(H)\subset P^*(G)$. The packing $\mathcal{Z}(H)$, therefore, must be max-min. Hence, $\nu(H)=j_1$. This leads to the contradiction. □

3.2. An A^*-Shortest Path Algorithm

For an even subgraph $H\subset G$, let $l^*(H)$ denote the length of the shortest cycle in *H*, then,

$$h^*(H)=l^*(G\setminus H)\cdot\left|E(G\setminus H)\right|,$$

is a *lower bound* for the max-min cycle value $L^*(G \setminus H)$. Moreover, the function $h^*(\cdot)$ is a monotonous node potential on $\vec{N}$ in the following sense

Lemma 3. *For* $j \geq 1$, *let* G_{j-1}, G_j *be two even subgraphs of G adjacent in* $\vec{N}$ *such that* $G_j = G_{j-1} \cup C_j$. *Then*

$$h^*(G_{j-1}) \leq l^2(C_j) + h^*(G_j).$$

Proof. Let l^*_{j-1} and l^*_j be the lengths of the shortest cycles in $G \setminus G_{j-1}$ and $G \setminus G_j$, respectively. Then

$$\begin{aligned} h^*(G_{j-1}) &= l^*_{j-1} \cdot \left(\left|E\left(G \setminus G_{j-1}\right)\right|\right) = l^*_{j-1} \cdot \left(|E| - \left|E\left(G_{j-1}\right)\right|\right) \\ &\leq l^*_{j-1} \cdot \left(|E| - \left|E\left(G_{j-1}\right)\right|\right) + \left|E\left(G_j \setminus G_{j-1}\right)\right|^2 - l^*_{j-1} \cdot \left|E\left(G_j \setminus G_{j-1}\right)\right| \\ &= \left|E\left(G_j \setminus G_{j-1}\right)\right|^2 + l^*_{j-1}\left[|E| - \left|E\left(G_{j-1}\right)\right| - \left|E\left(G_j \setminus G_{j-1}\right)\right|\right] \\ &= l^2(C_j) + l^*_{j-1} \cdot \left[|E| - \left(\left|E\left(G_{j-1} \cup \left(G_j \setminus G_{j-1}\right)\right)\right|\right)\right] \\ &= l^2(C_j) + l^*_{j-1} \cdot \left|E\left(G \setminus G_j\right)\right| \\ &\leq l^2(C_j) + l^*_j \cdot \left|E\left(G \setminus G_j\right)\right| = l^2(C_j) + h^*(G_j). \end{aligned}$$

The last inequality is true, since $G \setminus G_j \subset G \setminus G_{j-1}$. Hence, $l^*_{j-1} \leq l^*_j$. □

In [14], it is described how information of a monotonous node potential could be incorporated into a searching strategy for the shortest path procedure. Such an A^*-search algorithm constructs $\vec{N}$ successively and expands only such nodes that are candidates for a shortest path $P^*(G)$.

The A^* procedure essentially manages two sets of nodes in $\vec{N}$ and updates several quantities:

- *X*: even subgraphs of *G* that are candidates to determine $P^*(G)$.[2]
- *L*: subgraphs that are already expanded. For $H \in L$, a shortest path $P^*(H)$ is already constructed.
- $\nu(H) = |\mathcal{Z}(H)|$: index of the stage in $\vec{N}$ at which *H* is considered $H \in X_\nu$. If A^* terminates, $\nu(G)$ is the cardinality of a maximum cycle packing.
- $g^*(H) := L(\mathcal{Z}(H))$: (currently) the shortest length of a path $P(H)$. If $H \in L$, then $g^*(H) = L^*(H)$. If A^* stops, $g^*(G) = L^*(G)$.
- $h^*(H)$: monotonous node potential.

The scheme of such an A^*-search is outlined as follows.

Algorithmic scheme of A^* for the mmcp-problem

Input:	Even graph $G = (V, E)$, $\nu(G) \geq 2$ Labeling of E, i.e. $E = \{e_1, e_2, \ldots, e_m\}$ Monotonous node potential h^*
Output:	Max-min cycle packing $\mathcal{Z}^*(G), \nu(G), L^*(G)$
Initialisation:	Set L := ∅ , X := $\{C \mid C$ is a cycle in $G, i_1^*(C) = 1\}$. For each $H \in$ X: set $\mathcal{Z}(H) := \{H\}$, $\nu(H) := 1$, $g^*(H) := \lvert E(H)\rvert^2$. Go to step 1.

[2]We will write "$H \in X$", indicating that the node that corresponds to *H* belongs to *X*.

step 1: Selection of H to expand.
Determine $\bar{H} \in \mathrm{X}$ such that $g^*(\bar{H}) + h^*(\bar{H})$ is minimum.
Set $H := \bar{H}$.
Set $\mathrm{L} = \mathrm{L} \cup H$.
Go to step 2.

step 2: Expansion of H
Set $\mathrm{X} := \mathrm{X} \setminus \{H\}, \quad \mathrm{L} := \mathrm{L} \cup \{H\}$.
Determine
$\mathrm{C} := \{C_1, C_2, \ldots, C_r | C_j$ is a cycle in $G \setminus H$ s.t. $i_0^*(H) = i_1^*(C_j)\}$
Set $j := 1$.
Go to step 3

step 3: Update of X; elimination of superfluous nodes.
Set $H' := H \cup C_j$.
If $H' = G$,
set $\mathcal{Z}^*(G) := \mathcal{Z}(H) \cup C_j$, set $\nu(G) := \nu(H) + 1$, set $L^*(G) := L^*(H) + l^2(C_j)$
STOP.
Determine $\mathrm{H}_1 := \{\tilde{H} \in \mathrm{X} | H' \subset \tilde{H}\}$, $\mathrm{H}_2 := \{\tilde{H} \in \mathrm{X} | \tilde{H} \subset H'\}$

if $\mathrm{H}_1 = \emptyset$: Set $\mathrm{X} := \mathrm{X} \cup H'$

if $\mathrm{H}_1 \neq \emptyset$: For all $\tilde{H} \in \mathrm{H}_1$:
if $\nu(\tilde{H}) < \nu(H) + 1$, or if $\nu(\tilde{H}) = \nu(H) + 1$ and $\tilde{H} \neq H'$,
Set $\mathrm{X} := (\mathrm{X} \cup H') \setminus \tilde{H}$
Set $g^*(H') := g^*(H) + l^2(C_j)$, $\mathcal{Z}(H') := \mathcal{Z}(H) \cup C_j$, $\nu(H') := \nu(H) + 1$.

if $\mathrm{H}_2 \neq \emptyset$: For all $\tilde{H} \in \mathrm{H}_2$:
if $\nu(H) + 1 < \nu(\tilde{H})$, or if $\nu(\tilde{H}) = \nu(H) + 1$ and $\tilde{H} \neq H'$,
Set $\mathrm{X} := \mathrm{X} \setminus H'$

Go to step 4

step 4: If $j = r$,
go to step 1
If $j < r$
set $j := j + 1$,
go to step 3.

The determination of $\bar{H}$ in step 1 requires the determination of the girth of $G \setminus H$. This can efficiently be done by the shortest paths procedures. For the expansion of H in step 2, the value $i_0^*(H)$ and the set C of all cycles in $G \setminus H$ that contain $e_{i_0^*} = (u, v)$ must be generated. This makes it necessary to identify all simple paths between u and v in the graph $G \setminus H$. Typically, this subproblem is attacked by using DFS procedures. In general, it is NP-hard ([15]).

Step 3 incorporates the stopping rule ($H \cup C_j = G$) and the elimination of superfluous nodes (and sub-paths) according to Prop. 4.

Coming from step 2, A^* terminates in step 3 if G is expanded from some H for the first time. Since it is possible that the graph G may appear in $\vec{N}$ at different stages, it must be guaranteed that A^* doesn't stop at a "wrong" node that corresponds to G.

Proposition 5. *If* A^* *stops, then* $\mathcal{Z}^*(G)$ *is a max-min cycle packing of* G.

Proof. Assume, A^* terminates and $|\mathcal{Z}^*(G)| < \nu(G)$. In this case $G = G_s$ corresponds to a node at stage $s < \nu(G)$. Since the corresponding cycle packing $\mathcal{Z}^*(G_s) = \{C_1, C_2, \cdots, C_s\}$ is not maximum, $L(\mathcal{Z}^*(G_s)) > L^*(G)$. Let $P(G_s)$ be the path in $\vec{N}$ induced by $\mathcal{Z}^*(G_s)$ and let $G' := G_s \setminus C_s$ be the predecessor of G_s on that path.

The only node in $\vec{N}$ at stage $\nu(G)$ corresponds to G ($G_\nu := G$). For the shortest path $P^*(G_\nu)$, Theorem 1 gives $L^*(G) = g^*(P^*(G_\nu)) < g^*(P(G_s))$.

Let H^* be the last common node of the paths $P(G_s)$ and $P^*(G_v)$. Since H^* is expanded at some iteration of A^*, there must be a subgraph $\bar{H}$ on the subpath $P(H^*, G_v)$ of $P^*(G_v)$ that belongs to X when A^* terminates. Since this node is never selected in step 1 until A^* stops (otherwise it would have been deleted from X in step 2), the inequality $g^*(\bar{H}) + h^*(\bar{H}) \geq g^*(G') + h^*(G')$ must hold. But then

$$\begin{aligned} L^*(G) = L^*(G_v) &= L^*(\bar{H}) + L^*(G_v \setminus \bar{H}) \\ &\geq L^*(\bar{H}) + h^*(\bar{H}) = g^*(\bar{H}) + h^*(\bar{H}) \\ &\geq g^*(G') + h^*(G') = g^*(G') + l^2(C_s) = L(\mathcal{Z}^*(G)) \end{aligned}$$

is a contradiction. Hence, A^* terminates at stage $v(G)$, *i.e.* $\mathcal{Z}^*$ is maximum. □

Acknowledgements

We thank the Editor and the referees for their comments.

References

[1] Harant, J., Rautenbach, D., Recht, P., Schiermeyer, I. and Sprengel, E.-M. (2010) Packing Disjoint Cycles over Vertex Cuts. *Discrete Mathematics*, **310**, 1974-1978. http://dx.doi.org/10.1016/j.disc.2010.03.009

[2] Antonakopulos, S. and Zhang, L. (2011) Approximation Algorithms for Grooming in Optical Network Design. *Theoretical Computer Science*, **412**, 3783-3751.

[3] Bafna, V. and Pevzner, P.A. (1996) Genome Rearrangement and Sorting by Reversals. *SIAM Journal on Computing*, **25**, 272-289. http://dx.doi.org/10.1137/S0097539793250627

[4] Fertin, G., Labarre, A., Rusu, I., Tannier, É. and Vialette, S. (2009) Combinatorics of Genome Rearrangement. MIT Press, Cambridge. http://dx.doi.org/10.7551/mitpress/9780262062824.001.0001

[5] Kececioglu, J. and Sankoff, D. (1997) Exact and Approximation Algorithms for Sorting by Reversals with Application to Genome Rearrangement. *Algorithmica*, **13**, 180-210. http://dx.doi.org/10.1007/BF01188586

[6] Caprara, A. (1999) Sorting Permutations by Reversals and Eulerian Cycle Decompositions. *SIAM Journal on Discrete Mathematics*, **12**, 91-110.

[7] Caprara, A., Panconesi, A. and Rizzi, R. (2003) Packing Cycles in Undirected Graphs. *Journal of Algorithms*, **48**, 239-256. http://dx.doi.org/10.1016/S0196-6774(03)00052-X

[8] Krivelevich, M., Nutov, Z., Salvatpour, M.R., Yuster, J. and Yuster, R. (2007) Approximation Algorithms and Hardness Results for Cycle Packing Problems. *ACM Transactions on Algorithms*, **3**, Article No. 48.

[9] Harant, J., Rautenbach, D., Recht, P. and Regen, F. (2010.) Packing Edge-Disjoint Cycles in Graphs and the Cyclomatic Number. *Discrete Mathematics*, **310**, 1456-1462, http://dx.doi.org/10.1016/j.disc.2009.07.017

[10] Recht, P. and Sprengel, E.-M. (2015) Maximum Cycle Packing in Eulerian Graphs Using Local Traces. *Discussiones Mathematicae Graph Theory*, **35**, 121-132. http://dx.doi.org/10.7151/dmgt.1785

[11] Ore, O. (1951) A Problem Regarding the Tracing of Graphs. *Elemente der Mathematik*, **6**, 49-53.

[12] Baebler, F. (1953) Über eine spezielle Klasse Euler'scher Graphen. *Commentarii Mathema-*

tici Helvetici, **27**, 81-100. http://dx.doi.org/10.1007/BF02564555

[13] Recht, P. and Stehling, S. (2014) On Maximum Cycle Packings in Polyhedral Graphs. *Electronic Journal of Graph Theory and Applications*, **2**, 18-31. http://dx.doi.org/10.5614/ejgta.2014.2.1.2

[14] Hart, P.E., Nilsson, N. and Raphael, B. (1968) A Formal Basis for the Heuristic Determination of Minimum Cost Paths. *IEEE Transactions on Systems Science and Cybernetics*, **4**, 100-107. http://dx.doi.org/10.1109/TSSC.1968.300136

[15] Karger, D., Motwani, R. and Ramkumar, G.D.S. (1997) On Approximating the Longest Path in a Graph. *Algorithmica*, **18**, 82-98. http://dx.doi.org/10.1007/BF02523689